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Construction of an Exact Solution of Special Type for the 3D Problem of Thermosolutal Convection in Two Layered System

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Abstract. A three-dimensional joint flow of a liquid and a binary mixture with common interface is considered. It is assumed that the temperature field in the layers has a quadratic distribution. An exact solution of certian model problem is constructed, explicit expression for all the required function are obtained using a specific closing relation.

Keywords: Oberbeck–Boussinesq approximation, surface energy, binary mixture.

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The theory of the motion of liquid media with interfaces or with a free boundary attracts a lot of attention, due to the numerous technological applications. Thermocapillary flows induced by the surface tension forces arising at the interface can influence the movement of the liquid in the volume. To take into account various factors affecting the fluid dynamics, it is necessary to use new mathematical models and to formulate initial-boundary value problems. Therefore, there is a need to construct nontrivial exact solutions, to study stability issues, and to develop efficient numerical algorithms for such models.

Work [1] presents various formulations of problems on the motion of two immiscible liquids with a common interface. Possible generalizations and consequences of the formulations of the arising initial-boundary value problems are discussed. Exact solutions obtained in the frame of the different statements of the problem are the useful tool to study features of convection in fluidic systems. The works [2, 3] describe the construction of exact solutions of the classical convection equations which describe flows with evaporation, in the two-dimensional and three-dimensional cases. Basic characteristics obtained with the help of the exact solutions allowed one to analyze the impact of different factors affecting the convective regime structure. In [4], a three-dimensional flow of a viscous incompressible fluid in a single-layer system with a non-uniform temperature distribution at free boundaries was studied.

In this paper, we construct an exact solution to the problem describing a three-dimensional flow in a liquid-binary mixture system with a common interface. The Navier–Stokes equations in the Oberbeck–Boussinesq approximation are used as a mathematical model. Thermal and diffusion processes are described by heat and mass transfer equations.

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1. Problem statement and form of exact solution

We consider flow of two viscous incompressible media (liquid and binary mixture) filling the plane channel and having the common interface Γ. The domain occupied by the liquid is denoted by $\Omega_1=\{(x,y,z):|x|<\infty,|y|<\infty,-l_1<z<0\}$ and $\Omega_2=\{(x,y,z):|x|<\infty,|y|<\infty,0<z<l_2\}$ is domain filled by the binary mixture.

For description of the motion in regions Ω_j $(j=1,2)$ we use the Boussinesq approximation. Indexes $j=1$ and $j=2$ refer to the lower liquid and upper binary mixture, respectively. We assume that the temperature and the concentration slightly differ from constant mean values therefore the Oberbeck–Boussinesq approximation is valid. The state equation is taken in the following form

$$\rho_j=\rho_{0j}(1-\beta_j^{\theta}\theta-\beta_j^{c}c),$$

where ρ_{0j} is the characteristic density of j the medium corresponding to the mean values of the temperature and concentration in the layer, θ and c are the functions giving deviations of the temperature and concentration, respectively, from their mean values (c corresponds to the concentration of light component in the binary mixture), β_j^{θ} and β_j^{c} are the temperature and concentration expansion coefficients; $\beta_1^c=0$. Then, the equations describing the convective flows in the two-layer system incompressible media can be written in the form

$$\begin{gathered}\mathbf{u}_{jt}+(\mathbf{u}_j\cdot\nabla)\mathbf{u}_j=-\frac{1}{\rho_{0j}}\nabla p_j+\nu_j\Delta\mathbf{u}_j-\mathbf{g}(\beta_j^{\theta}(\theta_j-\theta_{0j})+\beta_j^{c}(c_j-c_{0j})),\\ \theta_{jt}+\mathbf{u}_j\cdot\nabla\theta_j=\chi_j\Delta\theta_j,\\ c_t+\mathbf{u}_2\cdot\nabla c=D\Delta c+\alpha D\Delta\theta_2,\\ \operatorname{div}\mathbf{u}_j=0.\end{gathered}\tag{1}$$

where $\mathbf{u}_j=(u_j,v_j,w_j)$ is the velocity vector, p_j is the pressure deviation from hydrostatic pressure, $\mathbf{g}=(0,0,-g)$ is vector of the gravity acceleration, $\nu_j=\mu_j/\rho_j$ is the kinematic viscosity, χ_j is the thermal diffusivity, D is the coefficient diffusion and α is the thermal diffusion parameter. All thermophysical parameters are assumed to be constant and correspond to the average values temperature and concentration.

The boundary conditions on solid walls are

$$z=-l_1:\quad \mathbf{u}_1=0,\quad \theta_1=\theta_{10}(x,y);$$

$$z=l_2:\quad \mathbf{u}_2=0,\quad \theta_{2n}=0,\quad c_n=0.$$

On the interface surface $z=0$ the following conditions are set:

$$\mathbf{u}_1=\mathbf{u}_2,\quad \mathbf{u}\cdot\mathbf{n}=V_{1n},\quad (P_2-P_1)\mathbf{n}=2\sigma H_1\mathbf{n}+\bigtriangledown_{11}\sigma;$$

$$\theta_1=\theta_2,\quad k_2\theta_{2n}-k_1\theta_{1n}=æ_1\theta\bigtriangledown_{11}\mathbf{u}_2,$$

$$c_n+\alpha\theta_{2n}=0,$$

where k_j is coefficient of thermal conductivity, $\mathbf{n}$ is unit vector normal to the interface Γ and it directed into the domain Ω_1 to Ω_2, V_{1n} is velocity of motion of the surface Γ by $\mathbf{n}$, $P_j=-p_jE+2\rho_j\nu_jD_j$ is the stress tensor, E is unit tensor, H_1 is the mean curvature of the surface Γ, $\bigtriangledown_{11}=\bigtriangledown-(\mathbf{n}\cdot\bigtriangledown)\mathbf{n}$ is the operator of surface gradient, $\sigma=\sigma(\theta,c)$ is the coefficient of surface tension at the interface. For most mixtures, the linear law provides a good approximation of this dependence $\sigma(\theta_1, c) = \sigma^0 - æ_1(\theta - \theta_0) - æ_2(c - c_0)$ with the constants $\sigma^0 > 0$, $æ_1 > 0$, $æ_2$.

Let us assume that solution of systems (1) has the form [4]

$$u_j = (f_j(z,t) + h_j(z,t))x, \quad v_j = (f_j(z,t) - h_j(z,t))y, \quad w_j = -2\int_{z_0}^{z} f_j(z_1,t)dz_1;$$
$$\theta_j = a_j(z,t)x^2 + b_j(z,t)y^2 + \bar{\theta}_j(z,t), \tag{2}$$
$$c = M(z,t)x^2 + N(z,t)y^2 + C(z,t).$$

Substitution of solution (2) into system of equations (1) leads to the following system of equations describing heat transfer in layers::

$$a_{jt} + 2a_j(f_j + h_j) - 2a_{jz}\int_0^z f_j(z_1,t)dz_1 = \chi_j a_{jzz},$$
$$b_{jt} + 2b_j(f_j - h_j) - 2b_{jz}\int_0^z f_j(z_1,t)dz_1 = \chi_j b_{jzz}, \tag{3}$$
$$\bar{\theta}_{jt} - 2\bar{\theta}_{jz}\int_0^z f_j(z_1,t)dz_1 = \chi_j\bar{\theta}_{jzz} + 2\chi_j(a_j + b_j).$$

From the equation of momentum and continuity, we obtain

$$f_{jt} + f_j^2 + h_j^2 - 2f_{jz}\int_0^z f_j(z_1,t)dz_1 + s_{j1}(t) =$$
$$= \nu_j f_{jzz} - g\int_0^z \Big(a_j(z_1,t) + b_j(z_1,t) + \delta_j(M(z_1,t) + N(z_1,t))\Big)dz_1,$$
$$h_{jt} + 2f_jh_j - 2h_{jz}\int_0^z f_j(z_1,t)dz_1 + s_{j2}(t) = \tag{4}$$
$$= \nu_j h_{jzz} - g\int_0^z \Big(a_j(z_1,t) - b_j(z_1,t) + \delta_j(M(z_1,t) - N(z_1,t))\Big)dz_1,$$

where $s_{j1}(t)$, $s_{j2}(t)$ are arbitrary functions. Physically, they represent additional pressure gradients. Here and below, we assume that $\delta_1 = 0$.

Equations for determining the functions describing the distribution of concentration in a layer with a binary mixture have the following form:

$$M_t + 2M(f_2 + h_2) - 2M_z\int_0^z f_2(z_1,t)dz_1 = DM_{zz} + \alpha D a_{2zz},$$
$$N_t + 2N(f_2 - h_2) - 2N_z\int_0^z f_2(z_1,t)dz_1 = DN_{zz} + \alpha D b_{2zz}, \tag{5}$$
$$C_t - 2C_z\int_0^z f_2(z_1,t)dz_1 = DC_{zz} + 2D(M + N) + \alpha D\Big(\bar{\theta}_{2zz} + 2(a_2 + b_2)\Big).$$

The pressure functions in the layers are determined by the formulas:

$$\frac{1}{\rho_j}P_j = x^2\Big(g\int_0^z(\beta_j^\theta a_j(z_1,t) + \delta_j\beta^c M(z_1,t))dz_1 + n_{j1}\Big) +$$
$$+ y^2\Big(g\int_0^z(\beta_j^\theta b_j(z_1,t) + \delta_j\beta^c N(z_1,t))dz_1 + n_{j2}\Big) -$$
$$-2\nu_j f_j - gz + g\int_0^z(\beta_j^\theta\bar{\theta}(z_1,t) + \delta_j\beta^c C(z_1,t))dz_1 - 2\Big(\int_0^z f_j(z_1,t)dz_1\Big)^2 + q_{j0},$$

q_{j0} are arbitrary constants.

Substituting solution (2) into the boundary conditions, we obtain the following relations being the result of the no-slip condition for velocities

$$f_1(-l_1) = h_1(-l_1) = 0, \quad f_2(l_2) = h_2(l_2) = \int_0^{l_2} f_2(z_1,t)dz_1 = 0. \tag{6}$$

We assume that the upper wall is thermally insulated and impenetrable, and at the lower boundary the temperature distribution has the form $a_1 = a_{10}x^2 + b_{10}y^2 + T_{10}$. Then,

$$\begin{aligned} z = -l_1: &\quad a_1 = a_{10}, \quad b_1 = b_{10}, \quad \bar\theta_1 = T_{10}, \\ z = l_2: &\quad a_{2z} = b_{2z} = \bar\theta_{2z} = 0, \quad M_z = N_z = C_z = 0. \end{aligned} \tag{7}$$

Boundary conditions at the interface $z = 0$ are written as:

$$f_1 = f_2, \quad h_1 = h_2, \quad a_1 = a_2, \quad b_1 = b_2, \quad \bar\theta_1 = \bar\theta_2,$$

$$\begin{aligned} \rho_2\nu_2 f_{2z} - \rho_1\nu_1 f_{1z} &= -æ_1\Big(a_2 + b_2\Big) - æ_2\Big(M + N\Big), \\ \rho_2\nu_2 h_{2z} - \rho_1\nu_1 h_{1z} &= -æ_1\Big(a_2 - b_2\Big) - æ_2\Big(M - N\Big), \end{aligned} \tag{8}$$

$$\begin{gathered} k_2 a_{2z} - k_1 a_{1z} = 2æ_1 a_1 f_1, \quad k_2 b_{2z} - k_1 b_{1z} = 2æ_1 b_1 f_1, \\ k_2\bar\theta_{2z} - k_1\bar\theta_{1z} = 2æ_1\bar\theta_1 f_1, \end{gathered} \tag{9}$$

$$M_z + \alpha a_{2z} = 0, \quad N_z + \alpha b_{2z} = 0, \quad C_z + \alpha\bar\theta_{2z} = 0.$$

The kinematic condition on the immovable and non-deformable interface is equivalent to the integral equality

$$\int_{-l_1}^0 f_1(z_1,t)dz_1 = 0.$$

For the complete definiteness of the problem posed, it is necessary to set additional integral conditions

$$\int_{-l_1}^0 h_1(z_1,t)dz_1 = 0, \quad \int_0^{l_2} h_2(z_1,t)dz_1 = 0. \tag{10}$$

2. Solution of a stationary problem

The stationary case of problem (3)–(10) is considered. We introduce the nondimensional variables $x = \xi l_j, \quad y = \eta l_j, \quad z = \zeta l_j,$

$$f_j = \frac{\chi_1}{l_1^2}F_j, \quad h_j = \frac{\chi_1}{l_1^2}H_j, \quad a_j = a^*A_j, \quad b_j = a^*B_j, \quad \bar\theta_j = \theta^*T_j,$$

$$s_{ij} = \frac{\chi_1^2}{l_1^4}S_{ij}, \quad M = \frac{\beta_2^\theta a^*}{\beta^c}K_1, \quad N = \frac{\beta_2^\theta a^*}{\beta^c}K_2, \quad C = \frac{\beta_2^\theta l_1^2 a^*}{\beta^c}K_3, \quad p_j = g\beta_j^\theta a^* l_1^3\rho_1\bar p_j.$$

Here, $a^* = \max\{|a_1(-1)|, |b_1(-1)|\} > 0$, θ^* is the characteristic temperature at the point $x = 0$, $y = 0$, $z = -1$.

Then, system of equations (3)–(5) in dimensionless variables takes the following form:

$$
\begin{aligned}
&2A_j(F_j+H_j)-2A_{j\zeta}\int_0^\zeta F_j(\zeta_1)d\zeta_1=\frac{\chi_j}{\chi_1}A_{j\zeta\zeta},\\
&2B_j(F_j-H_j)-2B_{j\zeta}\int_0^\zeta F_j(\zeta_1)d\zeta_1=\frac{\chi_j}{\chi_1}B_{j\zeta\zeta},\\
&-2T_{j\zeta}\int_0^\zeta F_j(\zeta_1)d\zeta_1=\frac{\chi_j l_1^2}{\chi_1 l_j^2}T_{j\zeta\zeta}+2\frac{\chi_j d_1}{\chi_1}(A_j+B_j).
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&F_j^2+H_j^2-2F_{j\zeta}\int_0^\zeta F_j(\zeta_1)d\zeta_1+S_{j1}=\frac{\nu_j l_1^2}{\chi_1 l_j^2}F_{j\zeta\zeta}-\mathrm{G}_j\int_0^\zeta\Big(A_j+B_j+\delta_j(K_1+K_2)\Big)d\zeta_1,\\
&2F_jH_j-2H_{j\zeta}\int_0^\zeta F_j(\zeta_1)d\zeta_1+S_{j2}==\frac{\nu_j l_1^2}{\chi_1 l_j^2}H_{j\zeta\zeta}-\mathrm{G}_j\int_0^\zeta\Big(A_j-B_j+\delta_j(K_1-K_2)\Big)d\zeta_1,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&2K_1(F_2+H_2)-2K_{1\zeta}\int_0^\zeta F_2(\zeta_1)d\zeta_1=\mathrm{Le}l^2\Big(K_{1\zeta\zeta}+\psi A_{2\zeta\zeta}\Big),\\
&2K_2(F_2-H_2)-2K_{2\zeta}\int_0^\zeta F_2(\zeta_1)d\zeta_1=\mathrm{Le}l^2\Big(K_{2\zeta\zeta}+\psi B_{2\zeta\zeta}\Big),\\
&-2K_{3\zeta}\int_0^\zeta F_2(\zeta_1)d\zeta_1=\mathrm{Le}\bigg(l^2K_{3\zeta\zeta}+2(K_1+K_2)+\psi\Big(\frac{l^2}{d_1}T_{2\zeta\zeta}+2(A_2+B_2)\Big)\bigg).
\end{aligned}
\tag{13}
$$

Boundary conditions in dimensionless form will be written as follows:

$$\zeta=-1:\quad F_1=0,\quad H_1=0,\quad \int_{-1}^0 F_1(\zeta_1)d\zeta_1=0,\quad \int_{-1}^0 H_1(\zeta_1)d\zeta_1=0; \tag{14}$$

$$A_1=\alpha_1,\quad B_1=\alpha_2,\quad T_1=\alpha_3; \tag{15}$$

$$\zeta=1:\quad F_2=0,\quad H_2=0,\quad \int_0^1 F_2(\zeta_1)d\zeta_1=0,\quad \int_0^1 H_2(\zeta_1)d\zeta_1=0; \tag{16}$$

$$A_{2\zeta}=0,\quad B_{2\zeta}=0,\quad T_{2\zeta}=0,\quad K_{1\zeta}=0,\quad K_{2\zeta}=0,\quad K_{3\zeta}=0; \tag{17}$$

$$\zeta=0:\quad F_1=F_2,\quad H_1=H_2,\quad A_1=A_2,\quad B_1=B_2,\quad T_1=T_2, \tag{18}$$

$$K_{1\zeta}+\psi A_{2\zeta}=0,\quad K_{2\zeta}+\psi B_{2\zeta}=0,\quad K_{3\zeta}+\frac{\psi}{d_1}T_{2\zeta}=0, \tag{19}$$

$$lA_{2\zeta}-kA_{1\zeta}=2\frac{æ_1\chi_1}{l_1k_2}A_1F_1,\quad lB_{2\zeta}-kB_{1\zeta}=2\frac{æ_1\chi_1}{l_1k_2}B_1F_1, \tag{20}$$

$$lT_{2\zeta}-kT_{1\zeta}=2\frac{æ_1\chi_1}{l_1k_2}T_1F_1, \tag{21}$$

$$lF_{2\zeta}-\rho\nu F_{1\zeta}=-2\mathrm{M}\rho\nu\big(A_2+B_2\big)-2\mathrm{M}\rho\nu\omega\big(K_1+K_2\big), \tag{22}$$

$$lH_{2\zeta}-\rho\nu H_{1\zeta}=-2\mathrm{M}\rho\nu\big(A_2-B_2\big)-2\mathrm{M}\rho\nu\omega\big(K_1-K_2\big). \tag{23}$$

Here, the following dimensionless complexes are introduced in the problem: the Prandtl number Pr_j, the Marangoni number M, the Grashof number G_j, the Lewis number Le and other parameters and relationships determined by the formulas

$$\mathrm{Pr}_j=\frac{\nu_j}{\chi_j},\quad \mathrm{M}=\frac{æ_1a^*l_1^3}{\rho_1\nu_1\chi_1},\quad \mathrm{G}_j=\mathrm{MPr}_1\mathrm{L}_j,\quad \mathrm{L}_j=\frac{g\beta_j^\theta l_1l_j\rho_1}{æ_1},\quad \mathrm{Le}=\frac{D}{\chi_1},$$

$$\omega = \frac{æ_2\beta_2^\theta}{æ_1\beta^c},\quad d_1 = \frac{a^* l_1^2}{\theta^*},\quad \psi = \frac{\alpha\beta^c}{\beta_2^\theta},\quad l = \frac{l_1}{l_2},\quad \nu = \frac{\nu_1}{\nu_2},\quad \rho = \frac{\rho_1}{\rho_2},\quad k = \frac{k_1}{k_2}.$$

To reveal the characteristic features of the thermocapillary flow, we consider an approximate analytical solution in each of the layers. To do this, we construct an asymptotic solution of the problem in the form of a power series in the parameter M << 1:

$$F_j = \mathrm{M}F_j^0 + \mathrm{M}^2 F_j^1,\quad H_j = \mathrm{M}H_j^0 + \mathrm{M}^2 H_j^1,$$

$$A_j = A_j^0 + \mathrm{M}A_j^1,\quad B_j = B_j^0 + \mathrm{M}B_j^1,\quad T_j = T_j^0 + \mathrm{M}T_j^1, \tag{24}$$

$$K_1 = K_1^0 + \mathrm{M}K_1^1,\quad K_2 = K_2^0 + \mathrm{M}K_2^1;\quad K_3 = K_3^0 + \mathrm{M}K_3^1,\quad S_{ij} = \mathrm{M}S_{ij}^0 + \mathrm{M}^2 S_{ij}^1.$$

Substituting (24) into (11)–(13) and neglecting the terms with the parameter M, we obtain a linear system of equations. The desired functions that determine the fields of velocities, temperatures, and concentrations are found by simple integration (here and below, we omit the upper index of 0 denoting the first term of the expansion). As a result, we have

$$A_1 = C_1\zeta + C_2,\quad B_1 = C_3\zeta + C_4,$$

$$T_1 = -\frac{d_1}{3}(C_1 + C_3)\zeta^3 - d_1(C_2 + C_4)\zeta^2 + C_5\zeta + C_6,$$

$$F_1 = \frac{\mathrm{L}_1}{24}(C_1 + C_3)\zeta^4 + \frac{\mathrm{L}_1}{6}(C_2 + C_4)\zeta^3 + \frac{S_{11}}{2\mathrm{Pr}_1}\zeta^2 + C_7\zeta + C_8,$$

$$H_1 = \frac{\mathrm{L}_1}{24}(C_1 - C_3)\zeta^4 + \frac{\mathrm{L}_1}{6}(C_2 - C_4)\zeta^3 + \frac{S_{12}}{2\mathrm{Pr}_1}\zeta^2 + C_9\zeta + C_{10};$$

$$A_2 = C_{11}\zeta + C_{12},\quad B_2 = C_{13}\zeta + C_{14},$$

$$T_1 = -\frac{d_1}{3l^2}(C_{11} + C_{13})\zeta^3 - \frac{d_1}{l^2}(C_{12} + C_{14})\zeta^2 + C_{15}\zeta + C_{16},$$

$$K_1 = C_{17}\zeta + C_{18},\quad K_2 = C_{19}\zeta + C_{20},$$

$$K_3 = -\frac{1}{3l^2}(C_{17} + C_{19})\zeta^3 - \frac{1}{l^2}(C_{18} + C_{20})\zeta^2 + C_{21}\zeta + C_{22},$$

$$F_2 = \frac{\nu\mathrm{L}_2}{24l^2}(C_{11} + C_{13} + C_{17} + C_{19})\zeta^4 + \frac{\nu\mathrm{L}_2}{6l^2}(C_{12} + C_{14} + C_{18} + C_{20})\zeta^3 + \frac{\nu S_{21}}{2\mathrm{Pr}_1 l^2}\zeta^2 + C_{23}\zeta + C_{24},$$

$$H_2 = \frac{\nu\mathrm{L}_2}{24l^2}(C_{11} - C_{13} + C_{17} - C_{19})\zeta^4 + \frac{\nu\mathrm{L}_2}{6l^2}(C_{12} - C_{14} + C_{18} - C_{20})\zeta^3 + \frac{\nu S_{22}}{2\mathrm{Pr}_1 l^2}\zeta^2 + C_{25}\zeta + C_{26}.$$

The unknown constants C_n, as well as the functions S_{ij} are found from the boundary conditions. Note that conditions (20), (21) taking into account the effect of energy on the interfaces under the above assumption will be rewritten in the following form:

$$lA_{2\zeta} - kA_{1\zeta} = 2\mathrm{E}d_1 A_1 F_1,\quad lB_{2\zeta} - kB_{1\zeta} = 2\mathrm{E}d_1 B_1 F_1, \tag{25}$$

$$lT_{2\zeta} - kT_{1\zeta} = 2\mathrm{E}d_1 T_1 F_1, \tag{26}$$

where $\mathrm{E} = \dfrac{æ_1^2\theta^*}{\rho_1\nu_1 k_2}$. Conditions (22), (23) take the form:

$$lF_{2\zeta} - \rho\nu F_{1\zeta} = -2\rho\nu\Big(A_2 + B_2 - \omega(K_1 + K_2)\Big), \tag{27}$$

$$lH_{2\zeta} - \rho\nu H_{1\zeta} = -2\rho\nu\Big(A_2 - B_2 - \omega(K_1 - K_2)\Big). \tag{28}$$

3. Algorithm for computing the integration constants

To determine the integration constants, we use boundary conditions (14)–(19), given conditions (25)–(28).

From (15), (17), (18) we obtain that $C_{11} = C_{13} = C_{17} = C_{19} = 0$, $C_{24} = C_8$, $C_{26} = C_{10}$, $C_{12} = C_2$, $C_{14} = C_4$, $C_{16} = C_6$, $C_1 = C_2 - \alpha_1$, $C_3 = C_4 - \alpha_2$.

The third condition in (15) determines the connection

$$C_5 = C_6 - \frac{2d_1}{3}(C_2 + C_4) - \frac{d_1}{3}(\alpha_1 + \alpha_2) - \alpha_3.$$

Taking $T_{2\zeta} = 0$ on the wall $\zeta = 1$, we have

$$C_{15} = \frac{2d_1}{l^2}(C_2 + C_4).$$

Further, we consider conditions (19). Two of them are fulfilled identically. The third condition gives relation $C_{21} = -\frac{2\psi}{l^2}(C_2 + C_4)$.

From the condition $K_{3\zeta}(1) = 0$ we define $C_{18} = -\psi(C_2 + C_4) - C_{20}$.

In the joint solution of equations (14), the following constants are determined:

$$C_7 = 4C_8 + \mathrm{L}_1\left(\frac{C_2 + C_4}{20} + \frac{\alpha_1 + \alpha_2}{30}\right);$$
$$C_{10} = \frac{C_9}{4} - \mathrm{L}_1\left(\frac{C_2 - C_4}{80} - \frac{\alpha_1 - \alpha_2}{120}\right);$$
$$S_{11} = 6C_8\mathrm{Pr}_1 + \mathrm{Pr}_1\mathrm{L}_1\left(\frac{7(C_2 + C_4)}{20} + \frac{3(\alpha_1 + \alpha_2)}{20}\right);$$
$$S_{12} = \frac{3}{2}C_9\mathrm{Pr}_1 + \mathrm{Pr}_1\mathrm{L}_1\left(\frac{11(C_2 - C_4)}{40} + \frac{\alpha_1 - \alpha_2}{10}\right).$$

Further, conditions (16) complete the following relation:

$$C_{23} = -4C_8 - \frac{\nu\mathrm{L}_2}{12l^2}(C_2 + C_4)(\psi - 1);$$
$$C_{25} = -4C_{10} - \frac{\nu\mathrm{L}_2}{12l^2}\Big(2C_{20} + \psi(C_2 + C_4) - (C_2 - C_4)\Big);$$
$$S_{21} = \frac{6l^2}{\nu}C_8\mathrm{Pr}_1 + \frac{1}{2}\mathrm{Pr}_1\mathrm{L}_2(\psi - 1)(C_2 + C_4);$$
$$S_{22} = \mathrm{Pr}_1\mathrm{L}_2C_{20} + \frac{6\mathrm{Pr}_1 l^2}{\nu}C_{10} + \frac{\mathrm{L}_2\mathrm{Pr}_1}{2}\Big(\psi(C_2 + C_4) - (C_2 - C_4)\Big).$$

Dynamic condition projection (28) results in relation

$$C_9 = -\frac{\nu(\mathrm{L}_2 + 24l\rho\omega)}{6l(l + \rho\nu)}C_{20} + \frac{2\rho\nu}{l + \rho\nu}\Big(C_2 - C_4 - \psi\omega(C_2 + C_4)\Big) +$$
$$+ \frac{l\mathrm{L}_1}{60(l + \rho\nu)}\Big(3(C_2 - C_4) + 2(\alpha_1 - \alpha_2)\Big) - \frac{\nu\mathrm{L}_2}{12l(l + \rho\nu)}\Big(\psi(C_2 + C_4) - (C_2 - C_4)\Big).$$

Condition of heat transfer (26) at the interface $\zeta = 0$ determines the constant C_6

$$C_6 = \frac{2d_1(3 + kl)(C_2 + C_4)}{3l(k + 2\mathrm{E}d_1C_8)} + \frac{kd_1(\alpha_1 + \alpha_2)}{3(k + 2\mathrm{E}d_1C_8)} + \frac{k\alpha_3}{k + 2\mathrm{E}d_1C_8}.$$

Finally, we determine the constants $C_2, C_4, C_8, C_{20}, C_{22}$ when substituting into boundary conditions (25), (27). These conditions are not enough. Therefore, we introduce the condition that determines the distribution of the average concentration in the layer Ω_2

$$\int_0^1 (K_1\xi^2 + K_2\eta^2 + K_3)d\zeta = \gamma_1\xi^2 + \gamma_2\eta^2 + \gamma_3.$$

From this, we obtain the following relations

$$C_{20} = \gamma_2, \quad C_{22} = \frac{2(\gamma_1+\gamma_2)}{3l^2} - \gamma_3, \quad C_2 = \frac{\alpha_1}{\alpha_2}C_4,$$
$$C_4 = -\frac{\alpha_2(\gamma_1+\gamma_2)}{\psi(\alpha_1+\alpha_2)}, \quad C_8 = -\frac{k}{2\mathrm{E}d_1}\left(1 + \frac{\psi(\alpha_1+\alpha_2)}{\gamma_1+\gamma_2}\right).$$

In addition, the following relationship between the physical parameters of the system results

$$\alpha_1 + \alpha_2 = -\frac{\gamma_1+\gamma_2}{2l\psi(60k\psi(l+\rho\nu) - \rho\nu\mathrm{EL}_1 d_1(\gamma_1+\gamma_2))}\Big(120kl\psi(l+\rho\nu) +$$
$$+5\nu d_1\mathrm{E}(\psi-1)(\gamma_1+\gamma_2) + 3\rho\nu l d_1\mathrm{E}(\mathrm{L}_1 + 40(\omega\psi-1))(\gamma_1+\gamma_2)\Big).$$

So, all constants have been defined. The functions that determine the field of velocities, temperatures, and concentrations (2) are written out explicitly.

Conclusion

In this paper, an exact solution of three-dimensional Oberbeck-Boussinesque equations is found that describes the stationary flow of a two-layer system of liquid media with a common interface in a channel bounded by solid walls. The construction of exact solutions is of particular value in the study of mathematical models of fluid dynamics in domains with interfaces. The solutions in the close formulas make it possible to determine the role of different mechanisms in the formation of certain types of flows.

The solution obtained can be used in the development of experiments to study joint flows of liquid media in a closed channel. In further work, it is planned to analyze the physical parameters of the system and to select fluids that meet the conditions of the problem, to elucidate the effect of thermocapillary, gravitational and other forces on the nature of the flow, and to consider special cases of heating a solid substrate. In particular, when there is radial heating at $a_{10} = b_{10}$ (7) or the case when $a_{10} + b_{10} = 0$. Moreover, the constructed solution is planned to be used as a test at finding of corresponding non-stationary solution during a modeling of evolution process of heat and mass transfer with parabolic temperature field. Also using the obtained solution as a test one, it is planned to simulate a non-stationary process.

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References

[1] V.K.Andreev, Yu.A.Gaponenko,O.N.Goncharova, V.V.Pukhnachov, Mathematical models of convection (de Gruyter studies in mathematical physics), Berlin/Boston: De Gruyter, 2012.

[2] V.B.Bekezhanova, O.N.Goncharova, I.A.Shefer, Analysis of an Exact Solution of Problem of the Evaporative Convection (Review). Part I. Plane Case, *J. of Siberian Federal University. Mathematics &Physics*, **11**(2018), no. 2, 178–190.

[3] V.B.Bekezhanova, O.N.Goncharova, I.A.Shefer, Analysis of an Exact Solution of Problem of the Evaporative Convection (Review). Part II. Three-dimensional Flows, *J. of Siberian Federal University. Mathematics &Physics*, **11**(2018), no. 3, 342–355.

[4] V.V.Pukhnachov, Model of a viscous layer deformation by thermocapillary forces, *J. of Applied Mathematics*, **13**(2002), 205–224.

Построение точного решения специального вида для трехмерной задачи термоконцентрационной конвекции в двухслойной системе

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Аннотация. Рассмотрено трехмерное совместное течение жидкости и бинарной смеси с общей границей раздела. Предполагается, что поле температуры в слоях имеет квадратичное распределение. Построено точное решение некоторой модельной задачи. Получены явные выражения для всех искомых функций с помощью определенного замыкающего соотношения.

Ключевые слова: приближение Обербека–Буссинеска, поверхностная энергия, бинарная смесь.