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Local Asymptotic Normality of Statistical Experiments in an Inhomogeneous Competing Risks Model

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Abstract. In this paper we consider an inhomogeneous competing risks model. For the likelihood ratio statistics (LRS), proved the theorem on the locally asymptotic normality of statistical experiment.

Keywords: competing risks model, random censoring, local asymptotic normality, likelihood ratio statistics.

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1. Introduction and preliminaries

In a inhomogeneous competing risks model (CRM) it's interesting to investigate the independent random variables (r.v.) $\{X_m,\, m \geqslant 1\}$ with a distribution function (d.f.) $\{H_m(x;\theta),\, m \geqslant 1\}$ with values in a measurable space $(\mathcal{X}_m, \mathcal{B}_m)$, $m \geqslant 1$, where $\mathcal{X}_m$ is set of possible values of r.v. X_m and $\mathcal{B}_m = \sigma(\mathcal{X}_m)$. D.f. of r.v. X_m also depends on the scalar parameter $\theta \in \Theta$, where Θ-parametrical space; is open set in R^1. In CRM with r.v. X_m pairwise disjoint and forming a complete group of events $\left\{A_m^{(1)}, \dots, A_m^{(k)}\right\}$ observed. We observe the sample of size n: $\left\{\left(X_m;\ \delta_m^{(0)}, \delta_m^{(1)}, \dots, \delta_m^{(k)}\right),\, m = 1, \dots, n\right\}$, where $\delta_m^{(i)} = I\left(A_m^{(i)}\right)$ is indicator of events $A_m^{(i)}$. Let $H_m^{(i)}(x;\theta) = P_\theta\left(X_m < x,\, \delta_m^{(i)} = 1\right)$ are subdistributions such that $H_m^{(1)}(x;\theta) + \dots + H_m^{(n)}(x;\theta) = H(x;\theta)$ for all $(x;\theta) \in R^1 \times \Theta$. By $h_m^{(i)}$ we define density of subdistributions $H_m^{(i)}$. Then note that there is a density $h_m^{(1)}(x;\theta) + \dots + h_m^{(k)}(x;\theta) = \dfrac{\partial H_m(x;\theta)}{\partial x} = h_m(x;\theta)$ for all $(x;\theta) \in R^1 \times \Theta$. We define $\Lambda_m^{(i)}(x;\theta) = \int\limits_{-\infty}^{x} (1 - H_m(u;\theta))^1\, dH_m(u;\theta)$ integral and $\lambda_m^{(i)}(x;\theta)$ density of failure rate of the i-th risk. Then

$$\lambda_m^{(1)}(x;\theta) + \dots + \lambda_m^{(k)}(x;\theta) = \frac{h_m(x;\theta)}{1 - H_m(x;\theta)},$$

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for all $(x;\theta) \in R^1 \times \Theta$.

We introduce the functional of the exponential intensity function [1,3]:

$$1 - F_m^{(i)}(x;\theta) = \exp(-\Lambda_m^{(i)}(x;\theta)),\ i = 1,\dots,k;\quad m = 1,\dots,n.$$

Note that this functional has all the properties of a d.f. [1].

By $f_m^{(i)}(x;\theta)$ we define density of $F_m^{(i)}(x;\theta)$. It is easy to see that

$$f_m^{(i)}(x;\theta) = \exp\left\{\Lambda^{(i)}(x;\theta)\right\}\frac{h_m^{(i)}(x;\theta)}{1 - H_m(x;\theta)},\ i = 1,\dots,k.$$

Denote by $\mathcal{F}_t^{(i)} = \sigma\left\{X_m I\left(X_m \leqslant t\right),\ \delta_m^{(i)} I\left(X_m \leqslant t\right),\ 1 \leqslant m \leqslant n\right\}$, flow of σ-algebra, generated by pairs of observations $\left\{\left(X_m, \delta_m^{(i)}\right),\ 1 \leqslant m \leqslant n\right\}$, $i = \overline{1,k}$. Let define a sequence of martingale processes for $m = \overline{1,\ n}$ and $i = \overline{1,k}$:

$$\mu_m^{(i)}(t) = I\left(X_m \leqslant t, \delta_m^{(i)} = 1\right) - \int_{-\infty}^{t} I\left(X_m > s\right) d\Lambda^{(i)}(s;\theta_0),\quad t \geqslant 0,$$

where θ_0 is true value of parameter θ, $\theta_0 \in \Theta$. These martingales have zero mea

$$E_{\theta_0}\mu_m^{(i)} = H_m^{(i)}(t;\theta_0) - \int_0^t \left(1 - H(s;\theta_0)\right) d\Lambda_m^{(i)}(s;\theta_0) =$$

$$= H_m^{(i)}(t;\theta_0) - \int_0^t \frac{1 - H_m(s;\theta_0)}{1 - H_m(s;\theta_0)} dH_m^{(i)}(s;\theta_0) = H_m^{(i)}(t;\theta_0) - H_m^{(i)}(t;\theta_0) = 0,$$

$i = \overline{1,k};\ m = \overline{1,n}$. They are members of class $\mathcal{M}^2(\mathcal{F}_t^{(i)})$, i.e. square-integrable martingales with a predictable quadratic characteristics [4]:

$$< \mu_m^{(i)}, \mu_m^{(j)} >= \begin{cases} \Lambda^{(i)}(t), & i = j, \\ 0\quad, & i \neq j. \end{cases}$$

Therefore, these martingales are orthogonal. According to this

$$\sum_{m=1}^{n} \mu_m^{(i)}(t) \in \mathcal{M}^2(\mathcal{F}_t^{(i)}),\quad \sum_{i=1}^{k}\sum_{m=1}^{n} \mu_m^{(i)}(t) \in \mathcal{M}^2(\mathcal{F}_t),$$

where $\mathcal{F}_t = \bigcap_{i=1}^{k} \mathcal{F}_t^{(i)}$. To prove local asymptotic normality (LAN) we need some regularity conditions for $f_m^{(i)}(x;\theta)$:

(C1) The supports $N_{f_m^{(i)}} = \{x : f_m^{(i)}(x;\theta) > 0\}$ are independent from θ, $i = \overline{1,k};\ m = \overline{1,n}$;

(C2) There exist and are continuous for all x derivatives $\dfrac{\partial^l}{\partial\theta^l} f_m^{(i)}(x;\theta)$, $l = 1,2;\ i = \overline{1,k}$; $m = \overline{1,n}$;

(C3) Fisher information $\sum_{m=1}^{n} I_m(\theta) = \sum_{m=1}^{n}\sum_{i=1}^{k} I_m^{(i)}(\theta)$ is finite and continuous in θ and positive at the point θ_0, where

$$I_m^{(i)}(\theta) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial\theta} \log \lambda^{(i)}(s;\theta)\right)^2 dH_m^{(i)}(s;\theta).$$

Such a representation of Fisher information in terms of intensity density $\lambda_m^{(i)}(x;\theta)$ was established by the authors of [2]. Let $\varphi(n) = \left(\sum\limits_{m=1}^{n} I_m(\theta_0)\right)^{-1/2}$. Let for every $u \in R^1$, $\theta_n = \theta_0 + u\varphi(n) \in \Theta$. We have a likelihood ratio statistics (LRS) of the sample $Z^{(n)} = (X_1, \ldots, X_n)$: $dQ_{\theta_n}^{(n)}\left(Z^{(n)}\right)/dQ_{\theta_0}^{(n)}\left(Z^{(n)}\right)$, where

$$Q_\theta^{(n)}(Z^{(n)}) = p_n(Z^{(n)};\theta) = \prod_{m=1}^{n}\prod_{i=1}^{k}\left\{ f_m^{(i)}(X_m;\theta)\cdot \prod_{\substack{j=1\\ j\neq i}}^{k}[1-F_m^{(i)}(X_m;\theta)]^{\delta_m^{(i)}}\right\}.$$

2. LAN for LRS.

The following theorem asserts LAN for LRS.

Theorem. *Under regularity conditions (C1)–(C3) for LRS we have representation*

$$\frac{dQ_{\theta_n}^{(n)}(Z^{(n)})}{dQ_{\theta_0}^{(n)}(Z^{(n)})} = \exp\left\{u\Delta_n(\theta_0) + \frac{u^2}{2} + R_n(u)\right\}, \tag{1}$$

where $\Delta_n = \varphi(n)\sum\limits_{m=1}^{n}\sum\limits_{i=1}^{k}\int\limits_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\log\lambda_m(s;\theta_0+u\varphi(n)\gamma)\right)d\mu_m^{(i)}(s)$, $0<\gamma<1$, $\mathrm{L}\left(\Delta_n/Q_{\theta_0}^{(n)}\right) \to N(0;1)$ *and* $R_n(u) \overset{Q_{\theta_0}^{(n)}}{\to} 0$ *at* $n\to\infty$.

Proof of Theorem. Let represent logarithm of LRS in terms of martingale-processes as follows:

$$L_n(u) = \log\left\{\frac{dQ_{\theta_n}^{(n)}(Z^{(n)})}{dQ_{\theta_0}^{(n)}(Z^{(n)})}\right\} = \log\left[\frac{p_n(Z^{(n)};\theta_n)}{p_n(Z^{(n)};\theta_0)}\right] =$$

$$= \sum_{m=1}^{n}\sum_{i=1}^{k}\delta_m^{(i)}\log\left\{\frac{f^{(i)}(X_m;\theta_n)\prod_{j=1}^{k}\left(1-F^{(j)}(X_m;\theta_n)\right)}{f^{(i)}(X_m;\theta_0)\prod_{j=1}^{k}\left(1-F^{(j)}(X_m;\theta_0)\right)}\right\} =$$

$$= \sum_{m=1}^{n}\sum_{i=1}^{k}\delta_m^{(i)}\log\left[\frac{\lambda_m^{(i)}(X_m;\theta_n)}{\lambda_m^{(i)}(X_m;\theta_0)}\right] + \sum_{m=1}^{n}\log\left[\frac{1-H_m(X_m;\theta_n)}{1-H_m(X_m;\theta_0)}\right] =$$

$$= \sum_{m=1}^{n}\sum_{i=1}^{k}\delta_m^{(i)}\log\left[\frac{\lambda_m^{(i)}(X_m;\theta_n)}{\lambda_m^{(i)}(X_m;\theta_0)}\right] + \sum_{m=1}^{n}\left(\Lambda_m(X_m;\theta_n)-\Lambda_m(X_m;\theta_0)\right) = \tag{2}$$

$$= \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[\frac{\lambda_m^{(i)}(s;\theta_n)}{\lambda_m^{(i)}(s;\theta_0)}\right]d\mu_m^{(i)}(s) +$$

$$+\left\{\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[\frac{\lambda_m^{(i)}(s;\theta_n)}{\lambda_m^{(i)}(s;\theta_0)}\right]I(X_m>s)d\Lambda_m^{(i)}(s;\theta_0) -\right.$$

$$\left.-\sum_{m=1}^{n}\left(\Lambda_m(X_m;\theta_n)-\Lambda_m(X_m;\theta_0)\right)\right\} = A_n(u)+R_n(u).$$

By the mean value theorem, from (2) we have

$$A_n(u) = \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left[\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)) - \log\lambda_m^{(i)}(s;\theta_0)\right]d\mu_m^{(i)}(s) =$$

$$= u\varphi(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\tfrac{\partial}{\partial\theta}\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)d\mu_m^{(i)}(s) = u\Delta_n(\theta_0),$$

where

$$\Delta_n(\theta_0) = \varphi(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\tfrac{\partial}{\partial\theta}\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)d\mu_m^{(i)}(s). \tag{3}$$

Integrating function in (3) is a continuous real-valued function of s, hence it is a predictable and is a square-integrable martingale: $\Delta_n \in \mathcal{M}^2(\mathcal{F}_t)$. We need to establish that

$$\mathcal{L}\left(\Delta_n(\theta_0)/Q_{\theta_0}^{(n)}\right) \to N\left(0;1\right). \tag{4}$$

It is easy to see that $E_{\theta_0}\Delta_n(\theta_0) = 0$ (since $E_{\theta_0}\mu_m^{(i)}(t) = 0$). Moreover, the quadratic variation of the martingale $\Delta_n(\theta_0)$ is

$$< \Delta_n(\theta_0), \Delta_n(\theta_0) >=$$

$$>= \varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2\left(1-H_m(s;\theta_0)\right)d\Lambda_m^{(i)}(s;\theta_0) =$$

$$= \left(\sum_{m=1}^{n} I_m(\theta_0)\right)^{-1}\sum_{m=1}^{n} I_m(\theta_0+u\varphi(n)\gamma) \to 1, \quad n\to\infty,$$

in view of condition (C3). Consider the Lindberg conditions

$$\varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2\times$$

$$\times I\left(\left|\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right| > \varepsilon\varphi(n)\right)d < \mu_m^{(i)}(t), \mu_m^{(j)}(t) >=$$

$$= \varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2\times$$

$$\times I\left(\left|\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right| > \varepsilon\varphi(n)\right)(1-H_m(s;\theta_0))d\Lambda_m^{(i)}(s;\theta_0) =$$

$$= \varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2\times$$

$$\times I\left(\left|\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right| > \varepsilon\varphi(n)\right)dH_m^{(i)}(s;\theta_0) \to 0, \quad n\to\infty,$$

where the convergence to zero of the integral follows from the requirement that the Fisher information $\sum\limits_{m=1}^{n} I_m(\theta)$ is finite in view of condition (C3). Consequently, weak convergence (4) follows from the central limit theorem for martingales [5]. Consider second addition in (2). Second addition in (2) converges in probability to $-\dfrac{u^2}{2}$: $R_n(u) \overset{Q_{\theta_0}^{(n)}}{\to} -\dfrac{u^2}{2}$, $n \to \infty$. Therefore, it remains to show that

$$R_n(u) + \frac{u^2}{2} \overset{Q_{\theta_0}^{(n)}}{\to} 0, \quad n \to \infty. \tag{5}$$

We investigate second addition inside the large curly bracket in (2). After elementary transformations, we have

$$\begin{aligned} &\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[\frac{\lambda_m^{(i)}(s;\theta_n)}{\lambda_m^{(i)}(s;\theta_0)}\right] I(X_m > s)d\Lambda_m^{(i)}(s;\theta_0) - \sum_{m=1}^{n}\left(\Lambda_m(X_m;\theta_n) - \Lambda_m(X_m;\theta_0)\right) = \\ &= \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[1 + \frac{\lambda_m^{(i)}(s;\theta_n) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0)}\right] I(X_m > s)d\Lambda_m(s;\theta_0) + \\ &+ \sum_{m=1}^{n}\int_{-\infty}^{+\infty}\left(\Lambda_m(X_m;\theta_n) - \Lambda_m(X_m;\theta_0)\right) dI(X_m > s) = \\ &= \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left(1 + \frac{\lambda_m^{(i)}(s;\theta_n) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0)}\right) - \\ &- \int_{-\infty}^{\infty} I(X_m > s)\frac{\lambda_m^{(i)}(s;\theta_n) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0)} d\Lambda_m(s;\theta_0) = \\ &= -\frac{1}{2}\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left[\frac{\lambda_m^{(i)}(s;\theta_0 + u\varphi(n)\gamma^*) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0 + u\varphi(n)\gamma^*)}\right] I(X_m > s)d\Lambda_m^{(i)}(s;\theta_0) = \\ &= -\frac{u^2\varphi^2(n)}{2}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\log\lambda(s;\theta_0 + u\varphi(n)\gamma^*)\right)^2 I(X_m > s)d\Lambda_m^{(i)}(s;\theta_0) = \\ &= -\frac{u^2}{2}\left(\sum_{m=1}^{n} I_m(\theta_0)\right)^{-1}\sum_{m=1}^{n} I_m(\theta_0 + u\varphi(n)\gamma^*)\left(1 + o_p(1)\right) \to -\frac{u^2}{2}, \;\; n \to \infty, \end{aligned} \tag{6}$$

where $0 < \gamma^* < 1$. Now (6) implies (5). The theorem is proved. □

Mutual contiguity of probability measures $Q_{\theta_n}^{(n)}$ and $Q_{\theta_0}^{(n)}$ follows from the theorem.

References

[1] A.Abdushukurov, Estimation of unknown distributions from incomplete observations and their properties, LAP. LAMBERT Academic Publishing, 2011 (in Russian).

[2] A.Abdushukurov, N.Nurmuhamedova, Locally asymptotically normality of the family of distributions by incomplete observations, *Journal of Siberian Federal University. Mathematics & Physics*, **7**(2014), 141–154.

[3] I.Ibragimov, R.Khas'minskii, Statistical Estimation: Asymptotic Theory, New York: Springer-Verlag, 1981.

[4] R.Liptser, A.Shiryayev, Theory of Martingales, Springer, 1989.

[5] R.Rebolledo, Central limit theorem for local martingales, *Z. Wahrscheinlich. verw. Gebiete*, **51**(1980), 269–280.

Локальная асимптотическая нормальность статистических экспериментов в неоднородной модели конкурирующих рисков

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Аннотация. В статье рассматривается неоднородная модель конкурирующих рисков. Для статистики отношения правдоподобия доказана теорема о локально асимптотической нормальности статистического эксперимента.

Ключевые слова: модель конкурирующих рисков, случайное цензурирование, локальная асимптотическая нормальность, статистика отношения правдоподобия.