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Maximal Abelian Ideals and Automorphisms of Nonfinitary Nil-triangular Algebras

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Abstract. We study mutually connected automorphisms and abelian ideals of nonfinitary nil-triangular algebras.

Keywords: Chevalley algebra, nil-triangular subalgebra, unitriangular group, finitary and nonfinitary generalizations, automorphisms, abelian ideal.

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Introduction

We study mutually connected automorphisms and abelian ideals of nonfinitary nil-triangular algebras.

Choose an arbitrary chain (or a linearly ordered set) Γ with the order relation $\leqslant$. The matrix product in the module $M(\Gamma, K)$ of all Γ-matrices $\alpha = ||a_{ij}||_{i,j\in\Gamma}$ over K is not defined for an infinite chain Γ. However, the submodule $NT(\Gamma, K)$ of nil-triangular Γ-matrices is an algebra exactly when $\Gamma = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z} \setminus \mathbb{N}$. [1–3]. See Sec. 1.

The Chevalley algebra over a field K is characterized by a root system Φ and a Chevalley basis consisting of elements e_r ($r \in \Phi$) and a base of suitable Cartan subalgebra, [4, Sec. 4.4]. The subalgebra $N\Phi(K)$ with the basis $\{e_r \mid r \in \Phi^+\}$ is said to be a nil-triangular subalgebra. The root automorphisms of the subalgebra $N\Phi(K)$ generate a unipotent subgroup $U\Phi(K)$ of the Chevalley group of type Φ over K, [5]. For nil-triangular subalgebras of Chevalley algebras classical types the nonfinitary generalizations $NG(K)$ of types $G = B_{\mathbb{N}}$, $C_{\mathbb{N}}$ and $D_{\mathbb{N}}$ were constructed in [1].

R. Slowik [6] investigated the automorphisms of the limit unitriangular group $UT(\infty, K)$ over a field K of order >2. By [2], this group was represented as the group $UT(\mathbb{N}, K)$ and also as the adjoint group of radical ring $NT(\mathbb{N}, K)$. For any ring K which has no zero divisors, the automorphism group of the associated Lie ring of $NT(\mathbb{N}, K)$ (i.e. type $A_{\mathbb{N}}$) were described in [2]; also, it coincides with the automorphism group of the adjoint group ($\simeq UT(\mathbb{N}, K)$).

Let K be an arbitrary domain. Our main Theorem 2.1 in Sec. 2 describes maximal abelian ideals of algebras $NG(K)$ of types $G = B_{\mathbb{N}}$, $C_{\mathbb{N}}$ and $D_{\mathbb{N}}$. See also a reduction of automorphisms in Sec. 3.

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1. Preliminaries

Unless otherwise specified, K denotes an associative commutative ring with a (nonzero) unity.

The Chevalley algebra over a field K is characterized by a root system Φ and a Chevalley basis consisting of elements e_r $(r \in \Phi)$ and a base of suitable Cartan subalgebra, [4, Sec. 4.4].

The subalgebra $N\Phi(K)$ with the base $\{e_r \mid r \in \Phi^+\}$ is said to be a nil-triangular subalgebra. For the type A_{n-1} it is represented by the Lie algebra $NT(n, K)$ of all nil-triangular $n \times n$-matrices over K with the matrix units e_{iv} $(1 \leqslant v < i \leqslant n)$.

In [5], the subalgebra $N\Phi(K)$ of other classical types B_n, C_n and D_n was represented by special matrices with a base of 'matrix units' e_{iv} with restrictions, respectively,

$$-i < v < i \leqslant n, \quad -i \leqslant v < i \leqslant n,\ v \neq 0, \quad 1 \leqslant |v| < i \leqslant n.$$

After appropriate numbering of roots $r = r_{iv}$ we obtain $e_r = e_{iv}$ and

$$e_{ij} * e_{uv} = 0 \quad (i \neq v,\ j \neq u, j \neq -v). \tag{1}$$

We represent any element $\alpha \in N\Phi(K)$ by the sum $\alpha = \sum a_{iv} e_{iv}$ and by Φ^+- matrix $||a_{iv}||$ which corresponds to the types. For example, the B_n^+- matrix has the form

$$\begin{matrix} & & & a_{10} & & & \\ & & a_{2,-1} & a_{20} & a_{21} & & \\ & & \cdots & \cdots & \cdots & & \\ a_{n,-n+1} & \cdots & a_{n,-1} & a_{n0} & a_{n1} & \cdots & a_{n,n-1}, \end{matrix}$$

Removing the zero column, we obtain the D_n^+- matrix.

The following lemma is proved in [5, Lemma 1.1].

Lemma 1.1. *The signs of the structural constants of the Chevalley basis can be chosen so that we have* $e_{ij} * e_{jv} = e_{iv}$ *and (1), and also*

$$\Phi = B_n, D_n : \quad e_{jv} * e_{i,-v} = e_{i,-j} \quad (i > j >| v |> 0);$$

$$\Phi = C_n : \ e_{jm} * e_{i,-m} = e_{im} * e_{j,-m} = e_{i,-j} \quad (i > j > m \geqslant 1);$$

$$\Phi = B_n : \ e_{i0} * e_{j0} = 2e_{i,-j}, \quad \Phi = C_n : \ e_{ij} * e_{i,-j} = 2e_{i,-i} \quad (i > j \geqslant 1).$$

Now we choose an arbitrary chain (a linearly ordered set) Γ with the order relation $\leqslant$. All Γ-matrices $\alpha = ||a_{ij}||_{i,j \in \Gamma}$ over K with the usual multiplication by scalars from K and matrix addition form a K-module $M(\Gamma, K)$. For an infinite chain Γ, matrix multiplication in the module $M(\Gamma, K)$ is not defined. Denote by $\mathbb{N}$ a chain of natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \cdots, n\}$ (or $\mathbb{N} = \{0, 1, 2, 3, \cdots, n\}$).

It is known ([1–3]) that the submodule $NT(\Gamma, K)$ of all nonfinitary (low) nil-triangular Γ-matrices with the usual matrix multiplication is an algebra if and only if $\Gamma = \mathbb{Z}$, $\mathbb{N}$ or $\mathbb{Z} \setminus \mathbb{N}$. Algebras $NG(K)$ for Lie type A_Γ (i.e. $NT(\Gamma, K)$) were studied in [1].

Also, they had been constructed for classical types $G = B_\mathbb{N}$, $C_\mathbb{N}$, and $D_\mathbb{N}$ in [1,7]. The matrix units $e_{im} \in NG(K)\,(i, m \in \Gamma)$ is determined with restrictions, respectively,

$$-i < m < i, \quad -i \leqslant m < i,\ m \neq 0, \quad 1 \leqslant |m| < i.$$

Note that each unit $e_{im} \in NG(K)$ is associated to the ideal

$$T_{im} = \langle \alpha = ||a_{uv}|| \in NG(K) \mid a_{uv} = 0 \text{ if } u < i \text{ or } v > m \rangle.$$

For any ideal J of K the congruence subring $NG(K, J)$ of all matrices over J from $NG(K)$ is determined, as usual.

2. Main Theorem

According to [1] and [2], the maximal abelian ideals and automorphisms of the Lie ring $NA_\Gamma(K)$ were described at $\Gamma = \mathbb{N}$. The aim of this section is to describe the maximal abelian ideals of nonfinitary algebras $NG(K)$ of other classical types $G = B_\mathbb{N}$, $C_\mathbb{N}$ and $D_\mathbb{N}$.

By Lemma 1.1, T_{im} for $m < 0$ is always an abelian ideal. Another way of constructing abelian ideals is known for type $D_\mathbb{N}$. Note that the centralizer of the ideal T_{21} in Lie ring $ND_\mathbb{N}(K)$ is $C(T_{21}) = T_{3,-2}$. Also, for any pair $(a, b) \neq (0, 0)$ over K and $t \in K$ the elements

$$ae_{i,-1} + be_{i1},\ t(ae_{j,-1} + be_{j1}) \in ND_\mathbb{N}(K)$$

commute when $2ab = 0$. Thus, we obtain the abelian ideal

$$\mathcal{M}(K, a, b) = T_{3,-2} + \sum_{i \in \mathbb{N}} K(ae_{i,-1} + be_{i1}).$$

For any domain K we denote its field of fractions by Q_K.

Theorem 2.1. *Let M be a maximal abelian ideal of the Lie ring $NG(K)$ over a domain K. Then the following statements are fulfilled.*

i) $G = B_\mathbb{N}$: $M = T_{10}$ for $2K = 0$ and $M = T_{2,-1}$ for $2K \neq 0$.

ii) $G = C_\mathbb{N}$: $M = T_{1,-1}$.

iii) $G = D_\mathbb{N}$: $M = ND_\mathbb{N}(K) \cap \mathcal{M}(Q_K, a, b)$ for $(a, b) \neq (0, 0)$.

Proof. Any matrix from M can be correctly represented as a sum (possibly infinite) of elementary matrices. In this sense, we can assume that the ideal M is generated by elementary matrices.

Let $\alpha = \|a_{uv}\| \in NG(K)$. Denote by T_α, the principal ideal (α) of $NG(K)$. We need

Lemma 2.1. *Let a_{uv} be a nonzero element and either $v \geqslant 1$ or $G = D_\mathbb{N}$, $v > 1$. Let J be the principal ideal Ka_{uv} of K. Then $T_{m,v} \cap NG(K, J) \subset T_\alpha$ for all $m \geqslant u + 2$.*

Proof. It is sufficiently to prove the case of the matrix α with zeros all coordinates which are $(u + 2, v)$–coordinates above or to the right. The multiplications of α by $Ke_{u+1,u}$ and then by $Ke_{u+2,u+1}$ give matrices α_1 having zeros all rows with numbers $\neq (u + 2)$ and their $(u + 2, v)$–coordinates run ideal $J = Ka_{uv}$.

Further, multiplying the matrix α_1 by the elements e_{vm} $(m < v)$ in succession, we obtain inclusion $Je_{u+2,p} \subset T_\alpha$ and hence $Je_{m,p} \subset T_\alpha$ for all $p < v$ and $m \geqslant u + 2$. Thus, we arrive at the inclusion in T_α of the congruence subring $T_{mv} \cap NG(K, J)$ required in the lemma of each ideal T_{mv} in $NG(K)$ for $m \geqslant u + 2$. Lemma 2.1 is proved. □

By Lemma 1.1, all enumerated in i) – iii) ideals are abelian.

Let M be an arbitrary maximal abelian ideal in $NG(K)$. Assume that there is a matrix $\alpha = \|a_{uv}\| \in M$ with at least one non-zero coordinate a_{uv} for $v \geqslant 1$. By Lemma 2.1, the principal ideal $T_\alpha = (\alpha)$ contains intersections $T_{mv}(J) := T_{mv} \cap NG(K, J)$ for $m \geqslant u + 2$.

For algebras $NC_{\mathbb{N}}(K)$ the condition that the ideal $T_{mv}(J)$ be abelian is obviously equivalent to the condition $a_{uv}^2 = 0$. When K is a domain, we obtain the equality $a_{mv} = 0$. The obtained contradiction proves the inclusion $M \subset T_{1,-1}$.

Consider the case $G = B_{\mathbb{N}}$. We prove the inclusion $M \subset T_{10}$. Assume the opposite: M contains a matrix α with nonzero coordinate a_{uv} for $v > 0$. By Lemma 2.1, by analogy with the type of $C_{\mathbb{N}}$, we obtain the equality $a_{uv}^2 = 0$, contradicting the choice of principal ring K.

Now suppose that $M \subset T_{10}$ and α exists in M with nonzero (u,0)-coordinate a. By shifting the u-th row of the matrix (as in the proof of Lemma 2.1), we find the matrices β, $\gamma \in M$ with conditions:

$$\beta = ae_{i0} \ mod \ T_{2,-1}, \quad \gamma = ae_{s0} \ mod \ T_{2,-1} \quad (s > i > u+1).$$

From Lemma 1.1 it follows that equality $\gamma * \beta = 2a^2 e_{s,-i}$ and, since M is abelian, the equality $2a^2 = 0$. This is possible only for $2K = 0$ in which case $M = T_{10}$. For $2K \neq 0$, the centralizer of the ideal T_{10} is equal to $C(T_{10}) = T_{2,-1}$, whence $M = T_{2,-1}$.

For the type $G = D_{\mathbb{N}}$ (by analogy with the type $C_{\mathbb{N}}$) by Lemma 2.1 we obtain inclusion $M \subset T_{21}$. By the equality $C(T_{21}) = T_{3,-2}$ from § 1, we obtain inclusion $T_{3,-2} \subset M$ and we find the matrix $\alpha = \|a_{uv}\|$ in M with the pair $(a_{i,-1}, a_{i1}) \neq (0,0)$. Then the ideal M contains

$$K(ae_{m,-1} + be_{m1}) \quad (a := a_{i,-1},\, b := a_{i1})$$

for all $m > i$. Since M is abelian, we immediately obtain the conditions $2ab = 0$. When the domain K is a field, i.e. coincides with the field of fractions Q_K, the proof is complete.

When K is a proper subring in its field of fractions Q_K, we construct abelian ideals $\mathcal{M}(Q_K, a, b)$ for various $a, b \in Q_K$. Then the intersections

$$M = ND_{\mathbb{N}}(K) \cap \mathcal{M}(Q_K, a, b)$$

give all maximal abelian ideals of the Lie ring $ND_{\mathbb{N}}(K)$.

Therefore, Theorem 2.1 is completely proved. □

Example 2.1 Suppose a ring K has a nonzero element x with $x^2 = 0$. Choose a principal ideal $J = Kx$. (For example, $K = Z_n$ is the residue ring of integers modulo n, where n is a multiple of 4.) Then the congruence subring $NG(K, J)$ in $NG(K)$ is an abelian ideal that do not belong to any of the ideals T_{im} $(i > m)$.

3. Remark on the reduction of automorphisms

It is clear that an automorphism of a ring always induces an automorphism of a quotient ring with respect to the characteristic ideal.

As a corollary of Theorem 2.1 we easily obtain

Proposition 3.1. *When K is a domain, the ideal T_{10} is characteristic in the Lie ring $NB_{\mathbb{N}}(K)$. The ideals $T_{1,-1}$ and T_{21} are characteristic in the Lie rings $NC_{\mathbb{N}}(K)$ and $ND_{\mathbb{N}}(K)$ respectively.*

Proof. In the Lie ring $NB_{\mathbb{N}}(K)$ the ideal T_{10} is characteristic for $2K = 0$ as the only maximal abelian ideal, and for $2K \neq 0$ as the centralizer $C(T_{2,-1}) = T_{10}$ of the characteristic ideal $T_{2,-1}$. The ideal $T_{1,-1}$ is the only (and therefore characteristic) maximal abelian ideal in the Lie ring $NC_{\mathbb{N}}(K)$.

By Theorem 2.1, the ideal T_{21} in the Lie ring $ND_{\mathbb{N}}(K)$ generates all maximal abelian ideals; the ideal $T_{2,-1}$ and its image with respect to the graph automorphism are sufficient. Hence, the ideal T_{21} and its centralizer $C(T_{21}) = T_{3,-2}$ are characteristic. This finishes the proof of the proposition. □

Note the following isomorphisms:

$$NB_{\mathbb{N}}(K)/T_{10} \simeq NB_{\mathbb{N}}(K)/T_{2,-1} \simeq NT(\mathbb{N}, K),$$

$$NC_{\mathbb{N}}(K)/T_{1,-1} \simeq NT(\mathbb{N}, K),$$

$$ND_{\mathbb{N}}(K)/T_{21} \simeq NT(\mathbb{N}, K).$$

The automorphisms of nonfinitary Lie rings $NG(K)$ of type $G = A_{\mathbb{N}}$, i.e., Lie rings $NT(\mathbb{N}, K)$, were interconnectedly studied with maximal abelian ideals earlier in [1, 7]. Along with the standard automorphisms, for Lie rings $NT(\Gamma, K)$ single out hypercentral and other non-standard automorphisms, [8–10].

In [2], standard automorphisms of Lie rings $NT(\mathbb{N}, K)$ are proved. On the other hand, for the Lie algebra $ND_{\mathbb{N}}(K)$ there are nonstandard automorphisms generalizing graph automorphisms.

In the group $SL(2, K)$ we choose a subgroup

$$S = \{A = ||a_{uv}|| \in SL(2, K) \mid 2a_{11}a_{12} = 2a_{21}a_{22} = 0\}.$$

Obviously, $S = SL(2, K)$ for $2K = 0$. By analogy with [10, 11] for any matrix $A = ||a_{uv}|| \in S$, we associate the automorphism $\widetilde{A}$ of the Lie algebra $ND_{\mathbb{N}}(K)$ according to the rule

$$\widetilde{A} : e_{2,-1} \to a_{11}e_{2,-1} + a_{12}e_{2,1}, \quad e_{2,1} \to a_{21}e_{2,-1} + a_{22}e_{2,1}, \quad e_{i+1,i} \to e_{i+1,i} \quad (i = 3, 4, \cdots). \quad (2)$$

The reduction of automorphisms leads to the following theorem.

Theorem 3.1. *Every automorphism of a nonfinitary algebra $NG(K)$ over a domain K is a standard automorphism of type $G = A_{\mathbb{N}}, B_{\mathbb{N}}$ or $C_{\mathbb{N}}$, and for type $G = D_{\mathbb{N}}$ there is a product of the suitable automorphism $\widetilde{A}$ and the standard automorphism.*

Remark 3.1 Theorem 2.1 and Theorem 3.1 transfer to Lie rings $NG(K)$ of classical types $G = A_{\Gamma}, B_{\Gamma}, C_{\Gamma}, D_{\Gamma}$ for $\Gamma = Z \setminus \mathbb{N}$; they are antiisomorphic to the Lie rings studied for $\Gamma = \mathbb{N}$.

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Максимальные абелевы идеалы и автоморфизмы нефинитарных нильтреугольных алгебр

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Аннотация. Для нефинитарных обобщений нильтреугольных подалгебр алгебр Шевалле в статье взаимосвязано исследуются автоморфизмы и максимальные абелевы идеалы.

Ключевые слова: алгебра Шевалле, нильтреугольная подалгебра, унитреугольная группа, финитарные и нефинитарные обобщения, автоморфизмы, абелев идеал.