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Flow Past Various Types of Vane Mechanisms by a Two-Phase Compressible Flow

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Abstract. The statement of boundary value problems for the flow around real wing-shaped profiles of supercavitation mechanisms near the separation boundary is stated. The problem under consideration is extremely important for the numerical study of the processes occurring in heating equipment, supercavitation devices, and heat and mass transfer technologies. The discussed algorithms have been implemented as computational programs for algebraic (ALFA) and integral (OMEGA) equations, ordinary (SIMP) and improper (SECOB) integrals, including the Cauchy integral (DSECOB) as well as summation programs for sequences and series (SHENKS, AYTKEN), including divergent ones (EULER).

Keywords: two-phase compressible flow, wing-shaped profiles of supercavitation mechanisms, supercavitation mechanisms, mass transfer technologies.

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Обтекание различных типов лопастных механизмов двухфазным сжимаемым потоком

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Аннотация. Сформулирована постановка краевых задач для обтекания реальных крылообразных профилей механизмов суперкавитации вблизи границы отрыва. Рассматриваемая задача чрезвычайно важна для численного исследования процессов, происходящих в нагревательной технике, суперкавитационных устройствах и технологиях тепломассообмена. Обсуждаемые алгоритмы реализованы в виде вычислительных программ для алгебраических (ALFA) и интегральных (OMEGA) уравнений, обыкновенных (SIMP) и несобственных (SECOB) интегралов, в том числе интеграла Коши (DSECOB), а также программ суммирования последовательностей и рядов (SHENKS)., АЙТКЕН), в том числе и расходящиеся (ЭЙЛЕР).

Ключевые слова: двухфазный сжимаемый поток, обтекание крылообразных профилей, суперкавитационные устройства, технологии тепломассообмена.

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1. Introduction

A whole series of production challenges have been resolved due to successful application of cavitation technologies. There is a great variety of supercavitation-based technical devices such as mixers, blenders, and reactors, to mention just a few. Studies on the flow in the vicinity of separation boundaries (solid walls and free surfaces) show that in a confined flow liquid turns into a bubble mixture of liquid and gas. This complicates the flow analysis and introduces additional losses resulting in impaired energy performance of the concerned mechanisms. In a general case the problem of a two-phase compressible flow around various types of vane mechanisms is substantially nonlinear (even under no-vortex flow assumption). The knowledge of hydrodynamic characteristics of a pterygoid profile as an element of hydrofoil cascade with certain geometric parameters (the angle of stagger, the pitch, etc.) is crucial for designing vane mechanisms. Quite often the challenge is of a methodological nature as to the adequate formulation of boundary-value problems for a flow around actual pterygoid profiles of supercavitating mechanisms near separation boundary.

2. Keywords: flow around actual shaped profiles wing, supercavitating mechanisms near separation boundary, potential theory, numerical method solution.

3. Efficient sequences summation methods

Discussed in the article the problem is extremely important for the numerical investigation of the processes occurring in thermaltechnological equipment, supercavitation vehicles and heat and mass transfer technologies [1–9].

Most of the problems in potential theory [10] can generally be reduced to solving integral or integral differential equations or sets of such equations [11–14].

In an operator form,

$$AU = f, \tag{1}$$

where A is the integral or integral differential operator, U is the vector of desired solutions, and f is the vector on the right-hand side.

When the perturbation method is used solution to Eq. (1) is sought in the form of a functional sequence $\{U_n\}$). Operator A can also be subjected to perturbation. So the original problem reduces to solving the equation

$$A_n U_n = f_n, \tag{2}$$

where A_n and f_n are expressed through the values obtained in the earlier approximations.

Since convergence of series and convergence of sequences are equivalent concepts, i.e. convergence of the $\sum_1^N U_n$ series to sum S is, by definition, convergence of the partial sums sequence $\left(S_n = \sum_1^n U_k\right)$ to limit S and vice versa, it suffices to consider transformation of sequences [15].

In general terms, the problem of acceleration of sequence convergence involves using the source sequence $\{S_n\}(n = 1, 2, \ldots; S_n \to S)$ to construct a new one

$$\sigma_{k,n} = A_k(S_1, S_2, \ldots S_n)(k = 1,2,\ldots), \tag{3}$$

to satisfy the following requirements:

1) the new sequence converges to the same limit as $\{S_n\}$ does;
2) $\sigma_{k,n}$ exhibits, in a sense, better convergence to S than $\{S_n\}$. The comparison criterion must be specified.

If A_k is a linear operator, then the linear methods of acceleration are governed by Eqs. (3). A

Consider a general approach to convergence acceleration [11]. Let X be a metric space with metric d, $\{S_n\} \in X$ is convergent: $\lim_{n\to\infty} d(S_n, S) = 0$, its limit being $S \in X$.

Suppose $A : \{S_n\} \to \{\bar{S}_n\}$, $\bar{S}_n \in X$.

We ask A to be:

1) absolutely regular, i.e. if $\lim_{n\to\infty} S_n = S \in X$, then $\lim_{n\to\infty} \bar{S}_n = S \in X$;
2) A method of acceleration conversion if A is regular and $\lim_{n\to\infty} d(\bar{S}_n, S)/d(S_n, S) = 0$;
3) *a)* A – a linear method if $A(\alpha\{S_n\}+\beta\{t_n\}) = \alpha A\{S_n\}+\beta\{t_n\}$,

   b) – A – nonlinear method if A satisfies the quasilinear conditions

$$A(\alpha\{S_n\}) = \alpha \cdot A\{S_n\},\ A(\alpha+\{S_n\}) = \alpha + A\{S_n\};$$

4) A – numerically stable algorithm free from big accumulated calculation errors.

Transform (3) is chosen such that it accelerates convergence of any sequence $\{S_n\} \in X$. Consider a set of sequences $\{S_n\}$ (numerical or functional), including convergent and divergent as well as monotonic and oscillating ones. These sequences, if presented graphically with the number n plotted on abscissa and S_n on ordinate and a smooth curve drawn through a discrete set of points, will yield graphs similar to those of transient processes in dynamic systems.

Taking S to be dependent on t, we have

$$S(t)=\bar{S}+\sum_{k=1}^{\nu} a_k e^{\alpha_k t}, \tag{4}$$

where α_k is an arbitrary complex number.

Nonlinear transformations being equally applicable to convergent and divergent sequences, the mentioned similarity between the type of a transient process (steady or unsteady, harmonic or aperiodic) and the type of summation process becomes apparent.

Let all $\mathrm{Re}(\alpha_k) < 0$ in (4). Under this assumption, we deal with a steady transient process. However, if there is at least one $\mathrm{Re}(\alpha_k) > 0$, the transient process is unsteady. Obviously, this interpretation enables us to apply harmonic analysis.

Replacing e^{α_k} by q_k ($|q_k|) < 1$ in (4) yields

$$S(t)=\bar{S}+\sum_{k=1}^{\nu} a_k q_k^t \left(q_k \in (0,1)\right) \tag{5}$$

The $S(t)$ quantity may or may not oscillate and it may be stable or unstable depending on the value of ν and coefficients α_k and q_k. In other words, in a general case it possesses the desired properties as specified above.

Within this approach, it is justified to represent certain sequences $\{S_n\}$ as «mathematical transient processes», i.e. as n-type functions:

$$S(n)=\bar{S}+\sum_{k=1}^{\nu} a_k q_k^n \left(q_k \in (0,1)\right). \tag{6}$$

By analyzing such a process we should be able to find both the order of magnitude of υ and the range of parameters a and q. If the sequence $\{S_n\}$ satisfies (6) and if $|q_k| < 1$, then $\bar{S}=\lim_{n\to\infty} S_n$.

If $\{S_n\}$ is a transient process and one or more $|q_k| \geq 1$, then $\{S_n\}$ fails to converge; however, one can say that $\{S_n\}$ diverges from $\bar{S}$. Therefore the $\bar{S}$ parameter is called an antilimit [13] of the sequence $\{S_n\}$.

The calculus for finding $\bar{S}$ can be classified as a method of summation of $\{S_n\}$ sequence or a technique to extract the main term thereof. There are few sequences appearing in applications that are essentially mathematically transient processes of some finite order ν. The rest are of an infinite order, i.e. $\nu \to \infty$ in (6).

All others also have a continuous spectrum [16]

$$S(n)=\bar{S}+\int_{q}^{q_1} a(q) q^n \mathrm{d}q. \tag{7}$$

The function $S(n)$ in (7) depends on the $2\nu + 1$ parameters of $\bar{S},\ a_k,\ q_k, (k=1,\ 2,\ ...,\ \nu)$. The first one, as noted above, is of importance. Dropping the parameters α_k, q_k [17], we get the Shanks-Schmidt transformation at $\nu = k$:

$$\sigma_{k,n} = \bar{S} = D_{k,n}\left(S_n\right) / D^*_{k,n}, \tag{8}$$

where $D_{k,n}\left(S_n\right) = \begin{vmatrix} S_{n-k}, S_{n-k+1}, \ldots, S_n \\ \Delta S_{n-k}, \Delta S_{n-k+1}, \ldots, \Delta S_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta S_{n-1}, \Delta S_n, \ldots \Delta S_{n+k-1} \end{vmatrix}$, $\Delta S_n = S_{n+1} - S_n$ and $D^*_{k,n}$ is derived from $D_{k,n}$ by replacing the first row with unity. Aitken's approximation is a special case of this transformation.

For $k = 1$

$$\sigma_{1,n} = \frac{S_{n-1}S_{n+1} - S_n^2}{S_{n-1} - 2S_n + S_{n+1}}. \tag{9}$$

For $k = 2$

$$\sigma_{2,n} = \frac{S_{n-2}\begin{vmatrix}\Delta S_{n-1}, & \Delta S_n \\ \Delta S_n, & \Delta S_{n+1}\end{vmatrix} - S_{n-1}\begin{vmatrix}\Delta S_{n-2}, & \Delta S_n \\ \Delta S_n, & \Delta S_{n+1}\end{vmatrix} + S_n\begin{vmatrix}\Delta S_{n-2}, & \Delta S_{n-1} \\ \Delta S_{n-1}, & \Delta S\end{vmatrix}}{\Delta^2 S_{n-2}\Delta^2 S_n - \left(\Delta^2 S_{n-1}\right)^2}.$$

Validity conditions for the transformation: $D^*_{k,n} \neq 0$.

To ensure computation stability, it is advantageous to use the apparatus of branched continued fractions (BCF) [18] capable of providing an efficient algorithm for machine computation:

a) In the case of Aitken's transformation we get an algorithm in the form of a branched continued fraction with two branches:

$$\sigma_{1,n} = S_n + \frac{1}{1/\Delta S_n - 1/\Delta S_{n-1}}. \tag{10}$$

Formula (10) is also valid for analytical calculus as the approximation $|\sigma_{1,n} - S_n| = 0(\Delta S_n)$ can be easily evaluated straightforward. Furthermore, with (5) and (8) it is possible to get a numerical estimate for the approximation $|\sigma_{k,n} - S| = \alpha|S_{n+k} - S|$, where $0 < \alpha \leq 1$.

For the Shanks-Schmidt transformation (8) ($k = 1, 2, \ldots$), the effective condition for convergence acceleration is to be verified

$$(-1)^k \Delta^k S_n > 0 \tag{11}$$

where $\Delta^k = \sum_{j=0}^{k}(-1)^j C_k^j$ is the operator of central difference of order k, C_k^j are the binominal coefficients:

For $k = 1$, from (11) we have $S_{n+1} - S_n < 0$ i.e. $S_{n+1} < S_n$.

The sign on the right-hand side of (11) reverses if $\{S_n\}$ is divergent. In that case upper partial sums must be chosen.

a) Recursive algorithms (BCF) for calculating the Shanks-Schmidt transformation.

Numerical simulation of transformation (8) is implemented by means of the Wynn algorithm [7], which is insensitive to rounding and accumulated errors:

$$\varepsilon_{k+1}(S_n) = \varepsilon_{k-1}(S_{n+1}) + 1 / [\varepsilon_k(S_{n+1}) - \varepsilon_k(S_n)], \tag{12}$$

where $k = 0, 1, 2 \ldots$; $n = 2^k$; $\varepsilon_{-1}(S_n) = 0$; $\varepsilon_0(S_n) = S_n$; $\varepsilon_{2k}(S_n) = \sigma_{k,n}$; $\varepsilon_{2k+1}(S_n) = 1 / \sigma_{k,n}(\Delta S_n)$.

This method gives the highest-order convergence $\lim_{n\to\infty}|\varepsilon_{2k} - S| / |S_n - S| = 0$. If the denominator in (12) turns to zero, this indicates that exactness has been achieved at the previous stage and no further improvement is possible with this particular method. It should be noted that the ε-algorithm is closely connected with Thiele's interpolation formula for S_n as a rational function of n [18]:

$$\varepsilon_r(S_n) = \frac{1}{2}\{\rho_r(n) + S_{n+1}\}$$

where $\rho_r = S_{n+1} + 2/[1/\Delta S_{n+r} - 1/\Delta S_{n+r-1}]$. For $k = 1$, successive decomposition of recursive formula (12) yields

$$\sigma_{1,n} = \varepsilon_2(S_n) = \varepsilon_0(S_{n+1}) + 1/[\varepsilon_1(S_{n+1}) - \varepsilon_1(S_n)] = S_{n+1} + 1/[1/\Delta S_{n+1} - 1/\Delta S_n]$$

and so on. One can see that the ε-algorithm is a branched continued fraction with two branches. For $k = 2$ we have:

$$\sigma_{2,n} = \varepsilon_4(S_n) = \varepsilon_3(S_{n+1}) + 1/[\varepsilon_4(S_{n+1}) - \varepsilon_4(S_n)];$$
$$\varepsilon_3(S_{n+1}) = \varepsilon_2(S_{n+1}) + 1/[\varepsilon_3(S_{n+1}) - \varepsilon_3(S_n)];$$
$$\varepsilon_2(S_{n+1}) = \sigma_{1,\, n+1}.$$

b) Brezinski's Θ-algorithm can essentially accelerate linear convergence of the sequence $\{S_n\}$ as well [16].

Suppose there is a sequence such that $\{S_n\}$: $\lim_{n\to\infty} S_n = S$. We seek to construct a new sequence $\{\Theta_k^{(n)}\}$ that would converge to S faster than $\{S_n\}$. The new sequence is calculated using a recursive algorithm defined as:

$\Theta_{-1}^{(n)} = 0,\ \Theta_0^{(n)} = S_n$ – the initial conditions;

$$\Theta_{2k+1}^{(n)} = \Theta_{2k-1}^{(n)} + 1/\Delta\Theta_{2k}^{(n)},\ \ k = 0,\ 1,\ 2,\ \ldots;\ \ n = 2^k;$$

$$\Theta_{2k+2}^{(n)} = \Theta_{2k}^{(n+1)} \cdot \Delta\Theta_{2k+1}^{(n+1)} / \Delta^2\Theta_{2k+1}^{(n)},$$

for which $\lim_{n\to\infty} \left|\Theta_{2k+2}^{(n)} - S\right| / |S_n - S| = 0$. Furthermore, even the Wynn algorithm is accelerated by this recursive algorithm.

c) Levin's nonlinear recursive algorithm. If $R_n = S_n - S = A_r \sum_{i=0}^{k-1} \gamma_i / r^i$ is explicitly dependent on n, where $r = n, n+1, n+k$, γ_i denotes arbitrary real numbers, and A_r is a function of S_r, then acceleration is best achieved by means of Levin's transformations [17] having the form

$$T_{k,\,n} = \Delta^k B_n / \Delta^k C_n, \tag{13}$$

where $C_n = n^{k-1}/A_n,\ B_n = n^{k-1}S_n/A_n,\ \Delta^k = \sum_{j=0}^{k}(-1)^j C_k^j$ is the difference operator.

Depending on the form of (13), $A_n = \Delta S_n$, $A_n = n\Delta S_n$, $A_n = \Delta S_n\ \Delta S_{n+1} / \Delta^2 S_n$ are referred to as *t*-, *w*-, and *v*-transformations, respectively. For $k = 1$, the *t*-transformation coincides with Aitken's transformation. Numerical implementation of this algorithm is based on the formulas

$$T_{k,\,n} = M_k^{(n)} / N_k^{(n)};\ k = 0,\ 1,\ 2,\ \ldots;\ n = 2^k \tag{14}$$

where $M_k^{(n)} = \left(M_{k-1}^{(n)} - M_{k-1}^{(n+1)}\right) / \left(\frac{1}{n} - \frac{1}{n+k+1}\right), M_{-1}^{(n)} = S_n / A_n;$

$$N_k^{(n)} = \left(N_{k-1}^{(n)} - N_{k-1}^{(n+1)}\right) / \left(\frac{1}{n} - \frac{1}{n+k+1}\right), N_{-1}^{(n)} = 1 / A_n.$$

This algorithm is fairly simple, computable and capable of convergence acceleration $\lim\limits_{n\to\infty}\left|T_{k,n} - S\right| / \left|S_n - S\right| = 0.$

4. The problem of flow around a foil near separation boundary

In the case of small perturbations, the solution to the problem comes down to solving the integral equation [12, 13]

$$\frac{1}{2\pi}\int_{-1}^{+1}\gamma(s)\left\{\frac{1}{\xi - s} - G_1\right\}\mathrm{d}s = -F_{\text{cp}}(\xi) - \frac{1}{2\pi}\int_{-1}^{+1}\chi(s) G_2 \mathrm{d}s, \tag{15}$$

where χ and F_{cp} are functions of the hydrofoil shape and kernels D_1 and G_2 are given by $G_1 = (\xi - s)/\Delta$; $G_2 = \varepsilon / \Delta$; $\Delta(\xi - s)^2 + \varepsilon^2$; $\varepsilon = 4\bar{h}(0)$. The foil shape is defined as $F = F_{\text{н}} \pm F_{\text{c}}$, where $F_{\text{н}}$ is the centerline equation and F_c is the thickness distribution. In the first approximation of the no-penetration boundary condition we obtain:

a) $F_{\text{cp}}^{(1)}(\xi) = -\alpha + \sum\limits_0^N a_n \xi^n$, if $F_{и} = \sum\limits_0^N \frac{a_n}{n+1}\xi^{n+1}$;

b) $\chi^{(1)}(\xi) = 2w(\xi)\sum\limits_0^M b_n \xi^n + 2w'(\xi)\sum\limits_0^M \frac{b_n}{n+1}\xi^{n+1}$ if $F_{\text{c}} = \sum\limits_0^M \frac{b_n}{n+1}\xi^{n+1} w(\xi)$, where $w = 1$ for a sharp leading edge and $w = \sqrt{1 - \xi^2}$ for a rounded one.

Equation (15) is an integral Fredholm equation of the first kind with a singularity in its kernel

$$\int_a^b k(\xi,\ s,\ \varepsilon)\gamma(s)\mathrm{d}s = f(\xi). \tag{16}$$

The solution γ of Eq. (16) is known [16] to be unstable even at small errors in $f(\xi)$ data and sensitive to errors in the kernel $k(\xi, s)$ and it is virtually independent of the solving technique.

The problem itself, (15), is essentially ill-defined and requires proper regularization techniques to enable its solution [20]. What makes it ill-posed when $\bar{h} \to 0$ is that integral equation (15) degenerates to a differential equation that is linear with respect to the higher derivative. By means of unsophisticated mathematics [12, 13] integral equation (15) of the flow boundary-value problem is rewritten as

$$\frac{\varepsilon^2}{2\pi}\sum_{n=1}^{n}\frac{(-1)^n}{n!}\gamma^{(n)}(\xi) W_\alpha^n(\xi,\ \varepsilon) + \frac{1}{2\pi}\gamma(\xi) W_1\langle\xi, s\rangle = \Phi(\xi), \tag{17}$$

where $\Phi(\xi) = -F_{\text{cp}}(\xi) - \frac{1}{2\pi}\chi(\xi) W_0(\xi,\ \varepsilon) - \frac{1}{2\pi}\sum\limits_{n+1}^{N}\frac{(-1)^n}{n!}\chi_{(\xi)}^{(n)} W_\delta^n(\xi,\ \varepsilon)$,

$$W_\alpha^n(\xi,\ \varepsilon) = \int_{-1+\xi}^{1+\xi}\frac{U^{n-1}\mathrm{d}U}{U^2 + \varepsilon^2}$$

$$W_1(\xi,\ \varepsilon) = \int_{-1}^{+1}\left\{\frac{1}{\xi - s} - G_1\right\}\mathrm{d}s = \ln\frac{1+\xi}{1-\xi}\sqrt{\frac{(1-\xi)^2 + \varepsilon^2}{(1+\xi)^2 + \varepsilon^2}}$$

$$W_0(\xi,\ \varepsilon) = \int_{-1}^{+1} G_2 \mathrm{d}s = \operatorname{arctg}\frac{1-\xi}{\varepsilon} + \operatorname{arctg}\frac{1+\xi}{\varepsilon};$$

$$W_{\delta}^{n}(\xi, \varepsilon)=\int_{-1}^{+1}(\xi-s)^{n} G_{2} \mathrm{d}s, \quad \varepsilon=4\bar{h}(0)$$

From Eq. (17) it follows that when $\bar{h} \to 0$, problem (15) reduces to a boundary-layer-type problem, which sheds light on why the problem is ill-posed. The physical conditions impose limitation on N.

This problem can be solved employing a hybrid approach. First, solutions to the exterior, (15), and interior, (17), problems are found [11, 12] and then these are mutually adjusted.

There is yet another way to tackle the problem. It starts with constructing a perturbed exterior solution and then this solution, which is ill-suited for $\bar{h} \to 0$, is transformed so that it is able to reveal the nature of singularity. The resultant solution thus becomes uniformly valid everywhere and provides good approximation to the true solution. The solution can be further improved quantitatively via higher-order approximations and, finally, nonlinear methods can be applied to accelerate convergence of the functional sequence [12, 13].

If the sequence is divergent, which depends on the class of the function f(ξ) in (15), then the linear transformations and algorithms discussed above allow the main term of the sequence to be extracted. Write the solution of integral equation (15) as γ = γ1 + γ2. The first term is associated with the influence of the centerline shape on circulation, while the second one is attributed to the dynamic curvature resulting from the flow around a profile near the separation boundary.

5. The functional parameter method

Represent γ_1 solution in the form of a $\tau=\sqrt{1+4\bar{h}^2}-2\bar{h}$ series obtained by mapping $\bar{h} \in [0,\infty]$ to $\tau \in [1,0]$:

$$\gamma_1^{(n)}=\sum_{m=0}^{N} \gamma_{1(2m)}^{(n)} \tau^{2m} \tag{18}$$

There is an expansion for the kernel with respect to parameter τ such that:

$$G_1=\sum_{m=1}^{N} k_{1m} \tau^{2m}, \tag{19}$$

with k_{1m} defined in [12].

Series (19) is convergent. Moreover, it is convergent over the entire actual range of variation of the parameter r. Convergence of series (18) remains questionable because it is not possible to construct a general term. It is however possible to evaluate a convergence domain for specific foil shapes. Substituting (18) and (19) into (15) and resolving the solution into two terms (terms of the same τ power are taken equal) yields a system of singular integral equations with a Cauchy-type kernel:

$$\int_{-1}^{+1} \gamma_{1(2m)}^{(n)}(s) \frac{\mathrm{d}s}{\xi-s}=\Phi_m^{(n)}(\xi), \; m=0,\ 1,\ 2,\ \ldots N; \quad n=1,\ 2,\ \ldots \tag{20}$$

Converting this equation into a_{∞}^{0} class functions we have

$$\gamma_{1(2m)}^{(n)}(\xi)=-\frac{1}{\pi^2}\sqrt{\frac{1-\xi}{1+\xi}} \int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \frac{\Phi_m^{(n)}(s)}{\xi-s} \mathrm{d}s. \tag{21}$$

The function $\Phi_m^{(n)}(m=1,\ 2,\ ...,\ N)$ is found from the boundary conditions via the solutions $\gamma_{1(2m)}^{(n)}(m=1,\ 2,\ ...,\ N-1)$. Let us now write the solutions for $N = 13$ and $n = 1$ for a plate when $F_{\text{cp}}^{(1)} = -\alpha$.

$$\Phi_0^{(1)} = -2\pi\alpha;\ \ \Phi_1^{(1)} = \int_{-1}^{+1} \gamma_{10}^{(1)} k_{11} \mathrm{d}s;$$
$$\Phi_2^{(1)} = \int_{-1}^{+1} \left\{ \gamma_{10}^{(1)} k_{12} + \gamma_{12}^{(1)} k_{11} \right\} \mathrm{d}s; \tag{22}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Phi_m^{(1)} = \int_{-1}^{+1} \left\{ \sum_1^m \gamma_{1[2(k-1)]}^{(1)} k_{1(m-k+1)} \right\} \mathrm{d}s.$$

Substituting (22) into (21) gives

$$\gamma_{10}^{(1)}(\xi) = 2\alpha\sqrt{\frac{1-\xi}{1+\xi}};\quad \gamma_{1(2m)}^{(1)} = \gamma_{10}^{(1)} W_{2m}. \tag{23}$$

where $m = 1, 2, \ldots, 13$;

$$W_2 = \frac{3}{2} + \xi;$$

$$W_4 = \frac{9}{8} - \frac{1}{2}\xi - \frac{5}{2}\xi^2 - \xi^3;$$

$$W_6 = \frac{15}{16} - \frac{9}{8}\xi - \frac{5}{4}\xi^2 + 3\xi^3 + \frac{7}{2}\xi^4 + \xi^5;$$

$$W_8 = \frac{105}{128} - \frac{1}{16}\xi + \frac{53}{16}\xi^2 + \frac{43}{8}\xi^3 - \frac{21}{8}\xi^4 - \frac{15}{2}\xi^5 - \frac{9}{2}\xi^6 - \xi^7;$$

$$W_{10} = \frac{255}{256} + \frac{217}{128}\xi + \frac{145}{32}\xi^2 - \frac{19}{4}\xi^3 - \frac{165}{8}\xi^4 - \frac{33}{4}\xi^5 + \frac{27}{2}\xi^6 + 14\xi^7 + \frac{11}{2}\xi^8 + \xi^9$$

$$W_{12} = \frac{1227}{1024} + \frac{471}{256}\xi - \frac{617}{256}\xi^2 - \frac{2553}{128}\xi^3 - \frac{889}{64}\xi^4 + \frac{305}{8}\xi^5 + \frac{415}{8}\xi^6 - \frac{3}{4}\xi^7 - \frac{275}{8}\xi^8 - $$
$$-\frac{45}{2}\xi^9 - \frac{13}{2}\gamma^{10} - \xi^{11};$$

$$W_{14} = \frac{2613}{2048} - \frac{263}{1024}\xi - \frac{5825}{512}\xi^2 - \frac{1631}{128}\xi^3 + \frac{11943}{256}\xi^4 + \frac{10773}{128}\xi^5 - \frac{837}{32}\xi^6 - \frac{553}{4}\xi^7 -$$
$$-\frac{1295}{16}\xi^8 + \frac{305}{8}\xi^9 + \frac{273}{4}\xi^{10} + 33\xi^{11} + \frac{15}{2}\xi^{12} + \xi^{13};$$

$$W_{16} = \frac{36297}{32768} - \frac{5617}{2048}\xi - \frac{20883}{2048}\xi^2 + \frac{27063}{1024}\xi^3 + \frac{99597}{1024}\xi^4 - \frac{6939}{256}\xi^5 - \frac{73617}{256}\xi^6 -$$
$$-\frac{20669}{128}\xi^7 + + \frac{32835}{128}\xi^8 + \frac{5325}{16}\xi^9 + \frac{1035}{16}\xi^{10} - \frac{101}{8}\xi^{11} - \frac{945}{8}\xi^{12} - \frac{91}{2}\xi^{13} - \frac{17}{2}\xi^{14} - \xi^{15};$$

$$W_{18} = \frac{4500439}{65536} + \frac{2374493}{32768}\xi + \frac{81825}{1024}\xi^2 + \frac{295437}{2048}\xi^3 + \frac{13037}{128}\xi^4 - \frac{100183}{512}\xi^5 - \frac{25677}{128}\xi^6 +$$
$$+\frac{165239}{256}\xi^7 + \frac{33753}{32}\xi^8 + \frac{2049}{8}\xi^9 - \frac{20049}{32}\xi^{10} - 594\xi^{11} + 77\xi^{12} + 294\xi^{13} + 187\xi^{14} +$$
$$+60\xi^{15} + \frac{19}{2}\xi^{16} + \xi^{17};$$

$$W_{20} = -\frac{12616275}{262144} - \frac{5179823}{65536}\xi - \frac{8336061}{65536}\xi^2 - \frac{5550025}{32768}\xi^3 - \frac{3159457}{8192}\xi^4 - \frac{745381}{1024}\xi^5 +$$
$$+\frac{149741}{1024}\xi^{13} - \frac{1885}{4}\xi^{14} - \frac{1151}{2}\xi^{15} - \frac{2223}{8}\xi^{16} - \frac{153}{2}\xi^{17} - \frac{21}{2}\xi^{18} - \xi^{19};$$

$$W_{22}=\frac{132266203353}{131072}+\frac{317275015003}{262144}\xi+\frac{158781624145}{131072}\xi^2+\frac{28124698217}{16384}\xi^3+$$
$$+\frac{112600162755}{65536}\xi^4+\frac{67212868461}{32768}\xi^5+\frac{420259071}{2048}\xi^6+\frac{2705060269}{2048}\xi^7+\frac{2688570425}{2048}\xi^8+$$
$$+\frac{80850475}{265}\xi^9+\frac{41139813}{128}\xi^{10}+\frac{6691095}{256}\xi^{11}+\frac{5094639}{256}\xi^{12}-\frac{299933}{64}\xi^{13}-\frac{66759}{16}\xi^{14}-$$
$$-\frac{1085}{2}\xi^{15}+\frac{20835}{16}\xi^{16}+\frac{8091}{8}\xi^{17}+\frac{1575}{4}\xi^{18}+95\xi^{19}+\frac{23}{2}\xi^{20}+\xi^{21};$$

$$W_{24}=\frac{17129979402909}{4194304}+\frac{1241572992573}{262144}\xi+\frac{1889838437643}{524288}\xi^2+\frac{1355616985531}{262144}\xi^3+$$
$$+\frac{1311925791219}{262144}\xi^4+\frac{396503371015}{65536}\xi^5+\frac{388978944203}{65536}\xi^6+\frac{106117200107}{32768}\xi^7+$$
$$+\frac{58073081641}{16384}\xi^8+\frac{116789059}{512}\xi^9+\frac{337015355}{512}\xi^{10}-\frac{94300235}{512}\xi^{11}-\frac{28915279}{512}\xi^{12}-$$
$$-\frac{4595813}{64}\xi^{13}-\frac{612715}{16}\xi^{14}-\frac{1086565}{16}\xi^{15}-\frac{9434469}{128}\xi^{16}-\frac{13413}{16}\xi^{17}-\frac{45135}{16}\xi^{18}-$$
$$-\frac{13185}{8}\xi^{19}-\frac{4301}{8}\xi^{20}-\frac{231}{2}\xi^{21}-\frac{25}{2}\xi^{22}-\xi^{23};$$

If we do not go beyond linearized formulation of the problem, then with (18) and (23) we can find the resultant aerodynamic characteristics – the lift force and pitching momentum coefficients – using the following formulas

$$C_Y^{(n)}=\int_{-1}^{+1}\gamma^{(n)}\mathrm{d}s;\quad C_{\mathrm{M}}^{(n)}=\int_{-1}^{+1}\gamma^{(n)}(s)s\mathrm{d}s. \tag{24}$$

In the first boundary conditions approximation, at n = 1 we get the functions of influence of the distance parameter $\bar{h}(0)$ on $C_Y^{(1)}$ and $C_{\mathrm{M}}^{(1)}$ in the form of τ-series:

$$\psi_Y^{(1)}=\frac{C_\gamma^{(1)}}{C_{\gamma\infty}^{(1)}}=1+\varepsilon+\frac{1}{2}\varepsilon^2+\frac{3}{4}\varepsilon^3+\frac{23}{32}\varepsilon^4+\frac{11}{16}\varepsilon^5+\frac{39}{64}\varepsilon^6+\frac{163}{256}\varepsilon^7+\frac{5491}{8192}\varepsilon^8+$$
$$+\frac{2244877}{32768}\varepsilon^9-\frac{5379225}{65536}\varepsilon^{10}+\frac{528804484515}{524288}\varepsilon^{11}+\frac{7641989002175}{2097152}\varepsilon^{12}+$$
$$+\frac{37205275391907}{16777216}\varepsilon^{13}+\ldots;$$

$$\psi_{\mathrm{M}}^{(1)}=\frac{C_{\mathrm{M}}^{(1)}}{C_{\mathrm{M}\infty}^{()}}=1+\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon^2+\frac{7}{16}\varepsilon^3+\frac{15}{32}\varepsilon^4+\frac{15}{32}\varepsilon^5+\frac{29}{64}\varepsilon^6+\frac{899}{2048}\varepsilon^7+$$
$$+\frac{3587}{8192}\varepsilon^8+\frac{192315}{16384}\varepsilon^9-\frac{1604301}{6536}\varepsilon^{10}+\frac{128261286621}{524288}\varepsilon^{11}+ \tag{25}$$
$$+\frac{2091064690271}{2097152}\varepsilon^{12}+\frac{4259769570263}{8388608}\varepsilon^{13}+\ldots,$$

where $\varepsilon=\tau^2$, $C_{\gamma\infty}^{(1)}=2\pi\alpha$, $C_{\mathrm{M}\infty}^{(1)}=-\pi\alpha$.

Under formal application of the discussed method, the solution γ given by (18)–(23) is unstable [20]. It is considered that a coarse solution can be obtained taking just a few terms of the series; a further increase in the number of terms will only enhance the instability and the resultant multi-term series will have nothing to do with the true solution of Eq. (15). If however this instability is treated as a transient process, then it can be asserted that increasing the number of approximation terms provides additional information on the behavior of the true solution.

Let $\psi(\varepsilon)$ be an analytical function, then expression (25) can be treated as a Taylor-series expansion. The asymptotic behavior of coefficients of this expansion $\psi = \Sigma a_n \varepsilon^n$ is determined by the type of the function singularity. So it is natural to look for a way to deduce singularity parameters from a limited sequence $\{a_n\}$of coefficients of the series [17].

The radius of convergence of the Taylor series is determined by the singularity of the function $\psi(\varepsilon)$ that is closest to the point around which expansion is performed. Farther away singularities may appear to influence the behavior of the series coefficients as well. If this is the case, one should find the spectrum of singularities.

The algorithms considered above offer a solution to the problem of $\psi(\varepsilon)$ synthesis.

a) Numerical solution of Eq. (15).

Integral equation (15) can be solved by reducing it to a set of algebraic equations (discrete singularities method, collocation method and others). The mentioned methods are strongly unstable [20] as they use conventional quadrature formulas. Attempts to improve the result by increasing the number of nodes when $\varepsilon \to 0$ only aggravate the situation.

Solution of singularity equation (18) relies on regularization techniques, hence a proper choice of regularizator can essentially enhance stability of the computation scheme.

The structure of the solution $\gamma(\xi)=\sqrt{(1-\xi)/(1+\xi)}\cdot\nu(\xi)$, where $\nu(\xi)$ is a non-zero regular function for $\xi = \pm 1$, is determined by a specific behavior of the edge flow. After substitution of this solution into (15) and some simple manipulations we have

$$+\frac{\varepsilon^2}{2\pi}\int_{-1}^{+1}\sqrt{\frac{1-s}{1+s}}\frac{\nu(s)\psi(\xi,\, s)\mathrm{d}s}{\xi-s}=\Phi(\xi),$$

where $\psi(\xi, s) = 1/\Delta$ and $\Phi(\xi)$ is the right-hand side of Eq. (15).

With the nodes, reference points and quadrature coefficients chosen by the formulas

$$\xi_i=\cos\theta_i\ ;\quad s_j=\cos\theta_j\ ;\quad \theta_i=\pi(2i-1)/(2N+1),\ \ (i=1,\ 2,\ ...,\ N)\ ;\quad \theta_j=2\pi j/(2N+1);$$
$$A_j=2\pi\sqrt{1-s_j^2}/(2N+1)\ ,\ (j=1,\ 2,\ ...,\ N)\ ,$$

we get the following set of algebraic equations

$$\sum_{j=1}^{N}A_{i,j}Y_j=b_i,\ (i, j = 1, 2, \ldots, N \text{ is the number of nodes}), \tag{26}$$

Where:

$$A_{ij}=\psi(\xi_i,s_i,\varepsilon)/(\xi_i-s_j)\ ;\quad Y_j=(1-s_j)\nu(s_j)/(2N+1)\ ;$$
$$b_i=-F_{cp}(\xi_i)\Big/\varepsilon^2-\sum_{J=1}^{N}\chi(s_i)\sqrt{1-s_j^2}\,\psi(\xi_i,s_j,\varepsilon)\Big/\varepsilon(2N+1)\ ;\quad \psi=1/\Delta\ ;\ \Delta=(\xi_{\mathrm{i}}-s_j)^2+\varepsilon^2.$$

Aerodynamic characteristics of a foil are derived from formulas (24). Switching to the quadrature formulas gives

$$C_Y\cong 2\pi\sum_{j=1}^{N}Y_j\ \text{ and }\ C_M\cong 2\pi\sum_{j=1}^{N}Y_js_j. \tag{27}$$

The BCF algorithm is a powerful tool to handle integrals and resolve sets of algebraic equations due to the quadrature and cubature formulas:

$$S=T\left(\sum_{k=1}^{n}\bar{P}_kS(\bar{x}_k)\right)+R \tag{28}$$

Where: $\bar{P}_k$ – weights, $\bar{x}_k$ – nodes, R – remainder and the Nedashkovsky-Skorobogatko BCF method [16] acting as regularizators in the process. The advantage of this technique [16, 18] is that it is inherently self-regularized because of mutual cancellation of computational errors and as such it is little sensitive to adverse changes in the coefficient matrix conditioning. A good result is obtained when the BCF method is employed to accelerate convergence or find an antilimit of sequences. To that end, a set of equations is constructed with matrixes of Toeplitz type [21]. So, for finding the Shanks-Schmidt transformation $\sigma_{k,\,n}$ we have

$$\begin{bmatrix} 1;\ \Delta S_n\ \ldots\ \Delta S_{n+k-1} \\ 1;\ \Delta S_{n+1}\ \ldots\ \Delta S_{n+k} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1;\ \Delta S_{n+k}\ \ldots\ \Delta S_{n+2k-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \ldots \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} S_n \\ S_{n+1} \\ \ldots \\ S_{n+k} \end{bmatrix}. \tag{29}$$

Once (29) has been solved, transformation follows the formula

$$\sigma_{k,\,n} = 1/(z_1 + z_2 + \ldots + z_{k+1}). \tag{30}$$

The $\mathrm{T}_{k,\,n}$ transformation is implemented using the following system

$$\begin{bmatrix} 1;\ A_n;\ A_n/n;\ \ldots;\ A_n/n^{k-1} \\ 1;\ A_{n+1};\ A_{n+1}/n+1;\ \ldots;\ A_{n+1}/(n+1)^{k-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 1;\ A_{n+k};\ A_{n+k}/n+k;\ \ldots;\ A_{n+k}/(n+k)^{k-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \ldots \\ \gamma_{k+1} \end{bmatrix} = \begin{bmatrix} S_n \\ S_{n+1} \\ \ldots \\ S_{n+k} \end{bmatrix}, \tag{31}$$

where $A_n = \Delta S_n$, $A_n = n\Delta S_n$, $A_n = \Delta S_n\, \Delta S_{n+1} / \Delta^2 S_n$ for Levin's t-, w-, and v-transformations, respectively. Actually, we can limit ourselves to finding only the first two components of solutions, z_1 and γ_1, as the rest k components are required for analysis of the transformation spectrum.

6. Transformation of divergent sequences and series

If solution of Eq. (15) has been obtained in the form of a series, the sequence γ1(2m) (m = 0, 1, 2, …), as a rule, is divergent. Summation of such series and sequences is done using nonlinear transformations. Since representation (4) holds for {Sn}, then by analogy with the Fourier-series, the Fire summation is applied:

$$\sigma_n = \frac{1}{n}\sum_{1}^{n} S_k. \tag{32}$$

For the series

$$\sigma_n = \sum_{i=0}^{n} \frac{n-i+1}{n} a_i \varepsilon^i. \tag{33}$$

Sometimes we know the location of the singularity [10], giving rise to divergence of the series, on the axis or in the domain of variation of the parameter ε. A divergent series can be converted into an absolutely or conditionally convergent one by introducing some other comparison function. The Euler transformation [12] $\bar{\varepsilon} = \varepsilon/(1+\varepsilon)\,(\bar{\varepsilon} \in [0, 0.5])$ applied to the series in (25) yields new series

$$\bar{\psi} = \sum_{k=0}^{N} (-1)^k \Delta^k €_0 \bar{\varepsilon}^k \cdot (1-\bar{\varepsilon}),$$

where $\Delta a_0 = a_1 - a_0$; $\Delta^2 a_0 = a_2 - 2a_1 + a_0$; ...; $\Delta^k a_0 = \sum_{j=0}^{k} \epsilon_{k-j} (-1)^j c_k^j$; Δ^k is the operator of finite differences of order k, and C_k^j are the binomial coefficients. Reiterating the procedure $\tilde{\varepsilon} = \bar{\varepsilon} / (1+\bar{\varepsilon})$ it is feasible to eventually obtain a uniform approximation for the solution at the extremes of the specified interval $\bar{h}$. The source series is to be represented in the form $\psi = \sum_{k=0}^{N} (-1)^k a_k \varepsilon^k$.

After grouping the terms with the same power $\bar{\varepsilon}$ and dropping the terms of order $O(\bar{\varepsilon}^N)$ we finally obtain the following approximating expressions for the foil lift force and pitching momentum as an influence function of the $\bar{h}$ interval at small angles of attack:

$$
\begin{aligned}
\bar{\psi}_\gamma &= 1 + \bar{\varepsilon} + \frac{3}{2}\bar{\varepsilon}^2 + \frac{11}{4}\bar{\varepsilon}^3 + \frac{175}{32}\bar{\varepsilon}^4 + \frac{177}{16}\bar{\varepsilon}^5 + \frac{1423}{64}\bar{\varepsilon}^6 + \frac{11323}{256}\bar{\varepsilon}^7 + \frac{715923}{8192}\bar{\varepsilon}^8 + \\
&+ \frac{7870829}{32768}\bar{\varepsilon}^9 + \frac{56910769}{65536}\bar{\varepsilon}^{10} + \frac{530324997219}{524288}\bar{\varepsilon}^{11} + \frac{30926120625699}{2097152}\bar{\varepsilon}^{12} \\
&+ \frac{1887973458979331}{16777216}\bar{\varepsilon}^{13} + \ldots ; \\
\bar{\psi}_M &= 1 + \frac{1}{2}\bar{\varepsilon} + \bar{\varepsilon}^2 + \frac{31}{16}\bar{\varepsilon}^3 + \frac{121}{32}\bar{\varepsilon}^4 + \frac{239}{32}\bar{\varepsilon}^5 + \frac{951}{64}\bar{\varepsilon}^6 + \frac{60675}{2048}\bar{\varepsilon}^7 + \frac{483543}{8192}\bar{\varepsilon}^8 + \\
&+ \frac{2108939}{16384}\bar{\varepsilon}^9 + \frac{20321375}{65536}\bar{\varepsilon}^{10} + \frac{128639927981}{524288}\bar{\varepsilon}^{11} + \frac{7737516436603}{2097152}\bar{\varepsilon}^{12} + \\
&+ \frac{240090786810183}{8388608}\bar{\varepsilon}^{13} + \ldots .
\end{aligned}
\tag{34}
$$

While these series are slow to converge for large $\bar{h}$, they are uniformly valid in a larger domain. Therefore when nonlinear methods of convergence acceleration are employed even Aitken's transformation can yield the best approximation, with an accuracy of up to 2–3 significant figures for $\bar{h} \in [0.02,\ 0.1]$. The accuracy improves for larger $\bar{h}$.

Let us now write an analytical expression for the sequence $\sigma_{1,n}$($n = 0, 1, 2, \ldots, N-1$) derived from the series $\psi = \sum_0^N a_m \varepsilon^m$. Constructing a sequence of partial sums and substituting it into (9) we get

$$
\sigma_{1,n} = \sum_0^n b_m \varepsilon^m \Big/ \left(a_n - a_{n+1}\varepsilon\right), \tag{35}
$$

where $b_0 = a_0 a_n$; $b_1 = a_1 a_n - a_0 a_{n+1}$; $b_m = a_m a_n - a_{m-1} a_{n+1}$.

If in (35) $a_n - a_{n+1}\varepsilon \equiv 0$, then the nominator of the fraction should be examined for zeroes. Suppose, there is $\varepsilon_{кр} = a_n/a_{n+1}$ among the polynomial zeroes; this defect can be readily cured. When $\varepsilon_{кр}$ is a pole, we should set $\sigma_{1,n} = \sigma_{1,n-1}$. Sometimes appearance of a pole indicates the limits of series convergence. Thus for the ψ_Y series from (25) these are $\varepsilon_{кр} = a_n/a_{n+1}$, which agrees with the results obtained by other techniques [12, 13].

The amount of information that can be derived for a given number of approximations does not appear to be enough for a transformation to ensure the best approximation over the entire domain of variation of parameter ε. We then have to approach the true solution via various transformations for each h value.

The Shanks-Schmidt-Levin transformations are based on rational approximation of the series $\sum a_k x^k$. They complement each other as their representation error is associated with the Loran-series expansion of the function

$$f(z)=\sum_{k=0}^{\infty} a_k z^k+\sum_{k=1}^{\infty} b_k z^{-k}. \tag{36}$$

The first term in (36) refers to errors under Shanks and second one under Levin transformations. In a general case, it is necessary to have available these transformations along with the recursive techniques for their evaluation to be able to automatically monitor the situation. If it appears that the Shanks transformation fails or is slow to converge for $n_i \to \infty$, i.e. $\left(\sigma_{k,\, n_i}-\sigma_{k,\, n_{i+1}}\right)>\varepsilon$, we should then switch to the Levin transformation.

If Fire summation (33) is applied to series (25) and (34) followed by summation for each $\overline{h}$ using Shanks-Schmidt and Levin's algorithms then the result we obtain will agree with that of numerical simulation. The best analytic approximations employing transformation of $\sigma_{k,\, n}$ series (34) are obtained with rational fractions $\sigma_{1,\,3}$; $\sigma_{1,\,6}$; $\sigma_{2,\,7}$; $\sigma_{3,\,5}$; $\sigma_{3,\,8}$. The figure shows the influence function ψ_Y as evaluated by (25). Also shown are the results of Fire summation applied to series (34) followed by summation using the Shanks-Schmidt-Levin algorithm. Computational results based on this algorithm are in very good agreement with the numerical result for integral equation (15) solved by the collocation technique using BCF apparatus as prescribed by the given algorithm. The four-term expansion ψ_Y from (25) is

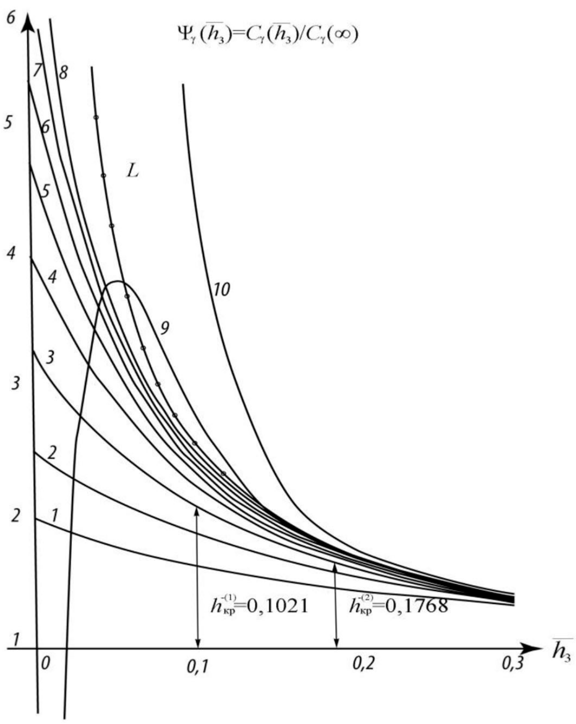

Fig. The function of influence of $\overline{h}$ interval on the lift force: L – influence function derived from (34) using Levin's transformation for each $\overline{h}$; o – Numerical simulation results for Eq. (15) obtained by the collocation method using the BCF apparatus uniformly valid over the interval $\bar{h} \in \left(\bar{h}_{kp}^{(1)},\ \bar{h}_{kp}^{(2)}\right)$, the quantitative result, however, being far from exact.

Optimal asymptotics contains nine terms, and still it rapidly deviates from the exact solution; however an approximation such as $\sigma_{3,\,8}$ for h < 0,01 already gives a relative error less than 1 %.

The discussed algorithms have been implemented as computational programs for algebraic (ALFA) and integral (OMEGA) equations, ordinary (SIMP) and improper (SECOB) integrals, including the Cauchy integral (DSECOB) as well as summation programs for sequences and series (SHENKS, AYTKEN), including divergent ones (EULER).

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