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A Nonlocal Problem for a Third Order Parabolic-Hyperbolic Equation with a Singular Coefficient

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Abstract. Non-classical problem with an integral condition for parabolic-hyperbolic equation of the third order is formulated and studied in this paper. The unique solvability of the problem was proved using the method integral equations. To do this the problem is equivalently reduced to a problem for a parabolic-hyperbolic equation of the second order with an unknown right-hand side. To study the obtained problem the formula of the Cauchy problem for hyperbolic equation with a singular coefficient and a spectral parameter was used. The solution of the first boundary value problem for the Fourier equation was also used.

Keywords: parabolic-hyperbolic equation, integral condition, uniqueness of the solution, existence of the solution, singular coefficient.

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Recently, problems with integral conditions for partial differential equations attract considerable interest. This is primarily due to the fact that problems with integral conditions have numerous applications in science and technology. Integral conditions appear when boundary conditions may not be available but the average value of the sought quantity is known. Conditions of this kind can appear in the mathematical modelling of phenomena in plasma physics, heat propagation, moisture transfer in capillary-porous media, demography processes and mathematical biology.

A problem with an integral condition was first considered by Cannon [1] and Kamynin [2] for the heat equation. Following these works, numerous problems with an integral condition for partial differential equations of the second order of parabolic, hyperbolic, elliptic types on the plane There are a number of works devoted to the study of problems with an integral condition for second order partial differential equations of mixed type. For example, problems with an integral condition for an elliptic-parabolic equation in a domain consisting of a rectangle and a semicircle were formulated and studied [26]. Problems for an elliptic-hyperbolic equation in a rectangular domain were considered [27].

Problems that are close to the subject of this work were considered in [3–14, 28–34]. Problems with an integral condition for a second order parabolic-hyperbolic equation with characteristic

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line of type changing were considered in a rectangular domain [28]. Problems with an integral condition for a second order parabolic-hyperbolic equation with non-characteristic line of type changing were considered [29, 30]. Parabolic-hyperbolic equations of the second order with characteristic line of type changing in the domain consisting of a rectangle and a characteristic triangle was considered and some problems with an integral condition in the domain of parabolicity of the equation were studied [31, 32]. Problems similar to equations of parabolic-hyperbolic type of the third order were studied [33, 34] where a model equation [33] and an equation with a spectral parameter [34] were considered.

A non-local problem with an integral condition for an equation of mixed parabolic-hyperbolic type of the third order with a singular coefficient in the hyperbolic part is formulated and studied in this paper.

1. Formulation of the problem

Let D be a finite simply connected domain bounded for $y > 0$ by lines $x = 0$, $x = 1$, $y = 1$ and for $y < 0$ it is bounded by straight lines $x + y = 0$, $xy = 1$, $D_1 = D \cap \{(x, y) : y > 0\}$, $D_2 = D \cap \{(x, y) : y < 0\}$, $D_0 = D \cap \{(x, y) : y = 0\}$.

Let us consider in domain D the following equation

$$(\partial/\partial x)\, Lu = 0, \tag{1}$$

where

$$L = \begin{cases} L_1 \equiv \left(\partial^2/\partial x^2\right) - \left(\partial/\partial y\right) - \lambda_1^2, & (x, y) \in D_1, \\ L_2 \equiv \left(\partial^2/\partial x^2\right) - \left(\partial^2/\partial y^{\,2}\right) - (2\beta/y)\,(\partial/\partial y) + \lambda_2^2, & (x, y) \in D_2, \end{cases}$$

β, λ_1 and λ_2 are given real numbers such that $0 < \beta < (1/2)$.

The equation $Lu = 0$ belongs to parabolic type in domain D_1 and it belongs to hyperbolic type in domain D_2 and segment D_0 is the line of type changing of the equation. The following problem with an integral condition for equation (1) is studied in domain D.

Problem 1. Find a function $u\,(x, y)$ with the following properties: 1) $u\,(x, y) \in C\left(\overline{D}\right)$, $u_x,\ u_y \in C\,(D \cup D_3)$; 2) $u\,(x, y)$ satisfies equation (1) in $D_1 \cup D_2$; 3) $u\,(x, y)$ satisfies the following conditions

$$u\,(0, y) = \varphi_1\,(y)\,, \quad u\,(1, y) = \varphi_2\,(y)\,, \quad 0 \leqslant y \leqslant 1; \tag{2}$$

$$\int_0^1 u\,(x, y)\, dx = \varphi_3\,(y)\,, \quad 0 \leqslant y \leqslant 1; \tag{3}$$

$$u\,(x, y)|_{D_3} = \psi_1\,(x)\,, \quad \left.\frac{\partial u}{\partial n}\right|_{D_3} = \psi_2\,(x)\,, \quad 0 \leqslant x \leqslant (1/2)\,; \tag{4}$$

$$\lim_{y \to +0} u_y\,(x, y) = \lim_{y \to -0} (-y)^{2\beta} u_y\,(x, y)\,, \quad 0 < x < 1, \tag{5}$$

where $D_3 = \{(x, y) : y = -x,\ 0 \leqslant x \leqslant 1/2\}$, n is the inner normal to D_3, and $\varphi_j\,(y)$, $\psi_1\,(x)$, $\psi_2\,(x)$ are given functions such that $\varphi_j\,(y) \in C^1\,[0, 1]$, $j = \overline{1, 3}$; $\psi_1(x) \in C^1\,[0, 1/2] \cap C^2\,(0, 1/2)$, $\psi_2\,(x) \in C\,[0, 1/2] \cap C^1\,(0, 1/2)$, $\psi_1(0) = \varphi_1\,(0)$ and $\psi_1''\,(x)$, $\psi_2'\,(x) \in L_1\,[0, 1/2]$.

2. Pleminaries

To study the considered problem the following operators are used [35], [36]:

$$D_{0x}^{\gamma}\left[f\left(x\right)\right]\equiv\begin{cases}\dfrac{1}{\Gamma\left(-\gamma\right)}\displaystyle\int_0^x\left(x-t\right)^{-\gamma-1}f\left(t\right)dt, & \gamma<0,\\ \dfrac{d}{dx}D_{0x}^{\gamma-1}f\left(x\right), & \gamma\in\left(0,1\right),\end{cases}$$

$$A_{0x}^{m,\lambda_2}\left[g\left(x\right)\right]\equiv g\left(x\right)-\int_0^x g\left(t\right)\left(\frac{t}{x}\right)^m\frac{\partial}{\partial t}J_0\left[\left|\lambda_2\right|\sqrt{x\left(x-t\right)}\right]dt,\quad m=\overline{0,1},$$

$$B_{0x}^{m,\lambda_2}\left[g\left(x\right)\right]\equiv g\left(x\right)+\int_0^x g\left(t\right)\left(\frac{x}{t}\right)^{1-m}\frac{\partial}{\partial x}J_0\left[\left|\lambda_2\right|\sqrt{t\left(t-x\right)}\right]dt,\quad m=\overline{0,1},$$

where $J_\nu(x)$ is the Bessel function of the first kind [37], $\Gamma(z)$ is the gamma function [38]. These operators have the following properties

Lemma 1 ([35]). *For all $f\left(x\right)\in C\left(0,1\right)\cap L\left[0,1\right]$ the following equality is valid:*

$$D_{0x}^{\gamma}D_{0x}^{-\gamma}f\left(x\right)=f\left(x\right),\quad\gamma>0. \tag{6}$$

Lemma 2 ([35]). *If $0<2\beta<1$ and $x^{-\beta}f\left(x\right)\in C\left(0,1\right)\cap L\left[0,1\right]$ then the equality*

$$D_{0x}^{\beta}x^{2\beta-1}D_{0x}^{\beta-1}x^{-\beta}f\left(x\right)=x^{\beta-1}D_{0x}^{2\beta-1}f\left(x\right) \tag{7}$$

is valid.

Lemma 3 ([36,39]). *For all functions $g(x)\in C(0,1)\cap L[0,1]$ the following equalities are hold:*

$$A_{0x}^{m,\lambda_2}\left\{B_{0x}^{m,\lambda_2}\left[g\left(x\right)\right]\right\}=g\left(x\right),\qquad B_{0x}^{m,\lambda_2}\left\{A_{0x}^{m,\lambda_2}\left[g\left(x\right)\right]\right\}=g\left(x\right),\quad m=\overline{0,1}. \tag{8}$$

Lemma 4 ([39,40]). *If $\nu\left(x\right)\in C^{(0,\alpha)}\left(0,1\right)$, $\alpha>\beta>0$, $\left[x\left(1-x\right)\right]^{-2\beta}\nu\left(x\right)\in L\left[0,1\right]$ then the following equality*

$$A_{0x}^{1,\lambda_2}\left\{x^{\beta-1}D_{0x}^{2\beta-1}x^{\beta}B_{0x}^{1,\lambda_2}\left[\nu\left(x\right)x^{-\beta}\right]\right\}=\frac{x^{\beta-1}}{\Gamma\left(1-2\beta\right)}\int_0^x\nu\left(t\right)\left(x-t\right)^{-2\beta}\bar{J}_{-\beta}\left[\left|\lambda_2\right|\left(x-t\right)\right]dt \tag{9}$$

is valid.

In addition to the above lemmas the following statements are also used.

Lemma 5. *Any solution of the equation $\left(\partial/\partial x\right)L_2u=0$ in domain D_2 can be represented in the form*

$$u\left(x,y\right)=v\left(x,y\right)+\omega\left(y\right), \tag{10}$$

where $v\left(x,y\right)$ is the general solution of the equation

$$v_{xx}-v_{yy}-\frac{2\beta}{y}v_y+\lambda_2^2v=0,\quad\left(x,y\right)\in D_2, \tag{11}$$

and $\omega\left(y\right)$ is an arbitrary function from the class $C\left[-1/2\,,0\right]\cap C^2\left(-1/2\,,0\right)$.

Proof. Let $u(x,y)$ be a solution of the equation $(\partial/\partial x)\,L_2u=0$. Integrating this equation in domain D_2 with respect to x, we obtain

$$u_{xx}-u_{yy}-\frac{2\beta}{y}u_y+\lambda_2^2u=\omega_0\left(y\right),\quad (x,y)\in D_2, \tag{12}$$

where $\omega_0\left(y\right)$ is an arbitrary function from the class $C\left(-1/2,0\right)\cap L\left[-1/2,0\right]$. It is easy to verify that any function of the following form

$$\omega\left(y\right)=\int_a^y\frac{f_1\left(y\right)f_2\left(\eta\right)-f_1\left(\eta\right)f_2\left(y\right)}{\Delta\left(\eta\right)}\omega_0\left(\eta\right)d\eta,\qquad a=const\in\left[-1/2,0\right],$$

satisfies equation (12) and $f_1\left(y\right)=\left(-y\right)^{1/2-\beta}J_{1/2-\beta}(-|\lambda_2|y)$, $f_2(y)=\left(-y\right)^{1/2-\beta}J_{\beta-1/2}(-|\lambda_2|y)$ are linearly independent solutions of the homogeneous equation

$$\omega''\left(y\right)+\left(2\beta/y\right)\omega'\left(y\right)-\lambda_2^2\omega\left(y\right)=0,\quad y\in\left(-1/2,0\right), \tag{13}$$

$\Delta\left(y\right)=f_1\left(y\right)f'_2\left(y\right)-f_2\left(y\right)f'_1\left(y\right)\neq 0$ is the Wronskian of functions $f_1(y)$ and $f_2(y)$.

Therefore, equality (10) is true.

Now, let the function $u(x,y)$ be representable in form (10). Then, substituting (10) into $(\partial/\partial x)\,L_2u$ and taking into account that $v(x,y)$ is a solution of equation (11), we immediately obtain the equality $(\partial/\partial x)\,L_2u=0$. Lemma 5 is proved. □

The following lemma can be proved in a similar way.

Lemma 6. *Any solution of the equation $(\partial/\partial x)\,L_1u=0$ in D_1 can be represented as*

$$u\left(x,y\right)=w\left(x,y\right)+\delta\left(y\right), \tag{14}$$

where $w\left(x,y\right)$ is the general solution of the equation

$$w_{xx}-w_y-\lambda_1^2w=0, \tag{15}$$

and $\delta\left(y\right)$ is an arbitrary function from the class $C\left[0,1\right]\cap C^1\left(0,1\right)$.

3. Study of the problem

Let us prove the unique solvability of the problem 1. To do this representation (10) of the solution of the equation $(\partial/\partial x)\,L_2u=0$ is used. Obviously, the function

$$\omega_1(y)=(-y)^{1/2-\beta}\left[AJ_{1/2-\beta}(-|\lambda_2|y)+BJ_{\beta-1/2}(-|\lambda_2|y)\right]$$

is a solution of equations (11) and (13) where A and B are arbitrary constants. Taking this into account when considering Problem 1, one can assume without loss of generality that arbitrary function $\omega(y)$ in representation (10) satisfies the following conditions

$$\omega\left(0\right)=0,\quad \lim_{y\to 0}\left(-y\right)^{2\beta}\omega'\left(y\right)=0. \tag{16}$$

Otherwise, rewriting function (10) in the form $u(x,y)=[v(x,y)+\omega_1(y)]+[\omega(y)-\omega_1(y)]$, one can distribute A and B so that the new function $\tilde{\omega}\left(y\right)=\omega\left(y\right)-\omega_1\left(y\right)$ satisfies conditions (16). Let $u\left(x,y\right)$ be a solution of Problem 1. Taking into account the conditions of the problem, the following notation and assumptions are we introduced

$$\lim_{y\to+0}u\left(x,y\right)=\lim_{y\to-0}u\left(x,y\right)=\tau\left(x\right),\quad 0\leqslant x\leqslant 1; \tag{17}$$

$$\lim_{y\to+0} u_y(x,y) = \lim_{y\to-0} (-y)^{2\beta} u_y(x,y) = \nu(x), \quad 0 < x < 1; \tag{18}$$

$$\tau(x) \in C^1[0,1] \cap C^3(0,1), \ \ \nu(x) \in C[0,1] \cap C^2(0,1), \ \ \nu'(x) \in L[0,1]. \tag{19}$$

If equalities (10), (16)–(19) are taken into account then function $v(x,y)$ in domain D_2 can be treated as a solution of the modified Cauchy problem for equation (11) [39, 41]:

$$v(x,y) = \gamma_1 \int_0^1 \tau[x + y(1-2t)] T^{2\beta-2} \overline{I}_{\beta-1}[-2|\lambda_2| yT] dt - \\ - \gamma_2 (-y)^{1-2\beta} \int_0^1 \nu[x + y(1-2t)] T^{-2\beta} \overline{I}_{-\beta}[-2|\lambda_2| yT] dt, \tag{20}$$

where $T=\sqrt{t(1-t)}$, $\gamma_1=\Gamma(2\beta)/\Gamma^2(\beta)$, $\gamma_2=\Gamma(1-2\beta)/\Gamma^2(1-\beta)$, $\overline{I}_\beta(x)=\Gamma(1+\beta)(x/2)^{-\beta} I_\beta(x)$, and $I_\beta(x)$ is the modified Bessel function [37].

Substituting the function $v(x,y)$ from (20) into (10), we find the function $u(x,y)$ as

$$u(x,y) = \gamma_1 \int_0^1 \tau[x + y(1-2t)] T^{2\beta-2} \overline{I}_{\beta-1}[-2|\lambda_2| yT] dt - \\ - \gamma_2 (-y)^{1-2\beta} \int_0^1 \nu[x + y(1-2t)] T^{-2\beta} \overline{I}_{-\beta}[-2|\lambda_2| yT] dt + \omega(y). \tag{21}$$

After satisfying condition $u(x,y)|_{D_3} = u(x,-x) = \psi_1(x)$, $x \in [0,1/2]$, we obtain

$$\gamma_1 \int_0^1 \tau(2xt) T^{2\beta-2} \overline{I}_{\beta-1}[2|\lambda_2| xT] dt - \\ - \gamma_2 x^{1-2\beta} \int_0^1 \nu(2xt) T^{-2\beta} \overline{I}_{-\beta}[2|\lambda_2| xT] dt + \omega(-x) = \psi_1(x), \quad x \in [0,1/2]. \tag{22}$$

Differentiating equality (22) with respect to x and using the equality $(d/dx)\overline{J}_\gamma(x) = -(x/2(\gamma+1))\overline{J}_{\gamma+1}(x)$, we obtain

$$\gamma_1 \int_0^1 \tau'(2xt) T^{2\beta-2} \overline{I}_{\beta-1}(2|\lambda_2| xT) 2t dt - \gamma_1 \int_0^1 \tau(2xt) T^{2\beta-2} \frac{2|\lambda_2| xT}{2\beta} \overline{I}_{-\beta}(2|\lambda_2| xT) 2|\lambda_2| T dt - \\ - \gamma_2 (1-2\beta) x^{-2\beta} \int_0^1 \nu(2xt) T^{-2\beta} \overline{I}_{-\beta}(2|\lambda_2| xT) dt - \gamma_2 x^{1-2\beta} \int_0^1 \nu'(2xt) 2t T^{-2\beta} \overline{I}_{-\beta}(2|\lambda_2| xT) dt + \\ + \gamma_2 x^{1-2\beta} \int_0^1 \nu(2xt) T^{-2\beta} \frac{2|\lambda_2| xT}{2-2\beta} \overline{I}_{1-\beta}(2|\lambda_2| xT) dt - \omega'(-x) = \psi_1'(x), \quad x \in [0,1/2]. \tag{23}$$

Let us calculate now $(\partial/\partial n)\, u|_{D_3}$. First, we find u_x and u_y:

$$u_x = \gamma_1 \int_0^1 \tau'[x + y(1-2t)] T^{2\beta-2} \overline{I}_{\beta-1}[-2|\lambda_2| yT] - \\ - \gamma_2 (-y)^{1-2\beta} \int_0^1 \nu'[x + y(1-2t)] T^{-2\beta} \overline{I}_{-\beta}(-2|\lambda_2| yT) dt,$$

$$u_y = \gamma_1 \int_0^1 \tau'[x + y(1-2t)](1-2t) T^{2\beta-2} \overline{I}_{\beta-1}(-2|\lambda_2| yT) dt -$$

$$
-\gamma_1 \int_0^1 \tau\left[x+y\left(1-2t\right)\right] T^{2\beta-2} \frac{\left(-2|\lambda_2|yT\right)}{2\beta} \overline{I}_{\beta}\left(-2|\lambda_2|yT\right)\left(-2|\lambda_2|T\right) dt+
$$

$$
+\gamma_2\left(1-2\beta\right)(-y)^{-2\beta} \int_0^1 \nu\left[x+y\left(1-2t\right)\right] T^{-2\beta} \overline{I}_{-\beta}\left(-2|\lambda_2|yT\right) dt-
$$

$$
-\gamma_2(-y)^{1-2\beta} \int_0^1 \nu'\left[x+y\left(1-2t\right)\right]\left(1-2t\right) T^{-2\beta} \overline{I}_{-\beta}\left(-2|\lambda_2|T\right) dt+
$$

$$
+\gamma_2(-y)^{1-2\beta} \int_0^1 \nu\left[x+y\left(1-2t\right)\right] T^{-2\beta} \frac{\left(-2|\lambda_2|yT\right)}{2\left(1-\beta\right)} \overline{I}_{1-\beta}\left(-2|\lambda_2|yT\right)\left(-2|\lambda_2|T\right) dt+\omega'\left(y\right).
$$

Then, according to the formula $\left(\partial/\partial n\right) u|_{D_3} = \left(u_x \cos\left(n,x\right)+u_y \cos\left(n,y\right)\right)|_{D_3} = \left(\sqrt{2}/2\right)\left(u_x+u_y\right)|_{D_3}$ and the second of boundary conditions (4), we obtain

$$
\gamma_1 \int_0^1 \tau'\left(2xt\right)\left(2-2t\right) T^{2\beta-2} \overline{I}_{\beta-1}\left(2|\lambda_2|xT\right) dt+
$$

$$
+\gamma_1 \int_0^1 \tau\left(2xt\right) T^{2\beta-2} \frac{\left(2|\lambda_2|xT\right)}{2\beta} \overline{I}_{-\beta}\left(2|\lambda_2|xT\right)\left(2|\lambda_2|T\right) dt+
$$

$$
+\gamma_2\left(1-2\beta\right) x^{-2\beta} \int_0^1 \nu\left(2xt\right) T^{-2\beta} \overline{I}_{-\beta}\left(2|\lambda_2|xT\right) dt-
$$

$$
-\gamma_2 x^{1-2\beta} \int_0^1 \nu'\left(2xt\right)\left(2-2t\right) T^{-2\beta} \overline{I}_{-\beta}\left(2|\lambda_2|xT\right) dt-
$$

$$
-\gamma_2 x^{1-2\beta} \int_0^1 \nu\left(2xt\right) T^{-2\beta} \frac{\left(2|\lambda_2|xT\right)}{2\left(1-\beta\right)} \overline{I}_{1-\beta}\left(2|\lambda_2|xT\right)\left(2|\lambda_2|T\right) dt+\omega'\left(-x\right)=
$$

$$
=\sqrt{2}\psi_2(x), \quad x\in[0,1/2].
$$

Combining this relation term by term with relation (23), we obtain

$$
2\gamma_1 \int_0^1 \tau'\left(2xt\right) T^{2\beta-2} \overline{I}_{\beta-1}\left(2|\lambda_2|xT\right) dt-2\gamma_2 x^{1-2\beta} \int_0^1 \nu'\left(2xt\right) T^{-2\beta} \overline{I}_{-\beta}\left(2|\lambda_2|xT\right) dt=
$$

$$
=\psi_1'(x)+\sqrt{2}\psi_2(x), \qquad x\in[0,1/2]. \tag{24}
$$

Using the change of variable $2x=z\in[0,1]$ in the last relation, we obtain

$$
\gamma_1 \int_0^1 \tau'\left(zt\right) T^{2\beta-2} \overline{I}_{\beta-1}\left(|\lambda_2|zT\right) dt-\gamma_2\left(\frac{z}{2}\right)^{1-2\beta} \int_0^1 \nu'\left(zt\right) T^{-2\beta} \overline{I}_{-\beta}\left(|\lambda_2|zT\right) dt=
$$

$$
=\psi_1'(z/2)+\sqrt{2}\psi_2(z/2), \ z\in[0,1].
$$

If we replace zt by ξ then $\xi\in[0,z], \ t=\xi/z, \ 1-t=(z-\xi)/z, \ dt=d\xi/z$. Then, taking into account $T=\sqrt{t\left(1-t\right)}=(1/z)\sqrt{\xi\left(z-\xi\right)}, \ \overline{I}_{\gamma}\left(|\lambda_2|zT\right)=\overline{I}_{\gamma}\left[|\lambda_2|\sqrt{\xi\left(z-\xi\right)}\right]$, we have

$$
\gamma_1 z^{1-2\beta} \int_0^z \tau'\left(\xi\right)\left[\xi\left(z-\xi\right)\right]^{\beta-1} \overline{I}_{\beta-1}\left[|\lambda_2|\sqrt{\xi\left(z-\xi\right)}\right] d\xi-
$$

$$
-\gamma_2 2^{2\beta-1} \int_0^z \nu'\left(\xi\right)\left[\xi\left(z-\xi\right)\right]^{-\beta} \overline{I}_{-\beta}\left[|\lambda_2|\sqrt{\xi\left(z-\xi\right)}\right] d\xi=\Phi\left(z\right), \ z\in[0,1], \tag{25}
$$

where $\Phi\left(z\right)={\psi'}_1\left(z/2\right)+\sqrt{2}\psi_2\left(z/2\right)$.

Let us denote the first and second integrals in the right-hand side of equality (25) by l_1 and l_2 and transform them. By virtue of the equality

$$(z-\xi)^{\beta-1}\,\overline{I}_{\beta-1}\left[|\lambda_2|\sqrt{\xi\,(z-\xi)}\right]=\frac{\partial}{\partial z}\int_{\xi}^{z}(z-t)^{\beta-1}\,J_0\left[|\lambda_2|\sqrt{\xi\,(\xi-t)}\right]dt,$$

which can be easily proved using the expansion of functions $\overline{I}_{\beta-1}(x)$ and $J_0(x)$ in power series, we rewrite l_1 as

$$l_1=\int_0^z\tau'(\xi)\xi^{\beta-1}\left\{\frac{\partial}{\partial z}\int_{\xi}^{z}(z-t)^{\beta-1}\,J_0\left[|\lambda_2|\sqrt{\xi\,(\xi-t)}\right]dt\right\}d\xi.$$

Integrating by parts the integral over t and performing the external operation $(\partial/\partial z)$, we have

$$l_1=\int_0^z\tau'(\xi)\xi^{\beta-1}\left\{(z-\xi)^{\beta-1}+\int_{\xi}^{z}(z-t)^{\beta-1}\,\frac{\partial}{\partial t}J_0\left[|\lambda_2|\sqrt{\xi\,(\xi-t)}\right]dt\right\}d\xi.$$

Hence, changing the order of integration in the integral and changing the specification of variables, we find

$$l_1=\int_0^z(z-\xi)^{\beta-1}\left\{\tau'(\xi)\xi^{\beta-1}+\int_0^{\xi}\tau'(t)t^{\beta-1}\frac{\partial}{\partial\xi}J_0\left[|\lambda_2|\sqrt{t\,(t-\xi)}\right]dt\right\}d\xi.$$

By virtue of notation D_{0x}^{γ} and B_{0x}^{m,λ_2} we obtain from the last relation

$$l_1=\Gamma(\beta)D_{0z}^{-\beta}B_{0z}^{1,\lambda_2}[\tau'(z)z^{\beta-1}]. \tag{26}$$

Similarly, we find

$$l_2=\Gamma(1-\beta)D_{0z}^{\beta-1}B_{0z}^{1,\lambda_2}[\nu'(z)z^{-\beta}]. \tag{27}$$

Due to (26) and (27), relation (25) can be rewritten as

$$\gamma_1\Gamma\left(\beta\right)x^{1-2\beta}D_{0x}^{-\beta}B_{0x}^{1,\lambda_2}\left[\tau'\left(x\right)x^{\beta-1}\right]-\gamma_2 2^{2\beta-1}\Gamma\left(1-\beta\right)D_{0x}^{\beta-1}B_{0x}^{1,\lambda_2}\left[\nu'\left(x\right)x^{-\beta}\right]=\Phi\left(x\right). \tag{28}$$

From here, applying the operator $A_{0x}^{1,\lambda_2}D_{0x}^{\beta}x^{2\beta-1}$ and taking into account (6)–(9), we have

$$\tau'\left(z\right)=\gamma_3\int_0^z\nu'\left(t\right)\left(z-t\right)^{-2\beta}\overline{J}_{-\beta}\left[|\lambda_2|\left(z-t\right)\right]dt+F\left(z\right), \tag{29}$$

where $\gamma_3=2^{2\beta-1}\Gamma\left(\beta\right)/\Gamma\left(1-\beta\right)\Gamma\left(2\beta\right)$, $F(x)=\Gamma\left(\beta\right)x^{1-\beta}A_{0x}^{1,\lambda_2}D_{0x}^{\beta}\left[x^{2\beta-1}\Phi\left(x\right)\right]/\Gamma\left(2\beta\right)$.

Integrating (29) with respect to z from 0 to x, we obtain

$$\tau(x)=\tau(0)+\gamma_3\int_0^x\nu'\left(t\right)M\left(x-t\right)dt+F_1\left(x\right), \tag{30}$$

where $F_1\left(x\right)=\int\limits_0^x F\left(z\right)dz$,

$$M\left(x-t\right)=\int_t^x\left(z-t\right)^{-2\beta}\overline{J}_{-\beta}\left[|\lambda_2|\left(z-t\right)\right]dz=\sum_{k=0}^{\infty}\frac{\Gamma\left(1-\beta\right)\left(-1\right)^k}{k!\Gamma\left(1+k-\beta\right)}\left(\frac{\lambda_2}{2}\right)^{2k}\frac{\left(x-t\right)^{1+2k-2\beta}}{1+2k-2\beta}.$$

Applying the integration by parts from relation (30), we find

$$\tau\left(x\right)=\psi_1\left(0\right)-\gamma_3\nu\left(0\right)M\left(x\right)+F_1\left(x\right)+\gamma_3\int_0^x\nu\left(t\right)\left(x-t\right)^{-2\beta}\overline{J}_{-\beta}\left[|\lambda_2|\left(x-t\right)\right]dt. \tag{31}$$

Multiplying both sides of (23) by $x^{2\beta}$ and then setting the limit as $x \to 0$, we obtain

$$\lim_{x\to 0} x^{2\beta}\psi_1'(x) = -\gamma_2(1-2\beta)\nu(0)\int_0^1 T^{-2\beta}dt = -\gamma_2(1-2\beta)\nu(0)\int_0^1 [t(1-t)]^{-\beta}dt = -\nu(0).$$

Taking the last relation into account, we obtain from (31) that

$$\tau(x) = \gamma_3\int_0^x \nu(t)(x-t)^{-2\beta}\bar{J}_{-\beta}[|\lambda_2|(x-t)]\,dt + F_2(x), \tag{32}$$

where $F_2(x) = \psi_1(0) + \gamma_3 M(x)\lim\limits_{x\to 0} x^{2\beta}\psi_1'(x/2) + F_1(x)$.

Introducing the notation $F_3(x) = \gamma_3^{-1}[\tau(x) - F_2(x)]$, we obtain from (32) an integral equation with respect to $\nu(x)$:

$$\int_0^x \nu(t)(x-t)^{-2\beta}\bar{J}_{-\beta}[|\lambda_2|(x-t)]\,dt = F_3(x).$$

Solving this integral equation [39], we obtain the relation between unknown functions $\tau(x)$ and $\nu(x)$ which is brought to D_0 from domain D_2

$$\nu(x) = \gamma_4 C_{0x}^{1,\lambda_2}[\tau(x) - F_2(x)], \quad 0 < x < 1, \tag{33}$$

where $\gamma_4 = \gamma_3{}^{-1}\Gamma^{-1}(1-2\beta) = 2^{1-2\beta}\Gamma(1-\beta)\Gamma(2\beta)/\Gamma(\beta)\Gamma(1-2\beta)$,

$$C_{0x}^{1,\lambda_2}[q(x)] \equiv \frac{1}{\Gamma(2\beta)}\left\{\frac{d}{dx}\int_0^x \frac{\bar{J}_\beta[|\lambda_2|(x-t)]}{(x-t)^{1-2\beta}}q(t)\,dt + \frac{\lambda_2^2}{4(\beta+\beta^2)}\int_0^x \frac{\bar{J}_{\beta+1}[|\lambda_2|(x-t)]}{(x-t)^{-2\beta}}q(t)\,dt\right\}.$$

Performing the same transformations that we do to obtain (28) from (24), we have from (22) that

$$\gamma_1\Gamma(\beta)x^{1-2\beta}D_{0x}^{-\beta}B_{0x}^{1,\lambda_2}\left[\tau(x)x^{\beta-1}\right] - \gamma_2 2^{2\beta-1}\Gamma(1-\beta)D_{0x}^{\beta-1}B_{0x}^{1,\lambda_2}\left[\nu(x)x^{-\beta}\right] =$$
$$= \psi_1(x/2) - \omega(-x/2), \quad 0 \leqslant x \leqslant 1. \tag{34}$$

Further, taking into account (8) and $x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}D_{0x}^{2\beta-1}x^\beta g(x) = D_{0x}^{\beta-1}g(x)$ (which can be verified by using the operator $D_{0x}^{\beta}x^{2\beta-1}$) and introducing the notation $x^\beta g(x) = f(x)$ (in this case it takes form (7)), we have

$$x^{1-2\beta}D_{0x}^{-\beta}B_{0x}^{1,\lambda_2}\left\{A_{0x}^{1,\lambda_2}x^{\beta-1}D_{0x}^{2\beta-1}x^\beta B_{0x}^{1,\lambda_2}\left[\nu(x)x^{-\beta}\right]\right\} =$$
$$= x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}D_{0x}^{2\beta-1}x^\beta B_{0x}^{1,\lambda_2}\left[\nu(x)x^{-\beta}\right] = D_{0x}^{\beta-1}B_{0x}^{1,\lambda_2}\left[\nu(x)x^{-\beta}\right]. \tag{35}$$

Then, substituting the function $\tau(x)$ from (32) into (34) and taking into account (9), (35) and (26), we find the unknown function $\omega(x)$ in the form

$$\omega\left(-\frac{x}{2}\right) = \psi_1\left(\frac{x}{2}\right) - \gamma_1 x^{1-2\beta}\int_0^x F_2(\xi)\,[\xi(x-\xi)]^{\beta-1}\bar{I}_{\beta-1}\left[|\lambda_2|\sqrt{\xi(x-\xi)}\right]d\xi, \quad 0 \leqslant x \leqslant 1.$$

Setting the limit at $y \to +0$ in equation (1) and in boundary conditions (2), (3) and taking into account notations (17), (18), we obtain the second relation between unknown functions $\tau(x)$ and $\nu(x)$, which is brought to D_0 from domain D_1, and conditions for the function $\tau(x)$:

$$\tau''(x) - \lambda_1^2\tau(x) - \nu(x) = k, \quad 0 < x < 1, \tag{36}$$

$$\tau(0) = \varphi_1(0), \quad \tau(1) = \varphi_2(0), \quad \int_0^1 \tau(x)\, dx = \varphi_3(0), \tag{37}$$

where k is an unknown number.

Substituting the expression for $\nu(x)$ from (33) into (36), we obtain integro-differential equation for the unknown function $\tau(x)$:

$$\tau''(x) - \lambda_1^2 \tau(x) - \gamma_4 C_{0x}^{1,\lambda_2}[\tau(x)] = k - \gamma_4 C_{0x}^{1,\lambda_2}[F_2(x)], \quad 0 < x < 1. \tag{38}$$

Therefore, the unknown function $\tau(x)$ is a solution of problem $\{(38),(37)\}$. From this problem we find the function $\tau(x)$. First, we prove uniqueness of the solution of problem $\{(38),(37)\}$. Let us consider the homogeneous problem

$$\tau''(x) - \lambda_1^2 \tau(x) - \gamma_4 C_{0x}^{1,\lambda_2}[\tau(x)] = k, \tag{39}$$

$$\tau(0) = 0, \tau(1) = 0, \quad \int_0^1 \tau(x)\, dx = 0. \tag{40}$$

Multiplying (39) by the function $\tau(x)$ and integrating the obtained relation over segment $[0,1]$, we obtain

$$\int_0^1 \tau(x)\tau''(x)\, dx - \lambda_1^2 \int_0^1 \tau^2(x)\, dx - \gamma_4 \int_0^1 \tau(x) C_{0x}^{1,\lambda_2}[\tau(x)]\, dx = k \int_0^1 \tau(x)\, dx.$$

Hence, integrating the first integral by parts and then taking into account (40) and $\tau'(x) \in C[0,1]$, we have

$$\int_0^1 [\tau'(x)]^2 dx + \lambda_1^2 \int_0^1 \tau^2(x)\, dx + \gamma_4 \int_0^1 \tau(x) C_{0x}^{1,\lambda_2}[\tau(x)]\, dx = 0. \tag{41}$$

Here, the notation

$$\Gamma^{-1}(1-2\beta) C_{0x}^{1,\lambda_2}[\tau(x)] = \mu(x). \tag{42}$$

is introduced. Hence, taking into account the conditions $\tau(0) = 0$ and $\tau'(x) \in C[0,1]$, we find the function $\tau(x)$ as follows [39]:

$$\tau(x) = \int_0^x (x-t)^{-2\beta} \bar{J}_{-\beta}[|\lambda_2|(x-t)]\, \mu(t)\, dt. \tag{43}$$

Substituting (42) and (43) into (41), we obtain

$$\int_0^1 \left\{[\tau'(x)]^2 + \lambda_1^2 \tau^2(x)\right\} dx + \gamma_3^{-1} \int_0^1 \mu(x) dx \int_0^x \frac{\bar{J}_{-\beta}[|\lambda_2|(x-t)]\, \mu(t)\, dt}{(x-t)^{2\beta}} = 0. \tag{44}$$

It was proved that the last integral in (44) is non-negative [39]. Then, this relation implies that $\tau'(x) = 0$, i.e., $\tau(x) = const$, $x \in (0,1)$. Taking into account that $\tau(x) \in C[0,1]$ and $\tau(0) = \tau(1) = 0$, we have $\tau(x) \equiv 0$, $x \in [0,1]$. Therefore, the homogeneous problem $\{(39),(40)\}$ has only a trivial solution. It follows from this that if there exists solution of problem $\{(37),(38)\}$ then it is unique.

Now, we prove the existence of the solution of this problem. We rewrite (38) as $\tau''(x) = p(x)$, where

$$p(x) = k + \lambda_1^2 \tau(x) + \gamma_4 C_{0x}^{1,\lambda_2}[\tau(x) - F_2(x)]. \tag{45}$$

The solution of this equation that satisfies the first two conditions of (37) is defined as follows [44]

$$\tau(x) = \varphi_1(0)(1-x) + \varphi_2(0)x + \int_0^1 p(t)\, G(x,t)\, dt, \tag{46}$$

where $G(x,t) = x(t-1)$ for $x \leqslant t$, $G(x,t) = t(x-1)$ for $x \geqslant t$.

Substituting (45) into (46) and then integrating the resulting relation over x by $[0,1]$ and taking into account the last of conditions (37) and $\int\limits_0^1 \int\limits_0^1 G(x,t)\, dx\, dt = -(1/12)$, we find the unknown number k

$$k = -12\varphi_3(0) + 6\varphi_1(0) + 6\varphi_2(0) + 12\int_0^1 \int_0^1 G(x,t)\left\{\lambda_1^2 \tau(t) + \gamma_4 C_{0t}^{1,\lambda_2}[\tau(t) - F_2(t)]\right\} dtdx.$$

Substituting k into (45) and (46), we obtain after some transformations that

$$\tau(x) = \int_0^1 Q(x,t)\left\{\lambda_1^2 \tau(t) + \gamma_4 C_{0t}^{1,\lambda_2}[\tau(t)]\right\} dt + p_1(x), \tag{47}$$

where $Q(x,t) = G(x,t) + 3xt(x-1)(t-1)$,

$$p_1(x) = \varphi_1(0)\left(1 - 4x + 3x^2\right) - \varphi_2(0)x(2-3x) + 6\varphi_3(0)x(1-x) - \gamma_4 \int_0^1 Q(x,t) C_{0t}^{1,\lambda_2}[F_2(t)]\, dt.$$

Taking into account the form of the operator C_{0t}^{1,λ_2} and the equality

$$\int_0^1 Q(x,t) \frac{d}{dt} \int_0^t \tau(t)(t-z)^{2\beta-1} \bar{J}_\beta[|\lambda_2|(t-z)]\, dzdt =$$

$$= -\int_0^1 \frac{\partial}{\partial t} Q(x,t)\, dt \int_0^t \tau(z)(t-z)^{2\beta-1} \bar{J}_\beta[|\lambda_2|(t-z)]\, dz =$$

$$= -\int_0^1 \tau(z)\, dz \int_z^1 (t-z)^{2\beta-1} \bar{J}_\beta[|\lambda_2|(t-z)] \frac{\partial}{\partial t} Q(x,t)\, dt,$$

we obtain an integral equation for the unknown function $\tau(x)$:

$$\tau(x) - \int_0^1 Q_1(x,z)\, \tau(z)\, dz = p_1(x), \quad x \in (0,1), \tag{48}$$

where

$$Q_1(x,z) = \lambda_1^2 Q(x,z) - \frac{\gamma_4}{\Gamma(2\beta)} \int_z^1 (t-z)^{2\beta-1} \bar{J}_\beta[|\lambda_2|(t-z)] \frac{\partial}{\partial t} Q(x,t)\, dt +$$

$$+ \frac{\gamma_4 \lambda_2^2}{2(1+\beta)\Gamma(1+2\beta)} \int_z^1 Q(x,t)(t-z)^{2\beta} \bar{J}_\beta[|\lambda_2|(t-z)]\, dt.$$

It is easy to verify that $Q_1(x,z) \in C(0 \leqslant x, z \leqslant 1) \cap C^2(0 < x, z < 1,\ x \neq z)$ and $p_1(x) \in C[0,1] \cap C^2(0,1)$. Therefore, (48) is the Fredholm integral equation of the second kind [45]. It is equivalent to problem $\{(37),(38)\}$. The homogeneous integral equation corresponding to equation (48) is equivalent to homogeneous problem $\{(39),(40)\}$. Since, the last problem has only a trivial solution the homogeneous integral equation corresponding to (48) has also only a trivial solution. Then, according to alternative of Fredholm [45], the solution of non-homogeneous integral equation (48) exists and it is unique.

Once the function $\tau(x)$ is found from (48), the function $\nu(x)$ can be found from (33). Substituting $\tau(x)$, $\nu(x)$ and $\omega(x)$ into (21), we find a solution of Problem 1 in domain D_2.

Now, we turn to the study of Problem 1 in domain D_1. Here, we have the problem $1'$: find the function $u(x,y)$ that satisfies equation (1) in domain D_1 and conditions (2), (3), $u(x,0) = \tau(x)$, $0 \leqslant x \leqslant 1$, where $\tau(x)$ is the function defined in (48).

We will prove the existence and uniqueness of the solution of problem $1'$. Let $u(x,y)$ be a solution of problem $1'$. To study this problem we use representation (14) of the solution of the equation $(\partial/\partial x)\, L_1 u = 0$. In this case, without loss of generality, one can assume that $\delta(0) = 0$. If we temporarily assume that $\delta(y)$ is a known function then problem $1'$, due to (14) and $\delta(0) = 0$, is equivalent to the problem of finding a solution of equation (15) in domain D_1 that satisfies the conditions

$$w(0,y) = \varphi_1(y) - \delta(y), \quad w(1,y) = \varphi_2(y) - \delta(y), \quad 0 \leqslant y \leqslant 1; \tag{49}$$

$$w(x,0) = \tau(x), \quad 0 \leqslant x \leqslant 1, \tag{50}$$

$$\int_0^1 w(x,y)\,dx = \varphi_3(y) - \delta(y), \quad 0 \leqslant y \leqslant 1. \tag{51}$$

Then function $w(x,y)$ is a solution of the first boundary value problem for equation (15) in domain D_1 with the boundary conditions (49) and (50), and it can be represented as [42]

$$w(x,y) = \int_0^1 \tau(\xi)\, e^{-\lambda_1^2 y} G(x,y;\xi,0)\,d\xi + \int_0^y [\varphi_1(\eta) - \delta(\eta)]\, e^{-\lambda_1^2(y-\eta)} G_\xi(x,y;0,\eta)\,d\eta -$$
$$- \int_0^y [\varphi_2(\eta) - \delta(\eta)]\, e^{-\lambda_1^2(y-\eta)} G_\xi(x,y;1,\eta)\,d\eta, \tag{52}$$

where $G(x,y;\xi,\eta)$ is the Green's function of the first boundary value problem [43] for the equation $w_{xx} - w_y = 0$:

$$G(x,y;\xi,\eta) = \frac{1}{2\sqrt{\pi(y-\eta)}} \sum_{n=-\infty}^{+\infty} \left\{ \exp\left[-\frac{(x-\xi+2n)^2}{4(y-\eta)} \right] - \exp\left[-\frac{(x+\xi+2n)^2}{4(y-\eta)} \right] \right\}. \tag{53}$$

Substituting $w(x,y)$ from (52) into condition (51), we obtain after some transformations that

$$\delta(y) - \int_0^1 \int_0^y \delta(\eta)\, e^{-\lambda_1^2(y-\eta)} \left[G_\xi(x,y;0,\eta) - G_\xi(x,y;1,\eta) \right] d\eta dx = g(y), \quad 0 \leqslant y \leqslant 1, \tag{54}$$

where

$$g(y) = \varphi_3(y) - \int_0^1 \left\{ \int_0^1 \tau(\xi)\, e^{-\lambda_1^2 y} G(x,y;\xi,0)\,d\xi + \right.$$
$$\left. + \int_0^y \varphi_1(\eta)\, e^{-\lambda_1^2(y-\eta)} G_\xi(x,y;0,\eta)\,d\eta - \int_0^y \varphi_2(\eta)\, e^{-\lambda_1^2(y-\eta)} G_\xi(x,y;1,\eta)\,d\eta \right\} dx.$$

Using (53), it is easy to verify that

$$\int_0^1 G_\xi(x,y;0,\eta)\,dx = -\int_0^1 G_\xi(x,y;1,\eta)\,dx = K(y,\eta), \tag{55}$$

where

$$K(y,\eta) = \frac{1}{\sqrt{\pi(y-\eta)}} + \frac{2}{\sqrt{\pi(y-\eta)}} \sum_{n=1}^{+\infty} \left\{ \exp\left[-\frac{n^2}{y-\eta} \right] - \exp\left[-\frac{(2n-1)^2}{4(y-\eta)} \right] \right\}.$$

After changing the order of integration over variables x and η, and then taking into account (55), we obtain from (54) the Volterra integral equation of the second kind with respect to $\delta(y)$:

$$\delta(y) - \int_0^y e^{-\lambda_1^2(y-\eta)} K_1(y,\eta)\,\delta(\eta)\,d\eta = g_1(y), \tag{56}$$

where $K_1(y,\eta) = 2K(y,\eta)$,

$$g_1(y) = \varphi_3(y) - \int_0^y [\varphi_1(\eta) + \varphi_2(\eta)]\, e^{-\lambda_1^2(y-\eta)} K(y,\eta) d\eta - \int_0^1 \int_0^1 \tau(\xi) e^{-\lambda_1^2 y} G(x,y;\xi,0) d\xi dx.$$

Obviously, the kernel $K_1(y,\eta)$ has a weak singularity. Using the properties of functions $\tau(x)$, $\varphi_1(y)$, $\varphi_2(y)$ and $\varphi_3(y)$, it is easy to show that $g_1(y) \in C[0,1] \cap C^1(0,1)$. Therefore, equation (56) has a unique solution in this class [45]. Solving it, we find function $\delta(y)$. Thus, function $w(x,y)$ is defined by (52) in domain D_1. Then solution of Problem 1 (Problem 1′) in domain D_1 is determined by expression (14). The study of Problem 1 is completed.

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Нелокальная задача для одного параболо-гиперболического уравнения третьего порядка с сингулярным коэффициентом

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Аннотация. В настоящей работе сформулирована и исследована неклассическая задача с интегральным условием для параболо-гиперболического уравнения третьего порядка. Методом интегральных уравнений доказана однозначная разрешимость поставленной задачи. При этом поставленная задача эквивалентно сведена к задаче для параболо-гиперболического уравнения второго порядка с неизвестной правой частью. При исследовании последней задачи использованы формулы решения задачи Коши для гиперболического уравнения, имеющего сингулярный коэффициент и спектральный параметр, а также решения первой краевой задачи для параболического уравнения Фурье.

Ключевые слова: параболо-гиперболическое уравнение, интегральное условие, единственность решения, существование решения, сингулярный коэффициент.