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On Ground States for the SOS Model with Competing Interactions

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Abstract. We study periodic and weakly periodic ground states for the SOS model with competing interactions on the Cayley tree of order two and three. Further, we study non periodic ground states for the SOS model with competing interactions on the Cayley tree of order two.

Keywords: Cayley tree, SOS model, periodic and weakly periodic ground states.

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Introduction

It is known that a phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure, (see [1, 2]). It leads us to investigate the problem of description of periodic and weakly periodic ground states. For the Potts model with competing interactions on the Cayley tree of order $k = 2$ periodic ground states are studied in [3] (see also [4]). The notion of a weakly periodic ground state is introduced in [5]. For the Ising model with competing interactions, weakly periodic ground states are described in [1, 5]. Such states for the Potts model for normal subgroups of index 2 are studied in [6, 7]. For the Potts model with competing interactions, such states for normal subgroups of index 4 are studied in [8] and in this work also studied periodic ground states for normal subgroups of index 4 (see also [9]). In [10] for the Potts model, with competing interactions and countable spin values, on a Cayley tree of order three periodic ground states are studied.

In [11] finite-range lattice models on Cayley trees with two basic properties: the existence of only a finite number of ground states and with a Peierls type condition are considered and the

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notion of a contour for the model on the Cayley tree is defined. Also using a contour argument the existence of different Gibbs measures is shown.

A q-component models on a Cayley tree is investigated in [12] and using a contour argument the existence of q different Gibbs measures for several q-component models is shown.

In [13] for the SOS model with $m = 2$ on the Cayley tree order of $k = a + b + 2$ the existence of at least two non periodic Gibbs measures is proved. In [14] an infinite system of functional equations for the Ising model with competing interactions and countable spin values $0, 1, \dots$ and non zero field on a Cayley tree of order two is investigated. In [15] the authors proved the existence of weakly periodic Gibbs measures for the Ising model on the Cayley tree of order $k = 2$ with respect to a normal divisor of index 4.

In this paper, we study periodic and weakly periodic ground states for the SOS model with competing interactions on a Cayley tree of order $k = 2$ and $k = 3$. Moreover, in the case $k = 2$ the existence of a countable set of non periodic ground states is proved.

1. Preliminaries

Let $\Gamma^k = (V, L)$ be the Cayley tree of order k, i.e., an infinite tree such that exactly $k + 1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of Γ^k. Let G_k denote the free product of $k + 1$ cyclic groups $\{e; a_i\}$ of order 2 with generators $a_1, a_2, a_3, \dots a_{k+1}$, i.e., let $a_i^2 = e$ (see [4]).

The group of all left (right) shifts on G_k is isomorphic to the group G_k. Each transformation S on the group G_k induces a transformation $\tilde{S}$ on the vertex set V of the Cayley tree Γ^k. In the sequel, we identify V with G_k.

The following assertion is quite obvious (see also [4]).

Theorem 1.1. The group of left (right) shifts on the right (left) representation of the Cayley tree is the group of translations.

By the group of translations we mean the automorphism group of the Cayley tree regarded as a graph. Recall (see, for example, [4]) that a mapping ψ on the vertex set of a graph G is called an automorphism of G if ψ preserves the adjacency relation, i.e., the images $\psi(u)$ and $\psi(v)$ of vertices u and v are adjacent if and only if u and v are adjacent.

For an arbitrary vertex $x_0 \in V$, we put

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \{x \in V | d(x, x^0) \leqslant n\},$$

where $d(x, y)$ is the distance between x and y in the Cayley tree, i.e., the number of edges of the path between x and y.

For each $x \in G_k$, let $S(x)$ denote the set of immediate successors of x, i.e., if $x \in W_n$ then

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x) = \{y \in G_k : \langle x, y\rangle \in L\}$. The set $S_1(x) \setminus S(x)$ is a singleton. Let $x_\downarrow$ denote the (unique) element of this set.

Let us assume that the spin values belong to the set $\Phi = \{0, 1, 2, \dots m\}$. A function $\sigma : x \in V \to \sigma(x) \in \Phi$ is called configuration on V. The set of all configurations coincides with the set $\Omega = \Phi^V$.

Consider the quotient group $G_k/G_k^* = \{H_1, H_2, \dots, H_r\}$, where G_k^* is a normal subgroup of index r with $r \geqslant 1$.

Definition 1.1. *A configuration $\sigma(x)$ is called G_k^*-periodic, if $\sigma(x) = \sigma_i$ for all $x \in G_k$ with $x \in H_i$. A G_k-periodic configuration is called translation invariant.*

The period of a periodic configuration is the index of the corresponding normal subgroup.

Definition 1.2. *A configuration $\sigma(x)$ is called G_k^*-weakly periodic, if $\sigma(x) = \sigma_{ij}$ for all $x \in G_k$ with $x_\downarrow \in H_i$ and $x \in H_j$.*

The Hamiltonian of the model SOS model with competing interactions has a form:

$$H(\sigma) = -J_1 \sum_{\langle x,y\rangle \in L} |\sigma(x) - \sigma(y)| - J_2 \sum_{\substack{x,y \in V: \\ d(x,y)=2}} |\sigma(x) - \sigma(y)|, \tag{1}$$

where $(J_1, J_2) \in \mathbb{R}^2$.

2. Ground states

In this section, we study ground states for the SOS model on a Cayley tree. For a pair of configurations σ and φ which coincide almost everywhere, i.e., everywhere except finitely many points, we consider the relative Hamiltonian $H(\sigma, \varphi)$ describing the energy differences of the two configurations σ and φ :

$$\begin{aligned} H(\sigma, \varphi) = &- J_1 \sum_{\langle x,y\rangle \in L} (|\sigma(x) - \sigma(y)| - |\varphi(x) - \varphi(y)|) - \\ &- J_2 \sum_{\substack{x,y \in V: \\ d(x,y)=2}} (|\sigma(x) - \sigma(y)| - |\varphi(x) - \varphi(y)|), \end{aligned} \tag{2}$$

where $(J_1, J_2) \in \mathbb{R}^2$.

Let M be the set of all unit balls with vertices in V, i.e. $M = \{\{x\} \cup S_1(x) : \forall x \in V\}$. A restriction of a configuration σ to the ball $b \in M$ is a bounded configuration and it is denoted by σ_b.

We define the energy of the configuration σ_b on b by the following formula

$$U(\sigma_b) \equiv U(\sigma_b, J_1, J_2) = -\frac{1}{2} J_1 \sum_{\substack{\langle x,y\rangle: \\ x,y \in b}} |\sigma(x) - \sigma(y)| - J_2 \sum_{\substack{x,y \in b: \\ d(x,y)=2}} |\sigma(x) - \sigma(y)|, \tag{3}$$

where $(J_1, J_2) \in \mathbb{R}^2$.

The following assertion is known (see [4]).

Lemma 2.1. *Relative Hamiltonian (2) has the form:*

$$H(\sigma, \varphi) = \sum_{b \in M} (U(\sigma_b) - U(\varphi_b)).$$

The existence of a countable set of non periodic ground states on the Cayley tree of order two

We consider the case $k = 2$.

Let $m = 2$. It is easy to see that $U(\sigma_b) \in \{U_i : i = 1, \ldots, 10\}$ for σ_b, where

$$U_1 = 0, \quad U_2 = -\frac{1}{2} J_1 - 2J_2, \quad U_3 = -J_1 - 2J_2, \quad U_4 = -\frac{3}{2} J_1, \quad U_5 = -J_1 - 4J_2,$$

$$U_6 = -2J_1 - 4J_2, \quad U_7 = -3J_1, \quad U_8 = -\frac{3}{2} J_1 - 4J_2, \quad U_9 = -2J_1 - 2J_2, \quad U_{10} = -\frac{5}{2} J_1 - 2J_2.$$

Definition 2.1. *The configuration φ is called the ground state for the Hamiltonian (1) if $U(\varphi_b) = \min\{U_1, U_2, U_3, \dots, U_{10}\}$ for any $b \in M$.*

Let
$$A_m = \{(J_1, J_2) \in R^2 \mid U_m = \min_{1 \leqslant k \leqslant 10}\{U_k\}\}.$$
It is easy to check that
$$\begin{aligned}
&A_1 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 \leqslant -\frac{1}{4}J_1\},\\
&A_2 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 = -\frac{1}{4}J_1\},\\
&A_3 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},\\
&A_4 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 \leqslant 0\},\\
&A_5 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 \geqslant -\frac{1}{4}J_1\},\\
&A_6 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 \geqslant \frac{1}{4}J_1\},\\
&A_7 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 \leqslant \frac{1}{4}J_1\},\\
&A_8 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 \geqslant 0\},\\
&A_9 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\}\},\\
&A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 = \frac{1}{4}J_1\}
\end{aligned}$$
and $\bigcup_{i=1}^{10} A_i = \mathbb{R}^2$.

In [16] periodic ground states are studied for SOS model on Cayley tree order of 2. In this subsection we shall prove the existence of a countable set of non periodic ground states on the Cayley tree of order two. The next subsection we study periodic and weakly periodic ground states for the model (1) on the Cayley tree of order three.

Let c_b denote the center of a unit ball b. We put
$$C_i = \{\sigma_b : U(\sigma_b) = U_i\}, i = \overline{1, 10},$$
$$B^{(i)} = \mid \{x \in S_1(c_b) : \varphi_b(x) = i\} \mid, \text{ for } i = 0, 1, 2$$
and $D_i = \Omega_i \cup \tilde{\Omega}_i$, where
$$\Omega_i = \{\sigma_b : \sigma_b(c_b) = 0,\ |x \in b\backslash\{c_b\} : \sigma_b(x) = 2| = i;\ \ |x \in b\backslash\{c_b\} : \sigma_b(x) = 1| = 0\},$$
$$\tilde{\Omega}_i = \{\tilde{\sigma}_b : |\tilde{\sigma}(x) - \sigma(x)| = 2, |x \in b\backslash\{c_b\} : \tilde{\sigma}_b(x) = 1| = 0, x \in b\},\ i = 0, 1, 2, 3, \text{ i.e.,}$$
$$\tilde{\sigma}(x) = \begin{cases} 2, \text{ if } \sigma(x) = 0 \\ 0, \text{ if } \sigma(x) = 2 \end{cases}.$$

For $A_i, A_j, i \neq j$ we have
$$A_i \cap A_j = \begin{cases} A_2 \text{ if } i = 1, j = 5, \\ A_4 \text{ if } i = 1, j = 7, \\ A_8 \text{ if } i = 5, j = 6, \\ A_{10} \text{ if } i = 6, j = 7. \end{cases} \tag{4}$$

Fix $J = (J_1, J_2) \in \mathbb{R}^2$ and denote

$$N_J(\sigma_b) = |\{j : \sigma_b \in C_j\}|.$$

Using (4) one can prove

Lemma 2.2. *For any $b \in M$ and σ_b we have*

$$N_J(\sigma_b) = \begin{cases} 10, & \text{if } J = (0;0) \\ 3, & \text{if } J \in A_i \setminus \{(0,0)\},\ i = 2,4,8,10 \\ 1, & \text{otherwise} \end{cases}. \tag{5}$$

Let GS(H) be the set of all ground states of the Hamiltonian (1).

Theorem 2.1. *(i) If $J = (0;0)$ then $GS(H) = \Omega$.*
(ii) If $J \in A_i \setminus \{(0,0)\}$, $i = 2,8,10$ then there exists a countable set of non periodic ground states.

Proof. The assertion (i) is trivial.
Prove (ii):

a) if $J \in A_2 \setminus \{(0,0)\}$ then the minimum points of $U(\sigma_b)$ would belong to the classes C_1, C_2 and C_5;
b) if $J \in A_8 \setminus \{(0,0)\}$ then the minimum points of $U(\sigma_b)$ would belong to the classes C_5, C_6 and C_8;
c) if $J \in A_{10} \setminus \{(0,0)\}$ then the minimum points of $U(\sigma_b)$ would belong to the classes C_6, C_7 and C_{10}.

Below we define the configurations of classes C_1, C_5, C_6 and C_7 which satisfying the condition $|x \in b\backslash\{c_b\} : \sigma_b(x) = 1| = 0$,

$$\begin{cases} \sigma_b^{(0)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(0)}(x) = 2| = 0 \text{ and} \\ \tilde{\sigma}_b^{(0)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(0)}(x) = 0| = 0,\ \sigma^{(0)}, \tilde{\sigma}^{(0)} \in C_1, \\ \sigma_b^{(1)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(1)}(x) = 2| = 1 \text{ and} \\ \tilde{\sigma}_b^{(1)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(1)}(x) = 0| = 1,\ \sigma^{(1)}, \tilde{\sigma}^{(1)} \in C_5, \\ \sigma_b^{(2)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(2)}(x) = 2| = 2 \text{ and} \\ \tilde{\sigma}_b^{(2)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(2)}(x) = 0| = 2,\ \sigma^{(2)}, \tilde{\sigma}^{(2)} \in C_6, \\ \sigma_b^{(3)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(3)}(x) = 2| = 3 \text{ and} \\ \tilde{\sigma}_b^{(3)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(2)}(x) = 0| = 3, \sigma^{(3)}, \tilde{\sigma}^{(3)} \in C_7 \end{cases}. \tag{6}$$

Thus any ground state $\varphi \in D_i$ must satisfy

$$\varphi_b \in \{\sigma_b^{(i)}, \tilde{\sigma}_b^{(i)}, \sigma_b^{(i+1)}, \tilde{\sigma}_b^{(i+1)}\}, \quad i = 0,1,2,\ b \in M. \tag{7}$$

Now we shall construct ground states $\varphi \in D_i$ which satisfying (7).

Note that the configurations σ_b and $\sigma_{b'}$ $(b, b' \in M)$ are the same up to a motion in G_k so we shall omit b. Thus configuration $\sigma^{(i)}$ is the configuration such that on any unit ball $b \in M$ the condition (6) is satisfied.

Suppose two unit balls b and b' are neighbors, i.e., they have a common edge. We shall then say that the two bounded configurations σ_b and $\sigma_{b'}$ are compatible if they coincide on the common edge of the balls b and b'. Denote by $B(b)$ the set of all neighbor balls of b.

Denote $\bar{\Omega}_i = \{\sigma^{(i)}, \tilde{\sigma}^{(i)}, \sigma^{(i+1)}, \tilde{\sigma}^{(i+1)}\},\ i = 0,1,2$. For any $\omega, \nu \in \bar{\Omega}_i$ denote by $n(\omega,\nu)$ the number of possibilities to set up the configuration ν as a compatible configuration (with ω) around (i.e., on neighboring balls of the ball on which ω is given) the configuration ω. Clearly $n(\omega,\nu) \in \{0,1,2,3\}$, for any $\omega,\nu \in \bar{\Omega}_i, i = 0,1,2$.

Denote

$$N_i = \begin{pmatrix} n(\sigma^{(i)},\sigma^{(i)}) & n(\sigma^{(i)},\tilde{\sigma}^{(i)}) & n(\sigma^{(i)},\sigma^{(i+1)}) & n(\sigma^{(i)},\tilde{\sigma}^{(i+1)}) \\ n(\tilde{\sigma}^{(i)},\sigma^{(i)}) & n(\tilde{\sigma}^{(i)},\tilde{\sigma}^{(i)}) & n(\tilde{\sigma}^{(i)},\sigma^{(i+1)}) & n(\tilde{\sigma}^{(i)},\tilde{\sigma}^{(i+1)}) \\ n(\sigma^{(i+1)},\sigma^{(i)}) & n(\sigma^{(i+1)},\tilde{\sigma}^{(i)}) & n(\sigma^{(i+1)},\sigma^{(i+1)}) & n(\sigma^{(i+1)},\tilde{\sigma}^{(i+1)}) \\ n(\tilde{\sigma}^{(i+1)},\sigma^{(i)}) & n(\tilde{\sigma}^{(i+1)},\tilde{\sigma}^{(i)}) & n(\tilde{\sigma}^{(i+1)},\sigma^{(i+1)}) & n(\tilde{\sigma}^{(i+1)},\tilde{\sigma}^{(i+1)}) \end{pmatrix}.$$

It is easy to see that

$$N_0 = \begin{pmatrix} 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \\ 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \end{pmatrix}.$$

Consider 3 sets $\mathbf{Q}_i = \{Q\},\ (i = 0,1,2)$ of matrices $Q = \{q(u,v)\}_{u,v\in\bar{\Omega}_i}$ such that

$$q(u,v) \in \{0,1,\ldots,n(u,v)\}, \quad \sum_{v\in\bar{\Omega}_i} q(u,v) = 3, \forall u \in \bar{\Omega}_i.$$

$q(u,\sigma^{(i)}) + q(u,\sigma^{(j)}) = n(u,\sigma^{(i)}),\ \ q(u,\tilde{\sigma}^{(i)}) + q(u,\tilde{\sigma}^{(j)}) = n(u,\tilde{\sigma}^{(i)})$, and $q(u,v) = 0$ if and only if $q(v,u) = 0,\ u,v \in \bar{\Omega}_i$.

Using matrices N_i we have

$$\mathbf{Q}_0 = \left\{ Q = \begin{pmatrix} a & 0 & 3-a & 0 \\ 0 & b & 0 & 3-b \\ c & 0 & 2-c & 1 \\ 0 & d & 1 & 2-d \end{pmatrix} \right\},$$

here $a,b \in \{0,1,2,3\}$; $c,d \in \{0,1,2\}$; $a = 3$ iff $c = 0$; $b = 3$ iff $d = 0$.

For $i = 1$ we get

$$\mathbf{Q}_1 = \left\{ Q = \begin{pmatrix} a_1 & b_1 & 2-a_1 & 1-b_1 \\ b_2 & a_2 & 1-b_2 & 2-a_2 \\ c_1 & d_1 & 1-c_1 & 2-d_1 \\ d_2 & c_2 & 2-d_2 & 1-c_2 \end{pmatrix} \right\},$$

here $a_1,a_2,d_1,d_2 \in \{0,1,2\}$; $b_1,b_2,c_1,c_2 \in \{0,1\}$; $a_1 = 2$ iff $c_1 = 0$; $a_2 = 2$ iff $c_2 = 0$; $b_1 = 0$ iff $b_2 = 0$; $b_1 = 1$ iff $d_2 = 0$; $b_2 = 1$ iff $d_1 = 0$; $d_1 = 2$ iff $d_2 = 2$.

For $i = 2$ we obtain

$$\mathbf{Q}_2 = \left\{ Q = \begin{pmatrix} 1 & a & 0 & 2-a \\ b & 1 & 2-b & 0 \\ 0 & c & 0 & 3-c \\ d & 0 & 3-d & 0 \end{pmatrix} \right\},$$

here $a, b \in \{0, 1, 2\}$; $c, d \in \{0, 1, 2, 3\}$; $a = 0$ iff $b = 0$; $a = 2$ iff $d = 0$; $b = 2$ iff $c = 0$; $c = 3$ iff $d = 3$.

For a given $\xi \in \bar{\Omega}_i$ and $Q = \{q(u, v)\}_{u,v \in \bar{\Omega}_i} \in \mathbf{Q}_i$ we recurrently construct a ground state $\varphi^{Q,\xi}$ by the following way: fix a ball $b \in M$ and put on b the configuration $\varphi^{Q,\xi} := \xi$. On balls taken from $B(b)$ we set exactly $q(\xi, \omega)$ copies of ω for any $\omega \in \bar{\Omega}_i$. Thus configurations $\varphi^{Q,\xi}_{b'}, b' \in B(b)$ are defined. Using these configurations, we define configurations on the balls $B(b') \setminus \{b\}, (b' \in B(b))$ putting $q(\varphi^{Q,\xi}_{b'}, v)$ copies of $v \in \bar{\Omega}_i \setminus \xi$ and $q(\varphi^{Q,\xi}_{b'}, \xi) - 1$ copies of ξ which are compatible with $\varphi^{Q,\xi}_{b'}$. Further, on the balls $B(b'') \setminus \{b'\}, (b'' \in B(b)), b' \in B(b)$ we set $q(\varphi^{Q,\xi}_{b''}, \tau)$ copies of $\tau \in \bar{\Omega}_i \setminus \{\varphi^{Q,\xi}_{b'}\}$ and $q(\varphi^{Q,\xi}_{b''}, \varphi^{Q,\xi}_{b'}) - 1$ copies of $\varphi^{Q,\xi}_{b'}$ which are compatible with $\varphi^{Q,\xi}_{b''}$. Repeating this construction one can obtain a ground state $\varphi^{Q,\xi}$ such that

$$\varphi^{Q,\xi}_b \in \bar{\Omega}_i, \quad |\{b' \in B(b) : \varphi^{Q,\xi}_b = \omega, \ \varphi^{Q,\xi}_{b'} = \nu\}| = q(\omega, \nu),$$

for any $b \in M$ and $\omega, \nu \in \bar{\Omega}_i$.

In general, the ground state $\varphi^{Q,\xi}$ is non periodic (see example below). It is easy to see that

$$\varphi^{Q_i,\sigma^{(j)}}_b \equiv \sigma^{(j)}, \ \ \varphi^{Q_i,\tilde{\sigma}^{(j)}}_b \equiv \tilde{\sigma}^{(j)}, \ \ j = i, \ i+1, \ i = 0, 1, 2,$$

where

$$Q_i = \begin{pmatrix} 3-i & i & 0 & 0 \\ i & 3-i & 0 & 0 \\ 0 & 0 & 2-i & i+1 \\ 0 & 0 & i+1 & 2-i \end{pmatrix}. \tag{8}$$

Now using the ground states $\varphi^{Q,\xi}$ we shall construct an infinite set of ground states by the following way: one can choose $\xi \neq \eta, \ \xi, \eta \in \bar{\Omega}_i$ and $Q_1, Q_2 \in \mathbf{Q}_1$ such that for configurations $\varphi^{Q_1,\xi}, \varphi^{Q_2,\eta}$ there are infinitely many $b \in M$ on which $\varphi^{Q_1,\xi}_b$ and $\varphi^{Q_2,\eta}_{b'}$ are compatible for some $b' \in B(b)$. Indeed it is sufficient to take $\xi \neq \eta$ such that $q_1(\xi, \eta) q_2(\xi, \eta) \neq 0$ (see example below).

Denote

$$M_1 \equiv M_1^{\xi\eta}(Q_1, Q_2) = \{b \in M : \ \varphi^{Q_1,\xi}_b$$

is compatible $\varphi^{Q_2,\eta}_{b'}$ for some $b' \in B(b)\}$;

$$\aleph_1 = \{n \in \{0, 1, 2, \dots\} : \ \exists b \in M_1 \text{ such that } |c_b| = n\};$$

$$V^{(y)} = \{z \in V : y < z\}.$$

Fix $m \in \aleph_1$ and denote

$$\tilde{W}_m = \{x \in W_m : \ \exists b \in M_1 \text{ such that } c_b = x\}.$$

Consider the configuration

$$\varphi^{Q_1,Q_2,\xi,\eta}_m(x) = \begin{cases} \varphi^{Q_1,\xi}(x) \text{ if } x \in V_m \cup \{V^{(y)}, y \in W_m \backslash \tilde{W}_m\} \\ \varphi^{Q_2,\eta}(x) \text{ if } x \in V^{(y)}, y \in \tilde{W}_m \end{cases}.$$

Clearly $\varphi_m^{Q_1,Q_2,\xi,\eta}, m \in \aleph_1$ is a ground state and the number of such ground states is infinite, since $|\aleph_1| = \infty$. This finishes the proof of Theorem 2.1. $\square$

Remark 2.1 Let $J \in A_4 \backslash (0,0)$. $\bar{\Omega}_3 = \{\sigma_b^{(0)}, \tilde{\sigma}_b^{(0)}, \sigma_b^{(3)}, \tilde{\sigma}_b^{(3)}\}$ are periodic ground states such that on any $b \in M$ the bounded configurations $\sigma_b^{(0)}, \tilde{\sigma}_b^{(0)} \in C_1$ and $\sigma_b^{(3)}, \tilde{\sigma}_b^{(3)} \in C_7$, i.e., $\sigma_b^{(0)}$, $\tilde{\sigma}_b^{(0)}$ are translation-invariant and $\sigma_b^{(3)}, \tilde{\sigma}_b^{(3)}$ are periodic with period 2. $\bar{\Omega}_3 = \{\sigma_b^{(0)}, \tilde{\sigma}_b^{(0)}, \sigma_b^{(3)}, \tilde{\sigma}_b^{(3)}\}$ and $\mathbf{Q}_3$ contains the unique matrix

$$Q_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Example. Take matrices

$$Q_2' = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}, \quad Q_2'' = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

and $\xi = \sigma^{(2)}$, $\eta = \sigma^{(3)}$. The configurations $\varphi^{Q_2',\xi}$, $\varphi^{Q_2'',\eta}$ and $\varphi^{Q_2',Q_2''\xi,\eta}$ are represented in Fig. 1 a), b) and c), respectively.

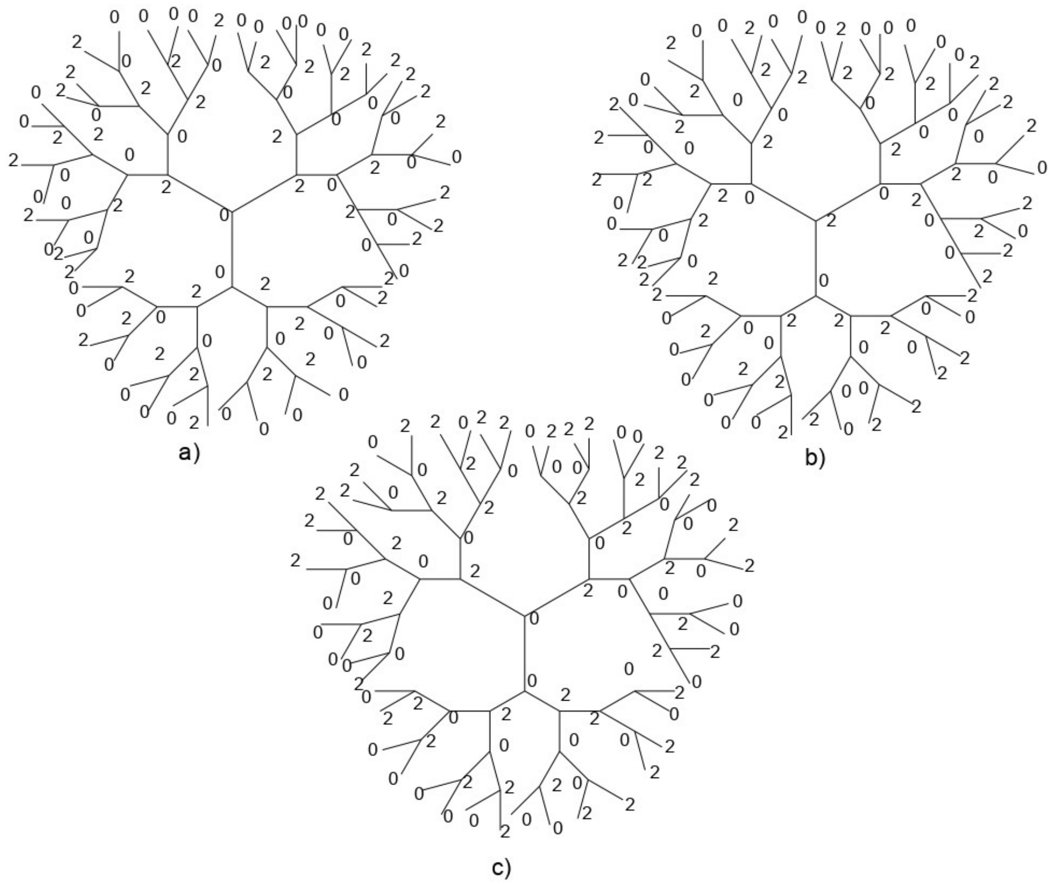

Fig. 1. Ground states

Periodic and weakly periodic ground states on the Cayley tree of order three

We consider the case $k = 3$.

Let $m = 2$. By (3) for any σ_b we have $U(\sigma_b) \in \{U_1, U_2, U_3, \ldots, U_{15}\}$, where

$$U_1 = 0, \quad U_2 = -\frac{1}{2}J_1 - 3J_2, \quad U_3 = -J_1 - 4J_2, \quad U_4 = -J_1 - 6J_2,$$

$$U_5 = -\frac{3}{2}J_1 - 3J_2, \quad U_6 = -2J_1 - 8J_2, \quad U_7 = -3J_1 - 6J_2, \quad U_8 = -2J_1 - 6J_2,$$

$$U_9 = -\frac{5}{2}J_1 - 7J_2, \quad U_{10} = -\frac{3}{2}J_1 - 7J_2, \quad U_{11} = -2J_1, U_{12} = -4J_1,$$

$$U_{13} = -\frac{7}{2}J_1 - 3J_2, \quad U_{14} = -\frac{5}{2}J_1 - 3J_2, \quad U_{15} = -3J_1 - 4J_2.$$

Definition 2.2. *The configuration φ is called the ground state for the Hamiltonian (1), if $U(\varphi_b) = \min\{U_1, U_2, U_3, \ldots, U_{15}\}$ for $\forall b \in M$.*

Let $A_m = \{(J_1, J_2) \in \mathbb{R}^2 \mid U_m = \min_{1\leqslant k\leqslant 15}\{U_k\}\}$. It is easy to check that

$$\begin{aligned}
&A_1 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 \leqslant -\frac{1}{6}J_1\},\\
&A_2 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 = -\frac{1}{6}J_1\},\\
&A_3 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},\\
&A_4 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; -\frac{1}{6}J_1 \leqslant J_2 \leqslant -\frac{1}{2}J_1\},\\
&A_5 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},\\
&A_6 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_2 \geqslant \frac{1}{2}|J_1|\},\\
&A_7 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; \frac{1}{6}J_1 \leqslant J_2 \leqslant \frac{1}{2}J_1\},\\
&A_8 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},\\
&A_9 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 = \frac{1}{2}J_1\},\\
&A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 = -\frac{1}{2}J_1\},\\
&A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 \leqslant 0\},\\
&A_{12} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 \leqslant \frac{1}{6}J_1\},\\
&A_{13} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0, J_2 = \frac{1}{6}J_1\},\\
&A_{14} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},\\
&A_{15} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\}
\end{aligned}$$

and $\bigcup_{i=1}^{15} A_i = \mathbb{R}^2$.

Let c_b be the center of a unit ball b. We put

$$C_i = \{\sigma_b : U(\sigma_b) = U_i\}, i = \overline{1, 15}$$

and

$$B^{(i)} =| \{x \in S_1(c_b) : \varphi_b(x) = i\} |,$$

for $i = 0, 1, 2$.

Let $H_A = \{x \in G_k : \sum_{i \in A} \omega_x(a_i) - \text{even}\}$, where $\omega_x(a_i)$ is the number of a_i in the word x.

Note, that H_A is a normal subgroup of index two (see [4]). Let $G_k/H_A = \{H_A, G_k \backslash H_A\}$ be the quotient group. Denote $H_0 = H_A$, $H_1 = G_k \backslash H_A$.

Periodic Ground States for the case $k = 3$

In this section, we shall study H_0-periodic ground states. We note that each H_0 periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, & \text{if } x \in H_0 \\ \sigma_2, & \text{if } x \in H_1 \end{cases}, \tag{9}$$

where $\sigma_i \in \Phi = \{0, 1, 2\}, i = 1, 2$.

Theorem 2.2. *Let $k = 3$. The configuration (9) is H_0-periodic ground state iff one of the following conditions holds:*

a) $|A| = 1$.

i) $| \sigma_1 - \sigma_2 |= 0$, and $(J_1, J_2) \in A_1$.

ii) $| \sigma_1 - \sigma_2 |= 1$, and $(J_1, J_2) \in A_2$.

iii) $| \sigma_1 - \sigma_2 |= 2$, and $(J_1, J_2) \in A_4$.

b) $|A| = 2$.

i) If $| \sigma_1 - \sigma_2 |= 1$,then there is not a H_0-periodic ground state;

ii) $| \sigma_1 - \sigma_2 |= 2$, and $(J_1, J_2) \in A_6$.

c) $|A| = 3$.

i) If $| \sigma_1 - \sigma_2 |= 1$, then there is not a H_0-periodic ground state;

ii) $| \sigma_1 - \sigma_2 |= 2$, and $(J_1, J_2) \in A_7$.

d) $|A| = 4$.

i) $| \sigma_1 - \sigma_2 |= 1$, and $(J_1, J_2) \in A_{11}$.

ii) $| \sigma_1 - \sigma_2 |= 2$, and $(J_1, J_2) \in A_{12}$.

Proof: a) i) Let us consider the following configuration

$$\varphi(x) = \begin{cases} i, & \text{if } x \in H_0 \\ i, & \text{if } x \in H_1 \end{cases},$$

where $i = 0, 1, 2$. We denote the center of $b \in M$ by c_b. Let $c_b \in H_0$, then we have

$$\varphi_b(c_b) = i, \ B^{(i)} = 4.$$

Hence, $\varphi_b(x) \in C_1$, i.e. if $(J_1, J_2) \in A_1$ then the corresponding configuration is a ground state.

ii) Now we consider the following configuration

$$\varphi(x) = \begin{cases} i, & \text{if } x \in H_0 \\ j, & \text{if } x \in H_1 \end{cases},$$

where $| i - j |= 1$.

1) Assume that $c_b \in H_0$

$$\varphi_b(c_b) = i, B^{(i)} = 3, B^{(j)} = 1.$$

Hence, $\varphi_b(x) \in C_2$.

2) Let $c_b \in H_1$, then one has

$$\varphi_b(c_b) = i, B^{(i)} = 3, B^{(j)} = 1.$$

Hence, $\varphi_b(x) \in C_2$.

We conclude that, if $(J_1, J_2) \in A_2$ then the corresponding periodic configuration $\varphi(x)$ is a H_0-periodic ground state.

iii) Let us consider the following configuration

$$\varphi(x) = \begin{cases} i, & \text{if } x \in H_0 \\ j, & \text{if } x \in H_1 \end{cases},$$

where $\mid i - j \mid = 2$.

1) Assume that $c_b \in H_0$

$$\varphi_b(c_b) = i, B^{(i)} = 3, B^{(j)} = 1.$$

Hence, $\varphi_b(x) \in C_4$.

2) Let $c_b \in H_1$, then one has

$$\varphi_b(c_b) = j, B^{(j)} = 3, B^{(i)} = 1.$$

Hence, $\varphi_b(x) \in C_4$.

We conclude that if $(J_1, J_2) \in A_4$ then the corresponding periodic configuration $\varphi(x)$ is a H_0-periodic ground state.

The proofs of assertions b), c) and d) of Theorem 2.2 are similar to the proof of assertion a). This finishes the proof of Theorem 2.2. □

Remark 2.2 In the case c), the H_0 periodic ground states coincides with the $G_k^{(2)}$-periodic ground states, where $G_k^{(2)} = \{x \in G_k : |x| \text{ is even}\}$.

Weakly Periodic Ground States for the $k = 3$

In this section, we describe H_A-weakly periodic ground states, where H_A is a normal subgroup of index two. Due to the definition of weakly periodic configuration, we infer that each H_A-weakly periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_{00}, & \text{if } x_\downarrow \in H_0, x \in H_0 \\ \sigma_{01}, & \text{if } x_\downarrow \in H_0, x \in H_1 \\ \sigma_{10}, & \text{if } x_\downarrow \in H_1, x \in H_0 \\ \sigma_{11}, & \text{if } x_\downarrow \in H_1, x \in H_1 \end{cases}, \tag{10}$$

where $\sigma_{ij} \in \Phi,\ i, j = 0, 1$.

In the sequel, we write $\sigma = (\sigma_{00}, \sigma_{01}, \sigma_{10}, \sigma_{11})$ for such a weakly periodic configuration $\sigma(x)$, $x \in G_k$.

Theorem 2.3. *Let $k = 3$ and $|A| = 1$. Then for the SOS model there is no H_A-weakly periodic (non periodic) ground state.*

Proof. Consider (10). If $\sigma_{00} = \sigma_{01} = \sigma_{10} = \sigma_{11}$, then corresponding configurations are translation-invariant. Translation-invariant ground states for this case are studied in Theorem 2.2. It is easy to see that in the case $\sigma_{00} = \sigma_{10}$ and $\sigma_{01} = \sigma_{11}$ the H_A-weakly periodic configurations (10) are periodic configurations which are studied in Theorem 2.2.

Now we consider the cases $\sigma_{00} \neq \sigma_{10}$ or $\sigma_{01} \neq \sigma_{11}$.

Let

$$\varphi(x) = \begin{cases} 0, \text{ if } x_\downarrow \in H_0, x \in H_0 \\ 0, \text{ if } x_\downarrow \in H_0, x \in H_1 \\ 1, \text{ if } x_\downarrow \in H_1, x \in H_0 \\ 0, \text{ if } x_\downarrow \in H_1, x \in H_1 \end{cases}.$$

Let $c_b \in H_0$, we have the following possible cases:

a) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 4$, $\varphi_b(c_b) \in C_1$,

b) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 3$, $B^{(1)} = 1$, $\varphi_b(c_b) \in C_2$,

c) $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 1$, then there is not any H_A-weakly periodic ground state,

d) $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 1$, $B^{(0)} = 4$, $\varphi_b(c_b) \in C_{11}$.

Let $c_b \in H_1$, we have the following possible cases:

a) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 4$, $\varphi_b(c_b) \in C_1$,

b) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 1$, then there is not any H_A-weakly periodic ground state,

c) $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 3$, $B^{(1)} = 1$, $\varphi_b(c_b) \in C_2$.

We conclude that the configuration φ is a ground state on the set

$$A_1 \cap A_2 \cap A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = J_2 = 0\}.$$

Therefore, if $J_1 \neq 0$ and $J_2 \neq 0$ then the weakly periodic configuration φ is not a weakly periodic ground state.

By similar way we can prove that all H_A-weakly periodic (non periodic) configurations are not ground states.

This finishes the proof of Theorem 2.3. □

Remark 2.3. 1) Theorem 2.3 shows that for the SOS model with competing interactions, every H_A-weakly periodic ground state is either H_A-periodic or translation-invariant.

2) The fact that for $k = 3$ there exists a set of countable non-periodic ground states can be proved in the same manner as in Theorem 2.1.

3) For the $k > 3$ by the same manner as in Theorem 2.1 periodic (and weakly periodic) ground states could be studied.

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Основные состояния для модели SOS с конкурирующими взаимодействиями

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Аннотация. В работе для нормального делителя индекса 2 изучены слабо-периодические основные состояния для модели SOS с конкурирующими взаимодействиями на дереве Кэли порядка 2 и порядка 3. Далее изучены непериодические основные состояния для модели SOS с конкурирующими взаимодействиями на дереве Кэли второго порядка.

Ключевые слова: дерево Кэли, SOS-модель, периодические и слабо-периодические основные состояния.