Journal of Siberian Federal University. Mathematics & Physics 2022, 15(1), 23–28

DOI: 10.17516/1997-1397-2022-15-1-23-28
УДК 512.761, 517.55

# Detailed Factorization Identities for Classical Discriminant

**Evgeny N. Mikhalkin**[*]
**Mearajuddin Nikzad**[†]
**Vitaly A. Stepanenko**[‡]
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 03.05.2021, received in revised form 30.06.2021, accepted 20.08.2021

**Abstract.** A general polynomial in one variable is considered and the explicit factorization formulas for the truncations of the discriminant with respect to coordinate faces of the polynomial Newton polytope are presented. As a result, the extension of the formulas presented by Gelfand–Kapranov–Zelevinsky is obtained.

**Keywords:** discriminant, Newton polytope, Horn–Kapranov parametrization.

**Citation:** E.N. Mikhalkin, M. Nikzad, V.A. Stepanenko, Detailed Factorization Identities for Classical Discriminant, J. Sib. Fed. Univ. Math. Phys., 2022, 15(1), 23–28.
DOI: 10.17516/1997-1397-2022-15-1-23-28.

## 1. Introduction and preliminaries

We consider a general polynomial of degree $n$:

$$f(y) = a_0 + a_1 y + \ldots + a_n y^n. \tag{1}$$

It is known that discriminant of this polynomial is an irreducible polynomial $\Delta_n = \Delta_n(a_0, a_1, \ldots, a_n)$ with integer coefficients that vanishes if and only if $f$ has multiple roots. Discriminants play a crucial role in mathematics ( [1,2]).

Let us recall that the Newton polytope $\mathcal{N}(\Delta_n)$ for the discriminant of polynomial (1) is the convex hull in $\mathbb{R}^{n+1}$ of the exponents set $(t_0, t_1, \ldots t_n)$ of the monomials involved in $\Delta_n$. The Newton polytope $\mathcal{N}(\Delta_n) \subset \mathbb{R}^{n+1}$ is known to be combinatorially equivalent to an $(n-1)$-dimensional cube [1]. Since such a cube has $2^{n-1}$ vertices, it is natural to encode vertices $\mathcal{N}(\Delta_n)$ with all possible subsets from the set $\{1, \ldots, n-1\}$. The polytope $\mathcal{N}(\Delta_n)$ has $n-1$ hyperfaces $\{h_k^0\}$ located in the coordinate hyperplanes $\{t_k = 0\}$, $k = 1, \ldots, n-1$ (assuming that we chose the coordinates $t = (t_0, t_1, \ldots, t_{n-1}, t_n)$ within the ambient space $\mathbb{R}^{n+1}$). Each face $h_k^0$ has $2^{n-2}$ vertices defined by subsets $I \subset \{1, \ldots, n-1\}$ that do not contain $k$. Let us denote by $h_k$ the face that is opposite to $h_k^0$ with vertices encoded by subsets of $I$ containing $k$. The formulas for the coordinates of the vertices $\mathcal{N}(\Delta_n)$ are given in Section 2.

We consider the truncations of the discriminant $\Delta_n$ with respect to the faces (including coordinate ones) of its polytope $\mathcal{N}(\Delta_n)$. Let us remind that *truncation* of a polynomial $\Delta$ with

[*]mikhalkin@bk.ru
[†]mearajuddin.nikzad3@gmail.com
[‡]v-stepanen@mail.ru

respect to the face $h$ of its polytope $\mathcal{N}(\Delta)$ is the sum of all monomials from $\Delta$ with indices belonging to $h$. Let us denote such truncation by $\Delta\big|_h$.

The formulas for the truncations $\Delta_n$ on noncoordinate hyperfaces

$$h_K := h_{k_1} \cap \ldots \cap h_{k_p}$$

were proved [4]. They were obtained by the intersection of $p$ non-coordinate hyperfaces ([5]). Here the multi-index $K = = \{k_1, \ldots, k_p\}$ defines a partition of the set $\{0, 1, \ldots, n\}$ into $p+1$ subsets (segments)

$$K_i = \{k_i, k_i + 1, \ldots, k_{i+1}\},\ i = 0, 1, \ldots, p\,,$$

where $k_0 = 0$, $k_{p+1} = n$. Let us denote the length of $K_i$ by $l_i := k_{i+1} - k_i$. Then

$$f_{K_i} := a_{k_i} + a_{k_i+1} y + \ldots + a_{k_{i+1}} y^{l_i}.$$

The result proved in [5] is the following:
*The truncation of $\Delta_n$ on the face $h_k$ is*

$$\Delta_n\big|_{h_K} = a_K^2 \prod_{i=0}^{p} \Delta_{l_i}(f_{K_i}), \tag{2}$$

*where $a_K^2 = a_{k_1}^2 \ldots a_{k_p}^2$, and $\Delta_{l_i}$ are the discriminants of polynomials $f_{K_i}$ of degree $l_i$.*

The generalization of formula (2) is presented in this paper. The truncations $\Delta_n$ are obtained by intersecting both non-coordinate and coordinate faces $\mathcal{N}(\Delta_n)$. To formulate the main result of this paper we denote the face of $\Delta_n$ obtained by the intersection of $p$ noncoordinate faces $h_{k_1}, \ldots h_{k_p}$ and $q$ coordinate faces $h^0_{j_1}, \ldots, h^0_{j_q}$ by

$$h_{K,J^0} := h_{k_1} \cap \ldots \cap h_{k_p} \cap h^0_{j_1} \cap \ldots \cap h^0_{j_q}.$$

The elements of the set $J := \{j_1, \ldots, j_q\}$ that define the coordinate faces are grouped as follows: $J_i := J \cap (k_i, k_{i+1})$. Now we define polynomials $f_{K_i,J^0}(z)$, $i = 1, \ldots, p$ of the factorization of the truncations. We omit the monomials with indices from $J_i$ for each of $f_{K_i}$ and change the variable $z = y^{d_i}$, where $d_i$ is the greatest common divisor of the exponents of the monomials remaining in $f_{K_i}$.

**Theorem 1.1.** *Using given above notations, the truncation $\Delta_n$ with respect to the face $h_{K,J^0}$ is*

$$\Delta_n\big|_{h_{K,J^0}} = a_K^2 \prod_{i=0}^{p} (-1)^{\frac{l_i(d_i-1)}{2}} d_i^{l_i} (a_{k_i} a_{k_{i+1}})^{d_i - 1} \Big(\Delta_{l_i/d_i}\big(f_{K_i,J^0}(z)\big)\Big)^{d_i}, \tag{3}$$

*where $a_K^2 = a_{k_1}^2 \ldots a_{k_p}^2$, and $\Delta_{l_i/d_i}$ are the discriminants of $f_{K_i,J^0}$ of polynomials of degree $l_i/d_i$.*

Thus, Theorem 1.1 gives complete information on the factorability of the truncations of the discriminant with respect to any faces of its Newton polytope.

Note that when set $\{h^0_{k_1}, \ldots, h^0_{k_q}\}$ is empty, which means we consider only the truncation of the discriminant with respect to noncoordinate faces when all $d_i$ in formula (3) are equal to 1, the discriminants of each of the polynomials $f_{K_i,J^0}$ and $f_{K_i}$ coincide, and we obtain formula (2).

As an example, we calculate the truncation of the polynomial $\Delta_7(a_0, \ldots, a_7)\big|_{h_{K,J^0}}$, where $h_{K,J^0} = h_3 \cap h^0_1 \cap h^0_2 \cap h^0_4 \cap h^0_6$. In our case $p = 1$ then there are two discriminants of polynomials in the product $\prod\limits_{i=0}^{p}$ from Theorem 1.1

$$f_{K_0} = a_0 + a_1 y + a_2 y^2 + a_3 y^3 \text{ and } f_{K_1} = a_3 + a_4 y + a_5 y^2 + a_6 y^3 + a_7 y^4.$$

Because there is one value $k_1 = 3$ among $k_i$ then there are two segments $(0, 3)$ and $(3, 7)$ among segments $(k_i, k_{i+1})$. Then sets $J_0$ and $J_1$ are $J_0 = \{1, 2\}$ and $J_1 = \{4, 6\}$, respectively. Hence polynomials $f_{K_0,J^0}(z)$ and $f_{K_1,J^0}(z)$ are

$$f_{K_0,J^0} = a_0 + a_3 z^3, \quad f_{K_1,J^0} = a_3 + a_5 z + a_7 z^2.$$

Using Theorem 1.1, we obtain the following factorization formula for the truncation

$$\Delta_7(a_0, \ldots, a_7)\big|_{h_{K,J^0}} = a_3^2 \Delta_3(f_{K_0,J^0}) \cdot 16a_3a_7(\Delta_2(f_{K_1,J^0}))^2 = -432a_0^2a_3^5a_7(a_5^2 - 4a_3a_7)^2.$$

## 2. Newton polytope for the discriminant

The theorem on the structure the Newton polytope of the discriminant is as follows.

**Theorem 2.1** ([1], Ch. 12). *The Newton polytope of the discriminant of polynomial (1) is combinatorially equivalent to an $(n-1)$-dimensional cube. It contains $2^{n-1}$ vertices which are in bijective correspondence with all possible subsets $I \subset \{1, 2, \ldots, n-1\}$.*

*The vertex $\mathrm{v}(I)$ corresponding to a subset $I = \{i_1 < i_2 < \ldots < i_s\}$ has the following coordinates*

$$\mathrm{v}_0 = i_1 - 1,\ \mathrm{v}_n = n - i_s - 1,$$

$$\mathrm{v}_{i_\nu} = i_{\nu+1} - i_{\nu-1} \quad for \quad i_\nu \in I,$$

$$\mathrm{v}_i = 0, \quad for \quad i \notin I \cup \{0, n\}.$$

*Let $l_\nu = i_{\nu+1} - i_\nu$ $(0 \leqslant \nu \leqslant s)$, $i_0 = 0$, $i_{s+1} = n$. Then the monomial $a^{\mathrm{v}(I)}$ appears in $\Delta_n$ with the coefficient*

$$C_{\mathrm{v}(I)} = C(I) = \prod_{\nu=0}^{s} (-1)^{\frac{l_\nu(l_\nu - 1)}{2}} l_\nu^{l_\nu}.$$

Thus, each vertex of the Newton polytope $\mathcal{N}(\Delta_n)$ for the discriminant of the polynomial (1) is determined by an appropriate partition of the segment $[0, n]$.

Considering the well-known fact that discriminants are bihomogeneous, the polytope $\mathcal{N}(\Delta_n)$ lies in the plane of $\mathbb{R}^{n+1}$ of codimension 2 defined by the following of equations

$$\sum_{j=0}^{n} t_j = 2(n-1),\ \sum_{j=1}^{n} j t_j = n(n-1).$$

Formulas defining $n-1$ noncoordinate hyperfaces of the polytope $\mathcal{N}(\Delta)$ were proved [5, 6]:

*In this plane, the polytope $\mathcal{N}(\Delta)$ is defined by the following inequalities:*

$$t_k \geqslant 0, \quad k = 1, \ldots, n-1,$$

$$\sum_{j=1}^{k} (n-k) j t_j + \sum_{j=k+1}^{n-1} k(n-j) t_j \leqslant nk(n-k), \quad k = 1, \ldots, n-1.$$

Thus, the hyperface $h_k$ is determined for each value of $k$.

# 3. Proof of the main result

To determine the truncation $\Delta_n\big|_{h_{K,J^0}}$, one need to determine the restrictions of each of the factors $\Delta_{l_i}(f_{K_i})$ from (2) to the coordinate faces from the set $J_i \subset J$. Each such restriction is obtained with all monomials from $\Delta_{l_i}(f_{K_i})$ that do not contain factors with indices from $J_i$. Thus, we need to calculate the discriminant of the so-called thinned polynomial.

**Lemma 1.** *The discriminant of the polynomial*

$$x_0 + x_1 y^{n_1} + \ldots + x_s y^{n_s} + x_{s+1} y^n \tag{4}$$

*can be written in the form*

$$(-1)^{\frac{n(d-1)}{2}} d^n \cdot (x_0 x_{s+1})^{d-1} \Big[\Delta(x_0 + x_1 z^{m_1} + \ldots + x_s z^{m_s} + x_{s+1} z^m)\Big]^d, \tag{5}$$

*where* $m_k := \dfrac{n_k}{d}$, $m = \dfrac{n}{d}$, $d = GCD(n_1, \ldots, n_s, n)$.

To prove Lemma 1 we need the following formula for factorization of the difference $a - b$ into linear factors with respect to $a^{\frac{1}{n}}$:

$$a - b = \prod_{\nu=0}^{n-1} (a^{\frac{1}{n}} - b^{\frac{1}{n}} e^{\frac{2\pi i}{n} k}). \tag{6}$$

This formula is obtained in the following way. Consider $g(a) := a - b$ as a polynomial with respect to $a^{\frac{1}{n}}$: $g(a) = (a^{\frac{1}{n}})^n - b$. Since

$$a^{\frac{1}{n}} = b^{\frac{1}{n}} e^{\frac{2\pi i}{n} k}, \quad k = 0, 1, \ldots, n-1$$

are $n$ roots of polynomial $g(a)$, we obtain (6).

*Proof of Lemma 1.* The discriminant of a polynomial is defined in terms of its roots. Let us remind that discriminant $\Delta_n(a_0, \ldots, a_n)$ of polynomial (1) is defined by the formula [7]

$$a_n^{2n-2} \prod_{i<j} (y_i - y_j)^2, \tag{7}$$

where $y_1, \ldots, y_n$ are the roots of the polynomial.

Considering $z = y^d$ in (4), we obtain the polynomial

$$x_0 + x_1 z^{m_1} + \ldots + x_s z^{m_s} + x_{s+1} z^m. \tag{8}$$

Let us assume that $z_p$ and $z_q$, $0 \leqslant p < q \leqslant m-1$ are arbitrary roots of polynomial (8). Let us compose two groups of primitive roots from them:

$$z_{p,k} = z_p^{\frac{1}{d}} e^{\frac{2\pi i}{d} k} \ \text{ and } \ z_{q,k} = z_q^{\frac{1}{d}} e^{\frac{2\pi i}{d} k}, \quad k = 0, 1, \ldots, d-1. \tag{9}$$

Since they are the roots of the equation $z = y^d$ then $y = z^{\frac{1}{d}}$ are $d$ roots of polynomial (4). Thus, the number of roots of form (9) is equal to $m \cdot d = n$. Hence, expressions (9) present the whole set of roots from (4). Then, according to (7), to determine the discriminant of polynomial (4), one need to find the squares of the products of the following differences:

$$\prod_{p<q} \prod_{k=0}^{d-1} (y_{p,k} - y_{q,k})(y_{p,k} - y_{q,k+1}) \ldots (y_{p,k} - y_{q,k+d-1}), \tag{10}$$

where the second index at $y_{q,t}$ is considered with respect to the absolute value of $d$: $0 \leqslant t \leqslant d-1$. These are the differences that are made up of the roots obtained from both $z_p$ and $z_q$, $p \neq q$. We also need to find the products of the differences obtained from each root $z_\nu$:

$$\prod_{\nu=0}^{m-1} \prod_{i<j} (y_{\nu,i} - y_{\nu,j}), \tag{11}$$

and then to find the product of their squares multiplied by $(x_{s+1})^{2n-2}$.

Let us find the product $\prod\limits_{k=0}^{d-1}$ in (10). It is easy to check that for each fixed $v$ the product

$$\prod_{k=0}^{d-1} (y_{p,k} - y_{q,k+v}) = (-1)^{d-1} (z_p^{\frac{1}{d}} - z_q^{\frac{1}{d}} e^{\frac{2\pi i}{d} v})^d.$$

Then, using formula (6), we obtain that

$$\prod_{k=0}^{d-1} (y_{p,k} - y_{q,k})(y_{p,k} - y_{q,k+1}) \ldots (y_{p,k} - y_{q,k+d-1}) = (z_p - z_q)^d. \tag{12}$$

Let us find the square of the inner product in (11), i.e. $\prod\limits_{i<j} (y_{\nu,i} - y_{\nu,j})^2$. Let us take into account that $y_{\nu,0}, \ldots, y_{\nu,d-1}$ are the roots of the equation $y^d - z_\nu = 0$, where $\nu = 0, \ldots, m-1$ are the roots of equation (8). Therefore, $\prod\limits_{i<j} (y_{\nu,i} - y_{\nu,j})^2$ is the discriminant of this binomial equation and it has the form $(-1)^{\frac{d(d-1)}{2}} d^d (-z_\nu)^{d-1}$ (see. [8]). Then the product $\prod\limits_{\nu=0}^{m-1} \prod\limits_{i<j} (y_{\nu,i} - y_{\nu,j})^2$ can be written as

$$\prod_{\nu=0}^{m-1} \prod_{i<j} (y_{\nu,i} - y_{\nu,j})^2 = (-1)^{\frac{md(d-1)}{2} + m(d-1)} d^{md} (z_0 \cdot \ldots \cdot z_{m-1})^{d-1}.$$

Let us note that $z_0, \ldots, z_{m-1}$ are the roots of equation (8). According to Vieta's formulas, their product is $(-1)^m \dfrac{x_0}{x_{s+1}}$. Taking this and relation $m \cdot d = n$ into account, we obtain the final representation for the product

$$\prod_{\nu=0}^{m-1} \prod_{i<j} (y_{\nu,i} - y_{\nu,j})^2 = (-1)^{\frac{n(d-1)}{2}} d^n \left( \frac{x_0}{x_{s+1}} \right)^{d-1}.$$

Now, using formula (7), namely, multiplying $x_{s+1}^{2n-2} = x_{s+1}^{(2m-2)d} x_{s+1}^{2d-2}$, the above obtained expression and the square of expression (12), we obtain the following representation for the discriminant of polynomial (8):

$$(-1)^{\frac{n(d-1)}{2}} d^n \cdot (x_0 x_{s+1})^{d-1} \Big( x_{s+1}^{2m-2} \prod_{p<q} (z_p - z_q)^2 \Big)^d.$$

This is equality (5). The Lemma 1 is proved. □

Now we can apply proved formula (5) to each factor from (2) to obtain formula (3). Thus, Theorem 1.1 is proved.

*This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement no. 075-02-2020-1534/1).*

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## Детализация факторизационных тождеств для классического дискриминанта


**Евгений Н. Михалкин**
**Меараюддин Никзад**
**Виталий А. Степаненко**
Сибирский федеральный университет
Красноярск, Российская Федерация



**Аннотация.** Рассматривается дискриминант многочлена одного переменного. Приводятся явные факторизационные формулы для срезок дискриминанта на координатные грани его многогранника Ньютона. Полученные формулы детализируют результаты известной книги Гельфанда–Капранова–Зелевинского.



**Ключевые слова:** дискриминант, многогранник Ньютона, параметризация Горна–Капранова.