The Euler-Maclaurin Formula in the Problem of Summation over Lattice Points of a Simplex

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Abstract. In the problem of summation over lattice points of a simplex we find an analog of the Euler–Maclaurin formula for discrete primitive function and the sum.

Keywords: summation of functions, a discrete primitive function, the Euler–Maclaurin formula

Introduction

Let $\mathbb{Z}_{\geqslant}$ be a set of integer non-negative numbers, $\mathbb{Z}_{\geqslant}^n = \underbrace{\mathbb{Z}_{\geqslant} \times \ldots \times \mathbb{Z}_{\geqslant}}_{n}$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and $\Delta(x) = \{t : t_1 + \cdots + t_n = x, \, t_j \geqslant 0, \, j = 1, 2, \ldots, n\}$ is a simplex in the n-dimensional space $\mathbb{R}^n$.

Find the sum

$$S(x) = \sum_{t \in \Delta(x) \cap \mathbb{Z}_{\geqslant}^n} \varphi(t) \tag{1}$$

for a given function $\varphi(t) = \varphi(t_1, \ldots, t_n)$ and $x \in \mathbb{Z}_{\geqslant}$.

In case of $n = 2$ the sum (1) can be written as

$$S(x) = \sum_{t_1 + t_2 = x} \varphi(t_1, t_2) = \sum_{t=0}^{x} \varphi(t, x - t) \tag{2}$$

and if the function φ does not depend on the second variable, then (2) is a classical problem of the 'indefinite' summation, i.e. the problem of finding in an explicit form the sum

$$S(x) = \sum_{t=0}^{x} \varphi(t). \tag{3}$$

Euler's approach to solving the problem is based on finding a discrete primitive function for $\varphi(t)$, i.e. any solution to a difference equation

$$f(t+1) - f(t) = \varphi(t), \tag{4}$$

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what yields a discrete analog of the Newton–Leibniz formula for the sum (2)

$$S(x) = f(x+1) - f(0). \tag{5}$$

Euler found a formula (1732–1733, published in 1738) in which a discrete primitive function for the function $\varphi(x)$ is expressed through derivatives $\varphi^{(\mu)}(x)$:

$$f'(x) = \sum_{\mu=0}^{\infty} \frac{B_\mu}{\mu!} \varphi^{(\mu)}(x), \tag{6}$$

where B_μ are Bernulli numbers (see, for example, [2]). Integrating equality (6) over the interval from 0 to $x+1$ we get a formula to solve the problem of summation:

$$\sum_{t=0}^{x} \varphi(t) = \int_0^{x+1} \varphi(t) \, dt + \sum_{\mu=1}^{\infty} \frac{B_\mu}{\mu!} \left[\varphi^{(\mu-1)}(x+1) - \varphi^{(\mu-1)}(0) \right].$$

Independently of Euler, Maclaurin found the same formula in 1742. Their derivation of the formula was not strict, the first serious investigation of the residual member was undertaken by Poisson in 1823 and its first rigorous proof was demonstrated by Jacobi in 1834 (see, for example, [1,2]). A discrete analog of the Euler summation formula was obtained in [3] .

A large number of works are devoted to various aspects of the problem of summation and its applications, among the relatively recent ones, we note [4,5].

The summation problem for the functions of several variables can be formulated in various ways. The problem of summation of polynomials of several variables over integer points of a rational convex polyhedron with variable faces was studied in the works [6,7], in particular, a multidimensional analog of the Euler–Maclaurin formula was obtained.

The problem of summation of arbitrary functions over lattice points of a rational parallelotope was studied in [8–10], where Euler's approach based on the concept of a discrete primitive function and a discrete analog of the Newton–Leibniz formula were used.

The connection of the classical summation problem with the problem of finding vector partition functions with weight was considered in [11]

1. Basic designations and definitions. Formulation of the main result

Let δ_j be the linear shift operator on j-th variable

$$\delta_j f(t_1, \ldots, t_n) = f(t_1, \ldots, t_{j-1}, t_j + 1, t_{j+1} \ldots, t_n), \ j = 1, \ldots n,$$

and consider the polinomial difference operator

$$W(\delta) = \prod_{1 \leqslant i < j \leqslant n} (\delta_j - \delta_i). \tag{7}$$

For a given function $\varphi(t)$, we call a discrete primitive function any solution $f(t)$ of the difference equation

$$W(\delta) f(t) = \varphi(t), \ t \in \mathbb{Z}_{\geqslant}^n. \tag{8}$$

For $n = 2$ we get $W(\delta) = \delta_2 - \delta_1$ and, if $f(t_1, t_2)$ is a solution of the equation (8), then for the sum (1) the formula $S(x) = f(0, x+1) - f(x+1, 0)$ holds true and it can be considered as

a discrete analog of the Newton–Leibniz formula. In case of arbitrary n we need to define the Newton–Leibniz operator.

Denote by π_j the projection operator along j-th coordinate axis, $\pi_j t = (t_1, \ldots, t_{j-1}, 0, t_{j+1}, \ldots, t_n)$, which acts on the function $f(t)$ according to the rule $\pi_j f(t) = f(\pi_j t)$, $j = 1, \ldots, n$.

Define the Newton–Leibniz operator as

$$W_{NL}\big(\delta, \pi\big) = \prod_{1 \leqslant i < j \leqslant n} (\pi_i \delta_j - \pi_j \delta_i). \tag{9}$$

If $n = 2$, then $W_{NL}(\delta, \pi) = \pi_1 \delta_2 - \pi_2 \delta_1$ and $W_{NL}(\delta, \pi) f(x, x) = f(0, x+1) - f(x+1, 0)$.

It is proved in the work [12], that if $f(t)$ is a discrete primitive function of the function $\varphi(t)$, then for the sum (1) the following discrete analog of the Newton–Leibniz formula holds true

$$S(x) = W_{NL}(\delta, \pi) f(x, \ldots, x). \tag{10}$$

One of the way to find a solution to the difference equation (8) (a discrete primitive function) is to construct the Todd operator. Consider a meromorphic function

$$T(\xi) = \frac{W(\xi)}{W(e^\xi)} = \frac{W(\xi_1, \ldots, \xi_n)}{W(e^{\xi_1}, \ldots, e^{\xi_n})} \tag{11}$$

which is holomorphic in a neighborhood $V_{2\pi} = \bigcup_{1 \leqslant i < j \leqslant n} \{|\xi_j - \xi_i| < 2\pi\}$ of the origin and, therefore, expands in a power series

$$T(\xi) = \sum_{\nu \in \mathbb{Z}_{\geqslant}^n} \frac{B_\nu}{\nu!} \xi^\nu, \tag{12}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is a multiindex, $\nu! = \nu_1! \ldots \nu_n!$ and $\xi^\nu = \xi_1^{\nu_1} \ldots \xi_n^{\nu_n}$.

The *generalized Bernoulli numbers* are coefficients B_ν of the expansion of the function $T(\xi)$ in power series (12). The *Todd operator* is a differential operator of infinite order constructed by substitution the differentiation operators by j-th variable ∂_j instead of variables ξ_j in (12)

$$Todd = T(\partial) = \sum_{\nu \in \mathbb{Z}_{\geqslant}^n} \frac{B_\nu}{\nu!} \partial^\nu, \tag{13}$$

where $\partial = (\partial_1, \ldots, \partial_n)$ and $\partial^\nu = \partial_1^{\nu_1} \ldots \partial_n^{\nu_n}$.

For $T(\xi) = \prod_{j=1}^{n} \dfrac{\xi_j}{e^{\xi_j} - 1}$, the Todd operator was defined in [13].

Define the space of function, where the Todd operator acts correctly. Denote by $Exp(\mathbb{C}^n)$ the set of entire functions $\varphi : \mathbb{C}^n \to \mathbb{C}$ of exponential type, which means that for any $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_j > 0$, $j = 1, \ldots, n$ and $C > 0$, satisfying $|\varphi(z)| < Ce^{\langle \sigma, |z| \rangle}$, where $|z| = (|z_1|, \ldots, |z_n|)$, $\langle \sigma, |z| \rangle = \sigma_1 |z_1| + \ldots + \sigma_n |z_n|$.

Also, we need an additional condition on the functions $\varphi \in Exp(\mathbb{C}^n)$ in terms of their derivatives. For $R = (R_1, \ldots, R_n)$, $R_j > 0$, $j = 1, \ldots, n$, denote $Exp_R(\mathbb{C}^n)$ is a subspace of the space $Exp(\mathbb{C}^n)$, whose elements φ satisfy the condition: there is constant $C > 0$ and $r = (r_1, \ldots, r_n)$, $r_j < R_j$, $j = 1, \ldots, n$ such that

$$\partial^\alpha \varphi(z) \leqslant Cr^\alpha e^{\langle r, |z| \rangle},$$

where multiindex α and points $z \in \mathbb{C}^n$ are arbitrary.

The Todd operator is defined correctly in the space $Exp_R(\mathbb{C}^n)$, i.e. if $\varphi \in Exp_R(\mathbb{C}^n)$, then $Todd\,\varphi \in Exp_R(\mathbb{C}^n)$ [see [14], Theorem 5.1].

Denote by $U_R = \{\xi : |\xi_j| < R_j, j = 1, \ldots, n\}$ the polycylinder with the maximal radius $R = (R_1, \ldots, R_n)$, which is contained in the space $V_{2\pi} = \bigcup_{1 \leqslant i < j \leqslant n} \{\xi : |\xi_i - \xi_j| < 2\pi\}$.

Next, we give the Euler–Maclaurin formula for a discrete primitive function.

Theorem 1. *Let $\varphi(t) \in Exp_R(\mathbb{C}^n)$, where $U_R \subset V_{2\pi}$, then for a discrete primitive function $f(t)$ the following analog of the Euler–Maclaurin formula*

$$W(\partial)\,f(t) = Todd\,\varphi(t) \tag{14}$$

holds.

For the differential operator $\partial_j - \partial_i$, $(1 \leqslant i < j \leqslant n)$, we define an 'inverse' operator P_{ij} as follows

$$P_{ij}f(t) = \int_0^{t_j} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, t_i + t_j, \ldots, t_n)\, d\tau.$$

It is directly checked that $(\partial_j - \partial_i) \circ P_{ij}f(t) = f(t)$ for $i, j = 1, 2, \ldots, n$ and $i < j$, therefore,

$$W(\partial) \circ Pf(t) = f(t), \tag{15}$$

where P is a polynomial differential operator of the form

$$P = \prod_{1 \leqslant i < j \leqslant n} P_{ij}. \tag{16}$$

We now formulate the following analog of the Euler–Maclaurin formula.

Theorem 2. *Let $\varphi(t) \in Exp_R(\mathbb{C}^n)$ and $U_R \subset V_{2\pi}$, then the analog of the Euler–Maclaurin formula for the sum (1)*

$$S(x) = W_{NL}(\delta, \pi) \circ P \circ Todd\,\varphi(t)\,|_{t_1 = \ldots = t_n = x}. \tag{17}$$

holds.

2. Proofs

Proof of the Theorem 1. For a function $A(\xi)$ holomorphic in the polycylinder $U_R = \{\xi \in \mathbb{C}^n : |\xi_j| < R_j, j = 1, \ldots, n\}$ we consider the differential operator $A(\partial)$ of infinite order, formally replacing the argument $\xi = (\xi_1, \ldots, \xi_n)$ by the symbol of differentiation $\partial = (\partial_1, \ldots, \partial_n)$. The function $A(\xi)$ is called the symbol of the differential operator $A(\partial)$. If $A(\xi) = \sum_{\alpha \geqslant 0} a_\alpha \xi^\alpha$ is an expansion of the holomorphic function into a series, then the action of the operator $A(\partial)$ on the function $\varphi(t) \in Exp_R(\mathbb{C}^n)$ is defined by the formula $A(\partial)\varphi(z) = \sum_{\alpha \geqslant 0} a_\alpha \partial^\alpha \varphi(z)$.

The set $\{A(\partial)\}$ of the operators of infinite order is an algebra with the domain of definition $Exp_R(\mathbb{C}^n)$, which is isomorhpic to the algebra of holomorphic in polycylinder U_R functions (see. [14], Theorem 5.2).

For the Todd operator (see formula (11)) it means that

$$W(e^\partial) \circ Todd = W(\partial). \tag{18}$$

It is directly checked that the connection between the polynomial difference operator $Q(\delta) = \sum_{\alpha \geqslant 0} c_\alpha \delta^\alpha$ and the operator of infinite order $Q(e^\partial)$ acting on the space $Exp_R(\mathbb{C}^n)$

$$Q(\delta) = Q(e^\partial) \tag{19}$$

holds true.

Thus, if $f(t)$ is a solution to the difference equation (8), formula (19) yields that $W(e^\partial) f(t) = \varphi(t)$. Acting by the Todd operator on the both parts of (19), we obtain (14). $\qquad \square$

Proof of the Theorem 2. To prove it, we need the discrete analog of the Newton-Leibniz formula in the problem of summation over lattice points of the simplex

$$S(x) = W_{NL}(\delta, \pi) f(x, x, \cdots, x),$$

which was proved in [12].

Application of the operator P to the equality (14) yields $f(t) = P(\delta) \circ Todd \, \varphi(t)$.

Acting by the Newton–Leibniz operator $W_{NL}(\delta, \pi)$ on the both parts of this equality, we obtain

$$S(x) = W_{NL}(\delta, \pi) \circ P \circ Todd \, \varphi(t)|_{t_1 = \ldots = t_n = x}. \qquad \square$$

Example 1. The problem is to find the sum

$$S(x) = \sum_{t_1 + t_2 = x} (t_1^2 + t_2^2 + 1).$$

According to Theorem 1 and Theorem 2 we obtain

$$(\partial_1 - \partial_2) f(t_1, t_2) = \sum_{\mu_1, \mu_2 = 0}^{\infty} \frac{B_{\mu_1 \mu_2}}{\mu_1! \mu_2!} \partial_1^{\mu_1} \partial_2^{\mu_2} (t_1^2 + t_2^2 + 1) = t_1^2 + t_2^2 - t_1 - t_2 + \frac{4}{3}.$$

and

$$f(t_1, t_2) = \frac{2}{3} t_1^3 + t_1^2 t_2 + t_1 t_2^2 - t_1 t_2 - t_1^2 + \frac{4}{3} t_1.$$

Applying the Newton–Leibniz formula we obtain the sum

$$\sum_{t_1 + t_2 = x} (t_1^2 + t_2^2 + 1) = (x + 1) \left(\frac{2}{3} x^2 + \frac{1}{3} x + 1 \right).$$

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Формула Ейлера-Маклорена в задаче суммирования по целым точкам рационального симплекса

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Аннотация. В задаче суммирования функции по целым точкам рационального симплекса найден аналог формулы Эйлера–Маклорена для дискретной первообразной и искомой суммы.

Ключевые слова: суммирование функций, дискретная первообразная, формула Эйлера–Маклорена