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On Some Decompositions of Matrices over Algebraically Closed and Finite Fields

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Abstract. Decomposition of every square matrix over an algebraically closed field or over a finite field into a sum of a potent matrix and a nilpotent matrix of order 2 is considered. This can be related to our recent paper, published in Linear & Multilinear Algebra (2022).

The question of when each square matrix over an infinite field can be decomposed into a periodic matrix and a nilpotent matrix of order 2 is also completely considered.

Keywords: nilpotent matrix, potent matrix, Jordan normal form, rational form, field.

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1. Introduction and conventions

Nilpotent and potent elements in matrix rings is mainly considered in this paper. Let us recall that an element q of an arbitrary ring R is said to be a *nilpotent* if there is an integer $n \geqslant 1$ that depends on q such that $q^n = 0$ (the minimal n with this property is called an *exponent* for q; in particular, if $n = 2$ the non-zero nilpotent is shortly called *square-zero*). Element $p \in R$ is said to be *potent* if there is a natural number $m \geqslant 2$ that depends on p and $p^m = p$ (p is called m-potent). If $m = 2$, this element is called *idempotent*. Common generalization of potent element is *periodic* element. An element t is said to be *periodic* if there are two different natural numbers m, n that depend on t and $t^m = t^n$.

Representation of an arbitrary matrix over a field as the sum of a nilpotent matrix and an idempotent matrix was considered in pioneering [3]. It was proved that this presentation is possible precisely when the field contains only two elements. This was further extended by showing in some cases the exact exponent of the nilpotent matrix [13], [12]. The valuable discussion on the decomposition of a matrix as the sum of an idempotent and a square-zero matrix was given [9]. On the other hand, as generalization of the aforementioned main fact from [3] it was proved that every matrix over any finite field of cardinality d is representable as the sum of a nilpotent matrix and a d-potent matrix [1]. Furthermore, this representation was refined by proving that if d is odd then the exponent of a nilpotent matrix is not more than 3 [2]. Moreover, it was constructed a 3×3 matrix over the field of three elements which is *not* presentable as the sum of a 3-potent matrix and a square-zero matrix [2, Example 6].

Hence the following intriguing problem is considered.

<u>*Question 1*</u>*: When every square matrix over a field K can be expressed as*

$$P + Q,$$

where P is a potent matrix and Q is a square-zero matrix?

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Let us consider below two situations, namely, algebraically closed fields (see Corollary 2.4) and finite fields (see Corollary 3.2). The results can be viewed and treated as the development of method and ideas presented in [8] and [6], respectively. Some closely related studies can also be found in [5].

It is well known that any element in finite rings is periodic, so any matrix over a finite ring is also periodic itself. This immediately rises the question on matrices over infinite rings. Attention will be concentrated only on infinite fields, so the following interesting problem will also be examined.

Question 2: *When each square matrix over an infinite field F can be expressed as*

$$T+Q,$$

where T is a periodic matrix and Q is a square-zero matrix?

It should be noted that a part of the established here results can be found in [7].

2. Decomposition into potent matrix and zero-square matrix over algebraically closed fields

As a first approach to the problem, it will be shown that all square matrices over an algebraically closed field admit decomposition into a diagonalizable matrix and a nilpotent matrix of order two. This decomposition will be significantly improved in the next section, where the same result is proved for non necessarily algebraically closed fields by using the rational canonical form. Nevertheless, in this section a simple argument is provided in terms of Jordan blocks. It is included here for the sake of completeness. Moreover, it provides a decomposition for nilpotent matrices over non necessarily algebraically closed fields.

The construction is based on Jordan blocks and roots of unity. Let us consider the explicit decomposition for a Jordan block (see Remarks 2.8 and 2.9 from [8]).

Lemma 2.1. *Let K be a field and let J be a Jordan block in $\mathbb{M}_n(K)$, $n \geqslant 3$ associated with $a \in K$*

$$J = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}.$$

(i) *Suppose that* $\mathrm{char}(K)$ *does not divide n. If K contains n (different) roots of the polynomial $x^n - 1 \in K[x]$ then J has the following decomposition*

$$J = \underbrace{(J + e_{1n})}_{D} + \underbrace{(-e_{1n})}_{Q}$$

where e_{1n} denotes the nilpotent matrix with 1 in the $(1n)$-entry and zero in the rest of entries, and matrix D is diagonalizable. Moreover, if $a = 0$ then $D^n = I$.

(ii) *Suppose that* $\mathrm{char}(K)$ *divides n. If K contains $n-1$ (different) roots of the polynomial $x^{n-1} - 1 \in K[X]$ then J has the following decomposition*

$$J = \underbrace{(J + e_{2n})}_{D} + \underbrace{(-e_{2n})}_{Q}$$

where e_{2n} denotes the nilpotent matrix with 1 in the $(2n)$-entry and zero in the rest of entries, and matrix D is diagonalizable. Moreover, if $a = 0$ then $D^n = D$ and D^{n-1} is similar to the diagonal matrix $\mathrm{diag}(1, \ldots, 1, 0)$.

Proof. (i) If $q = \mathrm{char}(K)$ does not divide n, then J can be written as

$$J = \underbrace{\begin{pmatrix} a & 0 & 0 & 0 & 1 \\ 1 & a & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{Q}$$

The minimal polynomial of D is $p(x) = (x - a)^n - 1$ and it has n different roots in K (by hypothesis K contains all roots of $p(x) = (x - a)^n - 1$ and they are all different because $p'(x) = n(x - a)^{n-1} \neq 0$ since $q \nmid n$). In particular, D is diagonalizable. Moreover, one can see that $Q^2 = 0$.

(ii) If $q = \mathrm{char}(K)$ divides n then

$$J = \underbrace{\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 1 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{Q}$$

The minimal polynomial of D is $p(x) = (x - a)^n - (x - a)$ and its n roots belong to K by hypothesis and they are all different because $p'(x) = -1 \neq 0$ (recall $q|n$). In particular, it follows that D is diagonalizable. Moreover, $Q^2 = 0$ as required. □

Remark 2.2. The decomposition of each Jordan block into $D + Q$ given in Lemma 2.1 has the following properties:

- Each D is diagonalizable with no multiple eigenvalues.
- $Q^2 = 0$ and $\mathrm{rank}(Q) \leqslant 1$.

Proposition 2.3. *Let K be an algebraically closed field. Then any matrix $A \in \mathbb{M}_n(K)$ can be written as $D + Q$, where D is a diagonalizable matrix and Q is a nilpotent matrix for which $Q^2 = 0$.*

Proof. Since K is algebraically closed and A is similar to a direct sum of Jordan blocks then it is sufficient to decompose each Jordan block. Let J be a Jordan block of size $m \times m$ for some $m \leqslant n$ and q is the characteristic of K. If $m \leqslant 2$ the decomposition is straightforward (see Section 1 of [8]). When $m \geqslant 3$ and if q does no divide m then decomposition of J is presented in Lemma 2.1(i). If q does divide m then decomposition of J is presented in 2.1(ii). □

The assumption of algebraic closeness of the field can be removed when dealing with nilpotent matrices over a field for which the decomposition into Jordan blocks always holds. Let us notice that this can be related to [10, Sec. 2] where minimal conditions for a nilpotent element in a ring are given to admit decomposition into Jordan blocks. As a consequence, any nilpotent matrix can be expressed as the sum of a potent matrix and a nilpotent matrix of zero square. This result can be related to [4, Corollary 8] where any nilpotent matrix is decomposed into an idempotent matrix and a nilpotent matrix.

Corollary 2.4. *Every nilpotent matrix over a field can be written as $D+Q$, where D is a potent matrix (i.e., $D^q = D$ for a certain $q \in \mathbb{N}$) and Q is a nilpotent matrix with $Q^2 = 0$.*

Proof. Let $A \in \mathbb{M}_n(K)$ be a nilpotent matrix over the field K. Then A is similar to the direct sum of Jordan blocks $J_1, \ldots, J_s$, each of them is associated with the eigenvalue 0. Any of these Jordan blocks $J_i \in \mathbb{M}_{m_i}(K)$ is decomposed as in Lemma 2.1: $J_i = D_i + Q_i$. Let us define

$$k_i := \begin{cases} m_i, & \text{if } \mathrm{char}(K) \text{ does not divide } m_i, \\ m_i - 1 & \text{if } \mathrm{char}(K) \text{ divides } m_i, \end{cases}$$

and let $q = lcm\{k_i \mid i = 1, \ldots, s\} + 1$. Then

$$(\bigoplus_{i=1}^{s} D_i)^q = \bigoplus_{i=1}^{s} D_i^q = \bigoplus_{i=1}^{s} D_i,$$

i.e., $\bigoplus_{i=1}^{s} D_i$ is q-potent. Finally, A is decomposed into $D + Q$ as in Proposition 2.3 and D is similar to $\bigoplus_{i=1}^{s} D_i$. □

It is also worth to notice that the last statement can be proved by using the rational (Frobenius) canonical form [4], [10].

In what follows, Corollary 2.4 and Remark 2.9 from [8] will be substantially generalized with the use of another approach.

Proposition 2.5. *Every nilpotent matrix over a von Neumann regular ring is decomposable as the sum of a potent matrix and a nilpotent matrix of order two.*

Proof. Let R be a von Neumann regular ring. Then A^s is a von Neumann regular matrix for all $s \in \mathbb{N}$ for any nilpotent matrix A over R. Hence A is decomposed into a direct sum of Jordan blocks (see, e.g., [10]). Each of these Jordan blocks can be represented as the sum of a potent matrix and a zero-square matrix. In particular, it is not too hard to verify that A itself can be represented as the sum of a potent matrix and a zero-square matrix, as asserted in Proposition 2.5. □

3. Decomposition into potent matrix and zero-square matrix over finite fields

In what follows the following approach is used [8].

Lemma 3.1. *Let K be a field, $n \geqslant 3$ and $A \in \mathbb{M}_n(K)$ is the companion matrix of a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. Then*

- *If $c_{n-1} = 0$ and $|K| \geqslant n$ then A admits decomposition into $D+Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with* $\mathrm{rank}(Q) \leqslant 1$.
- *If $c_{n-1} \neq 0$ and $|K| \geqslant n+1$ then A admits decomposition into $D+Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with* $\mathrm{rank}(Q) \leqslant 1$.

In this section the following assertion which is devoted to a non-trivial property of matrices over finite fields is considered (see Proposition 3.1 and Corollary 3.2 from [8]).

Corollary 3.2. *Let K be a finite field and $n \in \mathbb{N}$. Then every matrix in $\mathbb{M}_n(K)$ admits decomposition into the sum of an r-potent matrix, for certain $1 < r \in \mathbb{N}$, and a square-zero matrix.*

Proof. Let $\mathbb{F}_q$ be the finite field of q elements and assume that $A \in \mathbb{M}_n(\mathbb{F}_q)$. Let us consider the decomposition of A with respect to its invariant factors. Then A is similar to the direct sum of s companion matrices. Each companion matrix is of size $m_i \leqslant n$, $i = 1, \dots, s$. Let us take for each companion matrix an irreducible polynomial $q_i(x)$ of degree m_i with the same trace, and write this block as the sum of the companion matrix $C(q_i(x))$ and a nilpotent matrix of zero square (see Lemma 3.1 and its proof). The decomposition field F of $q_i(x)$ is an extension of degree m_i of $\mathbb{F}_q$, i.e., $F = \mathbb{F}_{q^{m_i}}$. Since matrix $C(q_i(x))$ is diagonalizable with different eigenvalues in F (finite fields are perfect), $C(q_i(x))$ is similar to a diagonal matrix $D_i \in \mathbb{M}_{m_i}(F)$. Therefore,

$$D_i^{q^{m_i}-1} = \begin{cases} I, & \text{if } q_i(0) \neq 0; \\ \operatorname{diag}(1, \dots, 1, 0), & \text{if } q_i(0) = 0. \end{cases}$$

Let us define $r = lcm\{q^{m_i} - 1 \mid i = 1, \dots, s\} + 1$. Since each $D_i^r = D_i$ one can express A as the sum of an r-potent matrix and a square-zero matrix. □

Actually, the more general question is whether or not every square matrix over an arbitrary (possibly infinite) field is presentable as the sum of a nilpotent matrix and a potent matrix. Let us notice that for finite fields this was settled independently in Corollary 3.2 and [1]. However, the answer seems to be definitely "not" as the next example illustrates but such matrix is rather the sum of a non-singular matrix and a nilpotent matrix (see, e.g., [11]). In particular, this fact surely implies that the matrix over the field $\mathbb{F}_4$ is the sum of a potent matrix and a nilpotent matrix (compare also with Corollary 3.2 and [1]).

Example 3.3. Let us consider matrix $A = 2\,\mathrm{Id} \in \mathbb{M}_n(\mathbb{R})$, and show that A cannot be expressed as the sum of a k-potent matrix and an r-nilpotent matrix. Otherwise, $A = Q + N$ with $Q^k = Q$ and $N^r = 0$ for some natural numbers k and r. On the one hand this surely implies that $Q = A - N$ satisfies the polynomial $X^k - X$ (because $Q^k = Q$). On the other hand, since $0 = N^r = (A - Q)^r = (2\,\mathrm{Id} - Q)^r$, matrix Q also satisfies the polynomial $(2 - X)^r$. This means that minimal polynomial of Q must divide both $X^k - X$ and $(X - 2)^r$ but these two polynomials have no common roots in $\mathbb{R}$. Then the minimal polynomial of Q is 1. This is a contradiction. The proof is completed.

4. Decomposition into periodic matrix and zero-square matrix over infinite fields

To begin with, it will be shown that Question 2 is *not* true even over algebraically closed fields. Let us consider the following example.

Example 4.1. Let $\mathbb{C}$ be the field of complex numbers, and consider the matrix $A = 2\mathrm{Id} \in \mathbb{M}_n(\mathbb{C})$. Let us assume that $A = T + N$, where $N^2 = 0$. Then $N = A - T$ and, therefore, $0 = N^2 = (A - T)^2 = 4\mathrm{Id} + T^2 - 4T$. This means that matrix T satisfies the polynomial $x^2 - 4x + 4 = (x - 2)^2$.

Furthermore, it is easy to verify that characteristic polynomial of T is of the form $(x - 2)^n$ for some $n \in \mathbb{N}$. Then the determinant of T is equal to $(-1)^n 2^n$. Consequently, T cannot be periodic because either $\det(P) = 0$, $\det(P) = 1$ or $\det(P) = -1$.

One can propose a complete answer to Question 2 as follows.

Proposition 4.2. *Let F be a field. Then the next three statements are valid:*

(1) If $\mathrm{char}(F) = 0$ *then the answer is NO.*

(2) If $\mathrm{char}(F) = p$ *and the extension of F over its prime field F_p is transcendental then the answer is NO.*

(3) If $\mathrm{char}(F) = p$ *and the extension of* F *over its prime field* F_p *is algebraic then the answer is YES.*

Proof. (1) The unique prime field of zero characteristic is precisely the field of rationals $\mathbb{Q}$. Hence the matrix 2Id cannot be decomposed into the sum of periodic matrix and zero-square matrix (compare with the stated above example).

(2) There exists an element $a \in F$ that is not algebraic over F_p. Let us consider then the matrix aId. Using the same argument as in (1), one can obtain that this matrix cannot be decomposed into the sum of periodic matrix and zero-square matrix.

(3) Indeed, square matrix can be decomposed even into the sum of potent matrix and zero-square matrix. Let $A \in \mathbb{M}_n(F)$. Let us consider the finite field L generated by F_p and by the entries of A. If L has more elements than the matrix size n then applying the main result from [8], the matrix A is decomposed into the sum of a diagonalizable matrix over L and a zero-square matrix. Otherwise, one can extend L by adding elements from F until some finite field L' with more elements than n is obtained. Since still $A \in \mathbb{M}_n(L')$ matrix A is decomposed into the sum of a diagonalizable matrix over L' and a zero-square matrix. Taking into account that a diagonalizable matrix over a finite field is always potent and hence periodic, the proof is complete. □

At the end of the paper let us consider the following interesting question.

Problem. Can any square matrix over the indecomposable ring $\mathbb{Z}_4$ be decomposed into the sum of a square-zero matrix and a potent matrix?

Let us note that it was established in [13] that every such matrix can be decomposed into the sum of a nilpotent matrix of order at most 8 and an idempotent matrix. So, it is rather realistic to replace the idempotent matrix by a potent matrix and thereby to expect that the order of the nilpotent matrix could be decreased to order 2 or, in a worse variant, to order 4.

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О некоторых разложениях матриц над алгебраически замкнутыми и конечными полями

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Аннотация. Мы доказываем, что каждая квадратная матрица над алгебраически замкнутым полем или над конечным полем разложима в сумму потентной матрицы и нильпотентной матрицы порядка 2. Это отчасти продолжает исследование из нашей недавней статьи, опубликованной в Linear & Multilinear Algebra (2022 г.).

Мы также полностью решаем вопрос, когда каждую квадратную матрицу над бесконечным полем можно разложить на периодическую матрицу и нильпотентную матрицу порядка 2.

Ключевые слова: нильпотентная матрица, потентная матрица, жорданова нормальная форма, рациональная форма, поле.