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On the Asymptotic Behavior of the Conjugate Problem Describing a Creeping Axisymmetric Thermocapillary Motion

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Abstract. In this paper the conditions for the law of temperature behavior on a solid cylinder wall describes, under which the solution of a linear conjugate inverse initial-boundary value problem describing a two-layer axisymmetric creeping motion of viscous heat-conducting fluids tends to zero exponentially with increases of time.

Keywords:the conjugate nonlinear inverse problem, interface, a crawling motion.
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1. Introduction and preliminaries

In work [1], the linear conjugate inverse initial boundary value problem describing a two-layer creeping motion of viscous heat-conducting fluids in a cylinder with a solid side surface $r = R_2 = \text{const}$ and interface $r = h(t)$, $0 < h(t) < R_2$ was considered

$$v_{1t} = \nu_1 \left(v_{1rr} + \frac{1}{r} v_{1r} \right) + f_1(t), \quad 0 < r < R_1, \tag{1}$$

$$v_{2t} = \nu_2 \left(v_{2rr} + \frac{1}{r} v_{2r} \right) + f_2(t), \quad R_1 < r < R_2, \tag{2}$$

$$v_1(R_1,t) = v_2(R_1,t), \quad \int_0^{R_1} r v_1(r,t)dr + \int_{R_1}^{R_2} r v_2(r,t)dr = 0, \tag{3}$$

$$\mu_1 v_{1r}(R_1,t) - \mu_2 v_{2r}(R_1,t) = -2æ a_1(R_1,t), \tag{4}$$

$$|v_1(0,t)| < \infty, \quad v_2(R_2,t) = 0, \tag{5}$$

$$v_1(r,0) = 0, \quad v_2(r,0) = 0, \tag{6}$$

$$\rho_1 f_1(t) = \rho_2 f_2(t) - \frac{2æ a_1(R_1,t)}{R_1} \tag{7}$$

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and the closed conjugate problem for functions $a_j(r,t)$ is described the following equations:

$$a_{jt} = \chi_j \left(a_{jrr} + \frac{1}{r}\, a_{jr} \right), \tag{8}$$

$$a_j(r,0) = a_j^0(r), \quad |a_1(0,t)| < \infty, \tag{9}$$

$$a_2(R_2,t) = \alpha(t), \tag{10}$$

$$a_1(R_1,t) = a_2(R_1,t), \quad k_1 a_{1r}(R_1,t) = k_2 a_{2r}(R_2,t). \tag{11}$$

The interface is described by the formula

$$h(t) = R_1[1 + \mathrm{M}\, h_1(t)], \quad h_1(t) = -\frac{1}{R_1} \int_0^t r v_1(R_1,t) dt. \tag{12}$$

Here $\mathrm{M} = \text{æ} a^1 R_1^3/\mu_1\chi_1$ is Marangoni number, $a^1 = \max\limits_{t\in[0,T]} |\alpha(t)|$. Note that $\mathrm{M} \to 0$ since the creeping motion considers in this paper.

In paper [1] the priori estimates were obtained for the functions $v_j(r,t)$, $a_j(r,t)$, $f_j(t)$. In this paper, it will be proved that under certain conditions which set for the temperature on the cylinder surface, the solution of the problem (1)–(11) tends to zero exponentially with increasing time.

2. The behavior of the solution under $t \to \infty$

A priori estimates for the function $a_j(r,t)$ satisfying the problem (8)–(11) have form [1]

$$|a_1(r,t)| \leqslant 2\left[\max_{t\in[0,T]} |\alpha(t)| + \frac{1}{(R_1^2 k_2 \rho_2 c_{\rho_2})^{1/4}} \max_{t\in[0,T]} (A(t)A_1(t))^{1/4}\right] + \max_{r\in[0,R_1]} |a_1^0(r)|, \tag{13}$$

$$|a_2(r,t)| \leqslant |\alpha(t)| + 2\left(\frac{1}{R_1^2 k_2 \rho_2 c_{\rho_2}} A(t)A_1(t)\right)^{1/4}, \tag{14}$$

where

$$A(t) \leqslant \left(\sqrt{A_0} + \frac{1}{2}\int_0^t G(\tau) e^{\eta\tau} d\tau\right)^2 e^{-2\eta t}, \tag{15}$$

$$A_1(t) = k_1 \int_0^{R_1} r(a_{1r}^0)^2\, dr + k_2 \int_{R_1}^{R_2} r(\bar{a}_{2r}^0)^2\, dr + \rho_2 c_{p_2} \int_0^t \int_{R_1}^{R_2} r g_2(r,t)\, dr\, dt. \tag{16}$$

Here A_0 is value of function $A(t)$ at $t = 0$ and

$$G(t) = \max_j \left(\frac{2}{\rho_j c_{p_j}}\right)^{1/2} \left(\int_{R_1}^{R_2} r g_2^2\, dr\right)^{1/2}, \tag{17}$$

$$\bar{a}_2(r,t) = a_2(r,t) - \frac{\alpha(t)(r-R_1)^2}{(R_2-R_1)^2}, \tag{18}$$

$$g_2(r,t) = \frac{2\chi_2\alpha(t)}{(R_2-R_1)^2}\left(2 - \frac{R_1}{r}\right) - \frac{\alpha'(r-R_1)^2}{(R_2-R_1)^2}. \tag{19}$$

If the function $\alpha(t)$ and its derivatives $\alpha'(t)$, $\alpha''(t)$, $\alpha'''(t)$ are defined for all $t \geqslant 0$, there is a question about the behavior of the problems solutions (1)–(11) at $t \to \infty$. From the definition of (19) the inequality is valid for the functions $g_2(r,t)$

$$\int_{R_1}^{R_2} r g_2^2 dr \leqslant \frac{2}{(R_2-R_1)^4} \int_{R_1}^{R_2} \left[4\chi_2^2 \left(2 - \frac{R_1}{r}\right)^2 \alpha^2(t) + \right.$$
$$\left. + \ (r-R_1)^4 (\alpha'(t))^2 \right] r dr \leqslant 2R_2(R_2-R_1)(\alpha'(t))^2 + \frac{32\chi_2^2\alpha^2(t)}{(R_2-R_1)^3}$$

(for integrals over r, an upper estimate is given but not their exact value, which can be quite cumbersome), so from (17) we have

$$G(t) \leqslant \left[\max_j \left(\frac{2}{\rho_j c_{\rho_j}}\right)\right]^{1/2} \left[2R_2(R_2-R_1)(\alpha'(t))^2 + \frac{32\chi_2^2\alpha^2(t)}{(R_2-R_1)^3}\right]^{1/2} \leqslant$$
$$\leqslant 2\left[\max_j \left(\frac{1}{\rho_j c_{\rho_j}}\right)\right]^{1/2} \left[\frac{4\chi_2}{(R_2-R_1)^{3/2}}|\alpha(t)| + \sqrt{R_2(R_2-R_1)}|\alpha'(t)|\right]. \tag{20}$$

So from (15) we obtain

$$A(t) \leqslant \left\{ \sqrt{A_0} + \left[\max_j \left(\frac{1}{\rho_j c_{\rho_j}}\right)\right]^{1/2} \left[\frac{4\chi_2}{(R_2-R_1)^{3/2}} \int_0^t |\alpha(\tau)| e^{\eta\tau} d\tau + \right.\right.$$
$$\left.\left. + \sqrt{R_2(R_2-R_1)} \int_0^t |\alpha'(\tau)| e^{\eta\tau} d\tau \right]\right\}^2 e^{-2\eta t}. \tag{21}$$

From (16) and (19) the estimate is valid

$$|A_1(t)| \leqslant k_1 \int_0^{R_1} r(a_{1r}^0)^2 dr + k_2 \int_{R_1}^{R_2} r(\bar{a}_{2r}^0)^2 dr +$$
$$+ \rho_2 c_{\rho_2} R_2 \left[\frac{4\chi_2}{R_2-R_1} \int_0^t |\alpha(\tau)| d\tau + (R_2-R_1) \int_0^t |\alpha'(\tau)| d\tau\right]. \tag{22}$$

We suppose that the following integrals converge

$$\int_0^\infty |\alpha(\tau)| e^{\eta\tau} d\tau, \quad \int_0^\infty |\alpha'(\tau)| e^{\eta\tau} d\tau, \tag{23}$$

then the expression for function modules $|\alpha(\tau)|$ and $|\alpha'(\tau)|$ have the form

$$|\alpha(\tau)| = \alpha_1(t) e^{-\eta\tau}, \quad |\alpha'(\tau)| = \alpha_2(t) e^{-\eta\tau} \tag{24}$$

with non-negative functions $\alpha_1(t)$, $\alpha_2(t)$, at that $\alpha_1(t) \to 0$, $\alpha_2(t) \to 0$ at $t \to \infty$ and the following estimate is valid

$$\int_0^\infty \alpha_k(\tau) d\tau < \infty, \quad k = 1, 2. \tag{25}$$

The convergence of integrals

$$\int_0^\infty |\alpha(\tau)| d\tau, \quad \int_0^\infty |\alpha'(\tau)| d\tau,$$

follows from (24), (25), so from (14), (21), (22) we obtain exponential convergence to zero of the function $a_2(r,t)$ $\forall r \in [R_1, R_2]$:

$$|a_2(r,t)| \leqslant \alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2\rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}, \tag{26}$$

where in the quality D we have designed the value of the expression in curly brackets (21) at $t = \infty$.

For $a_1(r,t)$ from the estimate (13)we find

$$|a_1(r,t)| \leqslant 2\left[\alpha_1(t)e^{-\eta t} + \left(\frac{A_1(\infty)D^2}{R_1^2 k_2\rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \max_{r\in[0,R_1]} |a_1^0(r)| \exp\left(-\frac{\chi_1\xi_1 t}{R_1}\right), \tag{27}$$

where $\xi_1 \approx 2.4048$ is the first roots of equation $J_0(\xi) = 0$ [2]. So there is

Lemma 2.1. *If the functions $\alpha(\tau)$, $\alpha'(\tau)$ satisfy conditions* (23)–(25), *then for the solutions of the initial-boundary value problems* (8)–(11) $a_j(r,t)$ *the following estimates are valid:* (26), (27), *from which it follows that these functions tend exponentially to zero with increasing time.*

The priori estimates for functions $v_j(r,t)$ and $f_j(t)$ have form [1]

$$|v_2(r,t)| \leqslant \frac{2æ}{\mu_2}|a_1(R_1,t)| \max_{r\in[R_1,R_2]} |P_4(r)| + \sqrt{\frac{2}{R_1}}\left(\frac{2}{\rho_2\mu_2}H_2(t)E(t)\right)^{1/4}. \tag{28}$$

$$\begin{aligned} |f_1(t)| \leqslant 2\nu_1 &\left[\left(\frac{1}{7}R_1^4 + \sum_{n=1}^{\infty}|h_n^2|\right) + 2R_1^2\sum_{n=1}^{\infty}\left(\frac{|h_n^1|}{\zeta_n^2} + \frac{|h_n^2|}{R_1^2}\right)\right] \max_{t\in[0,T]}|g(t)| + \\ &+ \frac{R_2^2 - R_1^2}{R_1^2}\left[\frac{2æ}{\mu_2}\max_{t\in[0,T]}|a_{1t}(R_1,t)| \max_{r\in[R_1,R_2]}|P_4(r)| + \right. \\ &\left. + \sqrt{\frac{2}{R_1}}\max_{t\in[0,T]}\left(\frac{2}{\rho_2\mu_2}H_3(t)E_1(t)\right)^{1/4}\right]. \end{aligned} \tag{29}$$

$$|v_1(r,t)| \leqslant R_1 \max_{t\in[0,T]}|v_2(R_1,t)| + \frac{2R_1}{\nu_1}\max_{t\in[0,T]}|f_1(t)|\sum_{n=1}^{\infty}\frac{1}{\xi_n^3|J_1(\xi_n)|}, \tag{30}$$

$$|f_2(t)| \leqslant \rho|f_1(t)| + \frac{2æ}{\rho_2 R_1}\max_{t\in[0,T]}|a_1(\rho_2 R_1,t)|. \tag{31}$$

Here $\rho = \rho_1/\rho_2$, ξ_n are the roots of the Bessel function $J_0(\xi_n)$=0, ζ_n are the positive roots of equation $J_2(\zeta) = 0$ [3], $h_n^1 = \beta_n^1/\zeta_n$ and $h_n^2 = \beta_n^2/\zeta_n$ (where β_n^1, β_n^2 are coefficients of Fourier series of functions $-15R_1 r$ and $3R_1(r^3 - 4R_1 r^2/7)$ when they are decomposed by function $J_2(R_1^{-1}\zeta_n r)$ [1]). Further we have

$$P_4(r) = \frac{1}{R_1^2(R_1 - R_2)}(r^2 - (R_1 + R_2)r + R_1R_2)(r^2 + C_1 r + C_2) \tag{32}$$

with constants

$$C_1 = -\frac{(R_1 + R_2)(2R_1^2 + 2R_2^2 + R_1R_2)}{(R_2 - R_1)(3R_2 + 2R_1)}, \quad C_2 = -R_1C_1 \tag{33}$$

and

$$E(t) \leqslant \left[\sqrt{E(0)} + \int_0^t H_1(\tau)e^{\delta\tau}d\tau\right]^2 e^{-2\delta t}, \tag{34}$$

$$E(0) = \frac{2æ^2\rho_2}{\mu_2^2}(a_1^0(R_1))^2 \int_{R_1}^{R_2} rP_4^2(r)dr. \tag{35}$$

$$H_1(t) = \left[\sqrt{\frac{\rho_2}{2}} \left(\int_{R_1}^{R_2} rQ_2^2 dr\right)^{1/2} + \frac{æ}{\sqrt{\rho_1}}|a_1(R_1,t)|\right], \tag{36}$$

$$Q_2(r,t) = \frac{2æ}{\mu_2}\left[a_{2t}(R_1,t)P_4(r) - \nu_2\left(P_{4rr} + \frac{1}{r}P_{4r}\right)a_2(R_1,t)\right]. \tag{37}$$

$$H_2(t) = \mu_2\int_{R_1}^{R_2} r(\bar{v}_{2r}^0)^2 dr + \frac{\rho_2}{2}\int_0^t\int_{R_1}^{R_2} rQ_2^2(r,t)drdt + \frac{æ^2}{\rho_1}\int_0^t a_1^2(R_1,t)dt, \tag{38}$$

Below, in order to determine the behavior of $v_1(r,t)$ and $f_j(t)$ for large t, we need the estimate $|a_{2t}(r,t)|$. It was obtained in [1], that

$$|a_{2t}(r,t)| \leqslant |\alpha'(t)| + 2\left(\frac{1}{R_1^2 k_2\rho_2 c_{\rho_2}}A_2(t)A_3(t)\right)^{1/4}, \tag{39}$$

where

$$A_2(t) = \frac{\rho_1 c_{\rho_1}}{2}\int_0^{R_1} ra_{1t}^2(r,t)dr + \frac{\rho_2 c_{\rho_2}}{2}\int_{R_1}^{R_2} r\bar{a}_{2t}^2(r,t)dr,$$

$$A_{20} = A_2(0) = \frac{\chi_1^2\rho_1 c_{\rho_1}}{2}\int_0^{R_1} r\left(a_{1rr}^0 + \frac{1}{r}a_{1r}^0\right)^2 dr +$$
$$+\frac{\rho_2 c_{\rho_2}}{2}\int_{R_1}^{R_2} r\left[\chi_2\left(\bar{a}_{2rr}^0 + \frac{1}{r}\bar{a}_{2r}^0\right) + \frac{2\chi_2\alpha(0)}{(R_2-R_1)^2}\left(2 - \frac{R_1}{r}\right) - \frac{\alpha'(0)(r-R_1)^2}{(R_2-R_1)^2}\right]^2 dr,$$

$$\bar{a}_2^0(r) = a_2^0(r) - \frac{\alpha(0)(r-R_1)^2}{(R_2-R_1)^2};$$

$$A_3(t) = k_1\chi_1^2\int_0^{R_1} r\left(a_{1rr}^0 + \frac{1}{r}a_{1r}^0\right)^2 dr +$$
$$+k_2\int_{R_1}^{R_2} r\left[\chi_2\left(a_{2rr}^0 + \frac{1}{r}a_{2r}^0\right) - \frac{\alpha'(0)(r-R_1)^2}{(R_2-R_1)^2}\right]^2 dr \,+\, \rho_2 c_{\rho_2}\int_0^t\int_{R_1}^{R_2} rg_3(r,t)drdt, \tag{40}$$
$$g_3(r,t) = \frac{1}{(R_2-R_1)^2}\left[2\chi_2\alpha'(t)\left(2 - \frac{R_1}{r}\right) - \alpha''(t)(r-R_1)^2\right].$$

Therefore, for $A_2(t)$ we obtain inequality (21) with replacement A_0 by A_{10}, $\alpha(\tau)$ by $\alpha'(\tau)$ and $\alpha'(\tau)$ by $\alpha''(\tau)$. For the function $A_3(t)$ inequality form (22) is satisfied with the replacement

$$\int_0^{R_1} r\left(a_{1r}^0\right)^2 dr \quad \text{by} \quad \chi_1^2\int_0^{R_1} r\left(a_{1rr}^0 + \frac{1}{r}a_{1r}^0\right)^2 dr \equiv d_1,$$
$$\int_{R_1}^{R_2} r\left(\bar{a}_{2r}^0\right)^2 dr \quad \text{by} \quad \int_{R_1}^{R_2} r\left[\chi_2\left(a_{2rr}^0 + \frac{1}{r}a_{2r}^0\right) - \frac{\alpha'(0)(r-R_1)^2}{(R_2-R_1)^2}\right]^2 dr \equiv d_2$$

and $\alpha(\tau)$ by $\alpha'(\tau)$, $\alpha'(\tau)$ by $\alpha''(\tau)$.

In addition to (23)–(25) we assume the convergence of the integral

$$\int_0^\infty |\alpha''(\tau)|e^{\eta\tau}d\tau < \infty, \tag{41}$$

so that there is valid

$$|\alpha''(t)| = \alpha_3(t)e^{-\eta t}, \quad \int_0^\infty \alpha_3(\tau)d\tau < \infty, \quad \alpha_3(t) \to 0 \ \text{ at } \ t \to \infty. \tag{42}$$

Taking into account the above, we find from (14) that

$$|a_{2t}(r,t)| \leqslant \alpha_2(t)e^{-\eta t} + 2\left(\frac{A_3(\infty)D_1^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}, \tag{43}$$

where

$$\begin{gathered} A_3(\infty) = k_1 d_1 + k_2 d_2 + \frac{R_2 \rho_2 c_{p_2}}{R_2 - R_1}\left[4\chi_2 \int_0^\infty |\alpha'(\tau)| + (R_2 - R_1)^2 \int_0^\infty |\alpha''(\tau)| d\tau\right], \\ D_1 = \sqrt{A_{10}} + \max_j \left(\frac{1}{\rho_j c_{p_j}}\right)^{1/2} \left[\frac{4\chi_2}{(R_2 - R_1)^{3/2}} \int_0^\infty |\alpha'(\tau)| e^{\eta\tau} d\tau + \right. \\ \left. + \sqrt{R_2(R_2 - R_1)} \int_0^\infty |\alpha''(\tau)| e^{\eta\tau} d\tau\right]. \end{gathered} \tag{44}$$

We turn to inequality for $|a_{2tt}(r,t)|$ [1]. We have

$$|a_{2tt}(r,t)| \leqslant |\alpha''(t)| + 2\left(\frac{1}{R_1^2 k_2 \rho_2 c_{\rho_2}} A_4(t) A_5(t)\right)^{1/4}, \tag{45}$$

where

$$\begin{gathered} A_4(t) = \frac{\rho_1 c_{p_1}}{2} \int_0^{R_1} r a_{1tt}^2 dr + \frac{\rho_2 c_{p_2}}{2} \int_{R_1}^{R_2} r \bar{a}_{2tt}^2 dr, \\ A_{40} = \frac{\rho_1 c_{p_1}}{2} \int_0^{R_1} r(a_{1tt}^0(r))^2 dr + \frac{\rho_2 c_{p_2}}{2} \int_{R_1}^{R_2} r(\bar{a}_{2tt}^0(r))^2 dr. \end{gathered} \tag{46}$$

The initial data are found from equations (9) and replacement of (18):

$$\begin{gathered} a_{1tt}^0(r) = \chi_1\left[\left(a_{1rr}^0 + \frac{1}{r}\, a_{1r}^0\right)_{rr} + \frac{1}{r}\left(a_{1rr}^0 + \frac{1}{r}\, a_{1r}^0\right)_r\right], \\ \bar{a}_{2tt}^0(r) = \chi_2\left[\left(a_{2rr}^0 + \frac{1}{r}\, a_{2r}^0\right)_{rr} + \frac{1}{r}\left(a_{2rr}^0 + \frac{1}{r}\, a_{2r}^0\right)_r\right] - \frac{\alpha''(0)(r - R_1)^2}{(R_2 - R_1)^2}. \end{gathered} \tag{47}$$

Further we have

$$\begin{gathered} A_5(t) = k_1 \int_0^{R_1} r(a_{1tt}^0)^2 dr + k_2 \int_{R_1}^{R_2} r(\bar{a}_{2tt}^0)^2 dr + \\ + \frac{\rho_2 c_{\rho_2}}{(R_2 - R_1)^2} \int_0^t \int_{R_1}^{R_2} r\left[2\chi_2 \alpha''(\tau)\left(2 - \frac{R_1}{r}\right) - \alpha'''(\tau)\left(r - R_1\right)^2\right] dr. \end{gathered} \tag{48}$$

Similarly to function $A(t)$ the function $A_4(t)$ satisfies an estimate of type (15), and hence (21) with the replacement A_0 by A_{40}, $\alpha(t)$ by $\alpha''(\tau)$ and $\alpha'(\tau)$ by $\alpha'''(\tau)$.

If we require convergence of the integral

$$\int_0^\infty |\alpha'''(\tau)| e^{\eta t} d\tau < \infty, \tag{49}$$

$$|\alpha'''(t)| = \alpha_4(t)e^{-\eta t}, \quad \int_0^\infty \alpha_4(\tau)d\tau < \infty, \tag{50}$$

we obtain an estimate of the function $A_5(t)$ (we use the formula (22)

$$|A_5(t)| \leqslant k_1 \int_0^{R_1} r(a^0_{1tt})^2 dr + k_2 \int_{R_1}^{R_2} r(\bar{a}^0_{2tt})^2 dr + \\ + \rho_2 c_{\rho_2} R_2 \left[\frac{4\chi_2}{R_2 - R_1} \int_0^t |\alpha''(\tau)| d\tau + (R_2 - R_1) \int_0^t |\alpha'''(\tau)| d\tau \right], \tag{51}$$

where $a^0_{jtt}(r)$ are defined by formulas (24). By virtue of (41), (49) $|A_5(t)| \leqslant A_5(\infty)$ and, similarly to estimate (21), we obtain from (45)

$$|a_{2tt}(r,t)| \leqslant \alpha_4(t) e^{-\eta t} + 2 \left(\frac{A_5(\infty) D_2^2}{R_1^2 k_2 \rho_2 c_{\rho_2}} \right)^{1/4} e^{-\eta t/2}, \tag{52}$$

$$D_2 = \sqrt{A_{40}} + \left[\max_j \left(\frac{1}{\rho_j c_{\rho_j}} \right) \right]^{1/2} \left[\frac{4\chi_2}{(R_2 - R_1)^{3/2}} \int_0^\infty |\alpha''(\tau)| e^{\eta\tau} d\tau + \right. \\ \left. + \sqrt{R_2(R_2 - R_1)} \int_0^\infty |\alpha'''(\tau)| e^{\eta\tau} d\tau \right].$$

We proceed to elaboration the estimates of the functions $v_j(r,t)$, $f_j(t)$, when $\alpha(\tau)$, $\alpha'(\tau)$, $\alpha''(\tau)$ and $\alpha'''(\tau)$ satisfy conditions (23)–(25), (41), (42). In this case everywhere we replace $a_1(R_1,t)$, $a_{1t}(R_1,t)$ by $a_2(R_1,t)$, $a_{2t}(R_1,t)$ according to the first equation (11). We begin with the function $v_2(r,t)$, for which inequality (28) is proved. The quantity $E(t)$ entering the right-hand side of this inequality has estimate (34), where $H_1(t)$ is given by (36) than from (37) we obtain

$$\int_{R_1}^{R_2} r Q_2^2(r,t) dr \leqslant \frac{8æ^2}{\mu_2^2} \left[a_{2t}^2(R_1,t) \int_{R_1}^{R_2} r P_4^2(r) dr + \nu_1^2 a_2^2(R_1,t) \int_{R_1}^{R_2} r \left(P_{4rr} + \frac{1}{r} P_{4r} \right)^2 dr \right] \equiv \\ \equiv d_3 a_2^2(R_1,t) + d_4 a_{2t}^2(R_1,t). \tag{53}$$

So the inequality is valid

$$H_1(t) \leqslant \frac{æ}{\sqrt{\rho_1}} |a_2(R_1,t)| + \sqrt{\frac{\rho_2}{2}} \left(\sqrt{d_3} |a_2(R_1,t)| + \sqrt{d_4} |a_{2t}(R_1,t)| \right) = \\ = \left(\frac{æ}{\sqrt{\rho_1}} + \sqrt{\frac{\rho_2 d_3}{2}} \right) |a_2(R_1,t)| + \sqrt{\frac{\rho_2 d_4}{2}} |a_{2t}(R_1,t)|$$

and estimate (34) takes the form

$$E(t) \leqslant \left[\sqrt{E(0)} + \left(\frac{æ}{\sqrt{\rho_1}} + \sqrt{\frac{\rho_2 d_3}{2}} \right) \int_0^t |a_2(R_1,t)| e^{\delta\tau} d\tau + \right. \\ \left. + \sqrt{\frac{\rho_2 d_4}{2}} \int_0^t |a_{2t}(R_1,t)| e^{\delta\tau} d\tau \right]^2 e^{-2\delta t}. \tag{54}$$

According to estimates (26), (43) the integrals in (54) have the order $e^{(\delta-\eta)t}$ and $e^{(\delta-\eta/2)t}$ for large t, therefore we obtain

$$E(t) \leqslant \gamma(t), \quad \text{where} \quad \gamma(t) \equiv d_5 \begin{cases} e^{-2\delta t}, & \delta < \eta/2, \\ t e^{-2\delta t}, & \delta = \eta/2, \\ e^{-\eta t}, & \delta > \eta/2, \end{cases} \tag{55}$$

with positive constant d_5.

Defined by equality (38) with using (53) the function $H_2(t)$ is evaluated as follows:

$$H_2(t) \leqslant \mu_2 \int_{R_1}^{R_2} r(\bar{v}_{2r}^0)^2 dr + \left(\frac{\rho_2 d_3}{2} + \frac{æ^2}{\rho_1}\right) \int_0^t a_2^2(R_1,\tau)d\tau +$$
$$+ \frac{\rho_2 d_4}{2} \int_0^t a_{2\tau}^2(R_1,\tau)d\tau \leqslant D_2 = \text{const} > 0$$

by virtue of inequalities (26), (43).

So from (13), (54), (55) we find estimate

$$|v_2(r,t)| \leqslant \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \\ + \sqrt{2}\left(\frac{2d_5}{R_1^2 \nu_2} D_2 \gamma(t)\right)^{1/4} \tag{56}$$

and $v_2(r,t)$ approaches to zero uniformly over $r \in [R_1, R_2]$ with increasing time t.

Below we need the values $f_j(0)$. From (7) we obtain the connection between them

$$\rho_1 f_1(0) = \rho_2 f_2(0) - \frac{2æ}{R_1} a_1^0(R_1).$$

The other relation follows from the second equality (3) and equation (5) (we recall that $v_j(r,0) = 0$):

$$f_1(0) = -\frac{R_2^2 - R_1^2}{R_1^2} f_2(0).$$

Now we find

$$f_1(0) = \frac{2æ(R_2^2 - R_1^2)a_1^0(R_1)}{R_1^2 + \rho(R_2^2 - R_1^2)}, \quad f_2(0) = \frac{R_1^2}{R_2^2 - R_1^2} f_1(0). \tag{57}$$

Moreover the relations are valid

$$v_{1t}(r,0) = f_1(0), \quad \bar{v}_{2t}(r,0) = f_2(0) + \frac{2æ\chi_1}{\mu_2}\left(a_{1rrr}^0 + \frac{1}{r} a_{1rr}^0\right) P_4(r). \tag{58}$$

The second initial condition follows from the equations

$$|a_1(R_1,t)| = |a_2(R_1,t)| \leqslant |\alpha(t)| + 2\left(\frac{1}{R_1^2 k_2 \rho_2 c_{\rho_2}} A(t) A_1(t)\right)^{1/4}, \tag{59}$$

and (37) and replacement

$$v_2(r,t) = \bar{v}_2(r,t) - \frac{2æ a_1(R_1,t)}{\mu_2} P_4(r). \tag{60}$$

We consider the following inequality that was obtained in [1]

$$|v_{2t}(r,t)| \leqslant \frac{2æ}{\mu_2} |a_{1t}(R_1,t)| \max_{r\in[R_1,R_2]} |P_4(r)| + \sqrt{\frac{2}{R_1}} \left(\frac{2}{\rho_2 \mu_2} H_3(t) E_1(t)\right)^{1/4}. \tag{61}$$

The function $E_1(t)$ on the right-hand side of inequality (61) has the form

$$E_1(t) = \frac{\rho_1}{2} \int_0^{R_1} r v_{1t}^2 dr + \frac{\rho_2}{2} \int_{R_1}^{R_2} r \bar{v}_{2t}^2 dr,$$

$$E_1(0) = \frac{\rho_1 R_1^2}{4} f_1^2(0) + \frac{\rho_2}{2} \int_{R_1}^{R_2} r\bar{v}_{2t}^2(r,0)dr,$$

where $f_1(0)$ and $\bar{v}_{2t}(r,0)$ are defined by the first (57) and the second (58) equality respectively. There is the estimate form (54) for $E_1(t)$.

$$E_1(t) \leqslant \left[\sqrt{E_1(0)} + \left(\frac{æ}{\sqrt{\rho_1}} + \sqrt{\frac{\rho_2 d_3}{2}}\right) \int_0^t |a_{2t}(R_1,\tau)|e^{\delta\tau}d\tau + \right.$$
$$\left. + \sqrt{\frac{\rho_2 d_4}{2}} \int_0^t |a_{2tt}(R_1,\tau)|e^{\delta\tau}d\tau\right]^2 e^{-2\delta t}. \qquad (62)$$

Taking into account the obtained estimates (43), (51) from (53) we find using the constant d_6 the inequality

$$E_1(t) \leqslant d_6\gamma(t) \qquad (63)$$

and the function $\gamma(t)$ from inequality (55).

For the function $H_3(t)$, from the right-hand side of inequality (61) we have the expression

$$H_3(t) = \mu_1 \int_0^{R_1} r(v_{1tr}^0)^2 dr + \mu_2 \int_{R_1}^{R_2} r(\bar{v}_{2tr}^0)^2 dr +$$
$$+ \frac{\rho_2}{2} \int_0^t \int_{R_1}^{R_2} rQ_3^2(r,\tau)drd\tau + \frac{æ^2}{\rho_1} \int_0^t a_2^2(R_1,\tau)d\tau, \qquad (64)$$

where in our case

$$Q_3(r,t) = \frac{2æ}{\mu_2}\left[-\nu_2 a_{2t}(R_1,t)\left(P_{4rr} + \frac{1}{r}P_{4r}\right) + a_{2tt}(R_1,t)P_4(r)\right],$$
$$v_{1tr}^0(r) = 0, \quad \bar{v}_{2tr}^0 = \frac{2æ}{\mu_2} a_{2t}(R_1,0)P_{4r},$$
$$a_{2t}(R_1,0) = \chi_2\left[a_{2rr}^0(R_1) + \frac{1}{R_1}a_{2r}^0(R_1)\right].$$

It is clear that

$$\int_{R_1}^{R_2} rQ_3^2(r,t)dr \leqslant d_3 a_{2t}^2(R_1,t) + d_4 a_{2tt}^2(R_1,t)$$

with constant d_3, d_4 from (52). By virtue of the convergence of the integrals

$$\int_0^\infty (a_2^{(k)}(\tau))^2 d\tau, \quad k = 0,1,2$$

we obtain the inequality $H_3(t) \leqslant H_3(\infty)$ and estimate (61) takes the form for all $r \in [R_1, R_2]$

$$|v_{2t}(r,t)| \leqslant \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_2(t)e^{-\eta t} + 2\left(\frac{A_3(\infty)D_1^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] +$$
$$+ \sqrt{2}\left(\frac{2d_6}{R_1^2\nu_2} H_3(\infty)\gamma(t)\right)^{1/4}. \qquad (65)$$

The function $f_1(t)$ is the pressure gradient in the first fluid along the axis z. The function $g(t)$ on the right side of the inequality (29) has form

$$g(t) = R_1^2 v_2(R_1,t) + 2\int_{R_1}^{R_2} rv_2(r,t)dr$$

and, taking into account estimate (56), we find

$$|g(t)| \leqslant R_2^2 \left\{ \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \right.$$
$$\left. + \sqrt{2}\left(\frac{2d_5}{R_1^2\nu_2} H_2(\infty)\gamma(t)\right)^{1/4} \right\} \leqslant d_7 e^{-\omega t}, \qquad (66)$$

where $\omega = \min(\delta/2, \eta/4)$ (at $\delta = \eta/2$ in (66) there is $te^{-\omega t}$ instead of $e^{-\omega t}$ according to (54)).

Now from (29) using inequalities (65), (66) we obtain the estimate

$$|f_1(t)| \leqslant 2\nu_1 \left[S_1 d_7 e^{-\omega t} + S_2 d_7 \left| \exp\left(-\frac{\zeta_1^2 \nu_1}{R_1^2} t\right) - e^{-\omega t}\right| \right] + d_8 e^{-\omega t}, \qquad (67)$$

$$S_1 = \frac{1}{7} R_1^4 + \sum_{n=1}^{\infty} |h_n^2|, \quad S_2 = \nu_1 \sum_{n=1}^{\infty} \frac{1}{\nu_1 R_1^{-2}\zeta_n^2 - \omega}\left(|h_n^1| + \frac{\zeta_n^2}{R_1^2}|h_n^2|\right),$$

at that $S_1 < \infty$ and $S_2 < \infty$. The estimate $f_2(t)$ follows from (5), inequalities (26) and (67)

$$|f_2(t)| \leqslant \rho |f_1(t)| + 2æ \left[\alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right]. \qquad (68)$$

Remark 1. *From inequality* (30), *estimates* (56) *and* (67) *it follows that the function* $v_1(r,t)$ *tends exponentially to zero with increasing time.*

$$|v_1(r,t)| \leqslant R_1 \max_{t\in[0,T]} \left| \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2\rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \right.$$
$$\left. + \sqrt{2}\left(\frac{2d_5}{R_1^2\nu_2} D_2\gamma(t)\right)^{1/4} \right| + \frac{2R_1}{\nu_1} \max_{t\in[0,T]} \left| 2\nu_1 \left[S_1 d_7 e^{-\omega t} + \right.\right. \qquad (69)$$
$$\left.\left. + S_2 d_7 \left| \exp\left(-\frac{\zeta_1^2\nu_1}{R_1^2}t\right) - e^{-\omega t}\right| \right] + d_8 e^{-\omega t} \right| \sum_{n=1}^{\infty} \frac{1}{\xi_n^3 |J_1(\xi_n)|}.$$

For the function $h_1(t)$ from (12), taking into account the first relation (3) and the inequality (56) we have the estimate

$$|h_1(t)| \leqslant \frac{R_2^2 - R_1^2}{2R_1} \left\{ \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\int_0^t \alpha_1(\tau) e^{-\eta\tau} d\tau + \right.\right.$$
$$\left.\left. + \frac{4}{\eta}\left(\frac{A_1(\infty)D^2}{R_1^2 k_2\rho_2 c_{\rho_2}}\right)^{1/4} \left(1 - e^{-\eta t/2}\right)\right] + \sqrt{2}\left(\frac{2d_5}{R_1^2\nu_2} H_2(\infty)\right)^{1/4} \int_0^t \gamma^{1/4}(\tau)d\tau \right\} \qquad (70)$$

and $h_1(t)$ is limited at $t \to \infty$.

Thus, it is proofed

Theorem 2.1. *If the function* $\alpha(\tau)$, $\alpha'(\tau)$, $\alpha''(\tau)$, $\alpha'''(\tau)$ *satisfy conditions* (23)–(25), (41), (42), (49), *then the following estimates* (26), (27), (56), (67), (68), (69) *are valid for the functions* $a_j(r,t)$, $v_j(r,t)$, $f_j(t)$ *from which it follows that these functions tend exponentially to zero with increasing time.*

Remark 2. *Remark 6. Conditions* (23)–(25), (41), (42), (49) *physically mean that the thermal effects on the solid wall surface of cylinder at* $r = R_2$ *are very small and the braking of liquids occurs at* $t \to \infty$ *due to frictional forces.*

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Об асимптотическом поведении сопряженной задачи, описывающей ползущее осесимметричное термокапиллярное движение

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Аннотация. В работе указаны условия для закона поведения температуры на твердой стенке цилиндра, при которых решение линейной сопряженной обратной начально-краевой задачи, описывающей двухслойное осесимметрическое ползущее движение вязких теплопроводных жидкостей, с ростом времени экспоненциально стремится к нулю.

Ключевые слова: сопряженная нелинейная обратная задача, поверхность раздела, ползущее движение.