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Inverse Problem for Source Function in Parabolic Equation at Neumann Boundary Conditions

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Abstract. The second initial-boundary value problem for a parabolic equation is under study. The term in the source function, depending only on time, is to be unknown. It is shown that in contrast to the standard Neumann problem, for the inverse problem with integral overdetermination condition the convergence of it nonstationary solution to the corresponding stationary one is possible for natural restrictions on the input problem data.

Keywords: parabolic equation, inverse problem, source function, a priori estimate, nonlocal overdetermination condition.

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1. Introduction and preliminaries

We consider the parabolic equation with second-type boundary conditions

$$u_t = \nu u_{xx} + f(t) + g(x,t) \qquad \text{in} \quad Q_T = (0,l)\times[0,T]; \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in (0,l); \tag{2}$$

$$-u_x(0,t) = q_1(t), \quad u_x(l,t) = q_2(t), \tag{3}$$

$$\int_0^l u(x,t)dx = q_3(t), \quad t\in[0,T]. \tag{4}$$

In (1)–(4) the functions $g(x,t)$, $u_0(x)$, $q_i(t), i = 1,2,3$, and positive constants ν, T, l are assumed to be given. The problem of finding a pair $u(x,t)$ and $f(t)$ is called inverse one.

Definition. The pair $f(t)$ and $u(x,t)$ from the class $C[0,T]\times C^{2,1}(Q_T)\cap C^{1,0}(\bar{Q}_T)$, for which equation (1) and conditions (2)–(4) are satisfied, is called a classical solution of the posed inverse problem.

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It is clear that for existence of smooth solution the consistency conditions should be fulfilled. They are the following

$$-u_{0x}(0) = q_1(0), \quad u_{0x}(l) = q_2(0), \quad \int_0^l u_0(x)dx = q_3(0).$$

From the physical point of view the posed problem (1)–(4) allows to describe a motion of viscous fluid in the flat layer with two free boundaries. The function $u(x,t)$ is the velocity in this case, $f(t)$ is unknown pressure gradient, condition (4) means flow rate through the section of layer. The solution of the inverse problem in this case gives an answer on a question: what is pressure gradient needed for providing the given flow rate?

It is necessary to mention that there are many known results concerning to the inverse problems close to the posed problem. Among of them it can be distinguish the coefficient inverse problems (see, e. g. [1–3]), problems with unknown source function [4–6] and problems, where unknown function in the boundary condition occurs [7]. The authors, dealing with finding the source function, usually assume, that this function is included into the equation by multiplicative way (see, for example, [4]).

The overdetermination conditions can be nonlocal integral ones [1, 8, 9]. One-point and two-point overdetermination conditions are considered in [5, 10]. As a rule, in the cited papers and books the existence and uniqueness of solution are proved, some asymptotic methods of solution construction are described. It is common situation when existence and uniqueness of solution are proved in the Sobolev's spaces. Usually, the same questions are considered in the uniform metric for 1-dimensional problems only. Concerning to different kinds of inverse problems and qualitative properties of their solutions we should also mention the monographs authored by Prilepko *et al* [11], Alifanov [12] and Belov [13].

1.1. Some remarks on corresponding direct problem

If the function $f = 0$ in equation (1), and condition (4) is not taken into account, then we deal with standard Neumann problem for the function $u(x,t)$. It is well known that the direct initial boundary problem

$$u_t = \nu\Delta u + g(x,t), \quad x \in \Omega \subset \mathbb{R}^n, \quad t \in [0,T]; \tag{5}$$

$$u(x,0) = u_0(x), \quad x \in \Omega; \qquad \frac{\partial u}{\partial n} = \varphi(x), \quad x \in \partial\Omega, \quad t \in [0,T] \tag{6}$$

has unique solution if the functions $g(x,t)$, $u_0(x)$ and $\varphi(x)$ are smooth ones. The corresponding stationary problem

$$\nu\Delta u^s = -g^s(x), \quad x \in \Omega \subset \mathbb{R}^n, \qquad \frac{\partial u}{\partial n} = \varphi^s(x), \quad x \in \partial\Omega$$

has a countable number of solutions $u^s(x) + const$ if and only if the following condition

$$\frac{1}{\nu}\int_\Omega g^s d\Omega + \int_{\partial\Omega} \varphi^s d\Gamma = 0$$

is fulfilled. For the separation of unique solution it is necessary to give additional functional of $u^s(x)$. For example, it could be $u^s(x_0)$, where $x_0 \in \partial\Omega$.

It should be noted that if $g(x,t) \to g^s(x)$ and $\varphi(x,t) \to \varphi^s(x)$ in the uniform metric at $t \to \infty$ for all $x \in \bar{\Omega}$, then it is not difficult to prove that nonstationary solution $u(x,t)$ does not tend to $u^s(x)$ at $t \to \infty$. We confirm this fact using an example. For the problem in the space $\mathbb{R}^1$ we consider equation (5) with conditions $u_0(x,0) = x,\ u_x(0,t) = u_x(l,t) = 1$ and the right hand side in the form

$$g(x,t) = \frac{\cos\ln M}{M}, \qquad M = 1 + \frac{\nu}{l^2}t.$$

It has the solution

$$u(x,t) = x + \frac{l^2}{\nu}\sin\ln M,$$

which has no a limit at $t \to \infty$ while the corresponding stationary solution is $u^s(x) = x + const$ at $\varphi_1^s = \varphi_2^s = 1$ and $g^s(x) = 0$.

We should also mention that for the Dirichlet and Robin problems for multidimensional linear parabolic equation (5) the sufficient convergence conditions of solution of nonstationary problem to corresponding stationary one are described in [14]. According to the example above, for the Neumann problem there is no such convergence. However, it turn out well to show the convergence of nonstationary solution to the corresponding stationary one for the posed inverse problem (1)–(4).

Below, using the specific features of the considered problem and their 1-dimensionality, we derive sufficient conditions for the initial data for which the nonstationary solution tends to stationary one at $t \to \infty$ in the uniform metric.

2. Analysis of the inverse problem (1)–(4)

Integrating equation (1) by x from 0 till l and using condition (4), the function $f(t)$ can be found in the form

$$f(t) = \frac{1}{l}\left[\frac{\partial q_3}{\partial t} - \nu\left(q_1(t) + q_2(t)\right) - \int_0^l g(x,t)dx\right]. \tag{7}$$

Substitution expression (7) into equation (1) leads to direct problem for the function $u(x,t)$ with conditions (2) and (3). The solution of the obtained problem can be constructed as follows [15]

$$u(x,t) = \int_0^l u_0(y)G(x,y,t)dy + \int_0^t\int_0^l F(y,\tau)G(x,y,t-\tau)\,dy d\tau +$$
$$+\,\nu\int_0^t q_1(\tau)G(x,0,t-\tau)\,d\tau + \nu\int_0^t q_2(\tau)G(x,l,t-\tau)\,d\tau,$$

where G is the Green's function:

$$G(x,y,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi y}{l}\right)\exp\left(-\frac{\nu n^2\pi^2 t}{l^2}\right), \quad F(x,t) = f(t) + g(x,t).$$

The examples of construction of a priori estimate for the functions presented as series with the Green's functions are given in [16,17]. As it can be observed in those works, the deriving the estimate of the function $|u(x,t)|$, $x \in [0,l]$, $t \in [0,T]$ from this expression is a cumbersome task. We suggest another way and reduce problem (1)–(4) to the axillary problem with the first-type boundary conditions.

Differentiating equation (1) with respect to variable x, we obtain the initial boundary value problem for the function $w(x,t) = u_x(x,t)$:

$$\begin{gathered} w_t = \nu w_{xx} + g_x(x,t), \quad x \in (0,l), \quad t \in [0,T]; \\ w(x,0) = w_0(x) = u_{0x}(x), \quad x \in [0,l]; \\ w(0,t) = -q_1(t), \quad w(l,t) = q_2(t), \quad t \in [0,T]. \end{gathered} \tag{8}$$

The corresponding stationary problem has the solution

$$w^s(x) = -q_1^s + \frac{x}{l}\left[q_1^s + q_2^s + \frac{1}{\nu}\int_0^l g^s(y)dy\right] - \frac{1}{\nu}\int_0^x g^s(y)dy, \tag{9}$$

where q_1^s, q_2^s, $g^s(x)$ are given constants and function respectively.

Let the functions $q_{1,2}(t)$ be known for all $t \geqslant 0$ and the following inequalities be fulfilled

$$|q_j(t)| \leqslant N_j(1+\tau)^{-\alpha}, \quad j = 1,2, \quad |g(x,t)| \leqslant N_3(1+\tau)^{-\alpha}, \quad |g_x(x,t)| \leqslant N_4(1+\tau)^{-\alpha} \tag{10}$$

with some positive constants $N_1, \ldots, N_4$ and α for all $x \in [0,l]$, where $\tau = \nu t/l^2$ is the dimensionless time here and below. Then with respect to the results from [14] it can be concluded that the function $w(x,t)$ can be restricted as

$$|w(x,t| \leqslant N_5(1+\tau)^{-\alpha} \tag{11}$$

with constant $N_5 > 0$ at $x \in [0,l]$.

In order to find the stationary solution $u^s(x)$, expression (9) should be integrated

$$u^s(x) = -q_1^s x - \frac{1}{\nu}\left(f^s\frac{x^2}{2} + \int_0^x (x-y)g^s(y)dy\right) + C. \tag{12}$$

Here

$$f^s = -\frac{\nu}{l}\left(q_1^s + q_2^s + \frac{1}{\nu}\int_0^l g^s(y)dy\right), \tag{13}$$

$$C = \frac{q_3^s}{l} + \frac{q_1^s l}{3} - \frac{l q_2^s}{6} + \frac{1}{\nu l}\left(\int_0^l \int_0^x (x-y)g^s(y)dydx - l^2\int_0^l g^s(y)dy\right).$$

After that the stationary solution of the posed inverse problem is constructed, we can start obtaining a priori estimates of the corresponding nonstationary solution.

2.1. A priori estimates of the solution of problem (1)–(4)

It should be noted that if the following conditions are fulfilled

$$\frac{\partial q_3}{\partial t} \to 0, \quad q_j(t) \to q_j^s, \quad g(x,t) \to g^s(x)$$

at $t \to \infty$ and $x \in [0,l]$, then it can be concluded that

$$f(t) \to f^s \quad \text{at} \quad t \to \infty$$

is valid. The function $f(t)$ is from formula (7), and f^s is from (13).

According to the integral mean-value theorem there is the point $x_0 \in (0, l)$ such that $u(x_0, t) = l^{-1} q_3(t)$ (see (4)). That is why for every $x \in [0, l]$, $t \geqslant 0$ it follows that

$$u(x,t) = u(x_0,t) + \int_{x_0}^{x} u_y(y,t)dy = l^{-1}q_3(t) + \int_{x_0}^{x} w(y,t)dy.$$

Using estimate (11), it can be obtained that

$$|u(x,t)| \leqslant l^{-1}|q_3(t)| + \int_0^l |w(y,t|dy \leqslant l^{-1}|q_3(t)| + N_5 l(1+\tau)^{-\alpha}. \tag{14}$$

Let the following inequalities should be fulfilled

$$\begin{gathered} |q_j(t) - q_j^s| \leqslant D_j(1+\tau)^{-\alpha}, \quad j = 1,2,3; \\ \left|\frac{\partial q_3}{\partial t}\right| \to 0,\ t \to \infty, \quad |g(x,t) - g^s(x)| \leqslant D_4(1+\tau)^{-\alpha}, \\ |g_x(x,t) - g_x^s(x)| \leqslant D_5(1+\tau)^{-\alpha} \end{gathered} \tag{15}$$

with constants $D_i > 0\,(i = 1, \ldots, 5)$, $\alpha > 0$ for every $x \in [0, l]$. Then the following estimates can be provided

$$\begin{gathered} |u(x,t) - u^s(x)| \leqslant D_6(1+\tau)^{-\alpha}, \\ |u_x(x,t) - u_x^s| \leqslant D_7(1+\tau)^{-\alpha}, \\ |f(t) - f^s| \leqslant D_8(1+\tau)^{-\alpha}, \end{gathered} \tag{16}$$

where D_6, D_7, D_8 are positive constants.

For the deriving the estimates in (16) it needs to make a change $\tilde{u}(x,t) = u(x,t) - u^s(x)$, $\tilde{w}(x,t) = w(x,t) - w^s(x)$ and $\tilde{f}(t) = f - f^s$ in equation (1) and condition (4). The boundary conditions should be rewritten as $\tilde{q}_j(t) = q_j(t) - q_j^s,\ j = 1,2,3$, and $\tilde{g}(x,t) = g(x,t) - g^s(x)$ in this case. Applying estimates (14) and (11), formulas (7) and (13), using assumptions (15) we derive estimates (16). It concludes that the solution of inverse problem (1)–(4) converges to the corresponding stationary solution (12), (13) in the class $C[0,\infty] \times C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$.

Conclusion

For the conclusion some remarks can be made. The first one is following. If the right hand sides of inequalities (15) are restricted by exponent function ($\exp(-\alpha\tau)$, $\alpha > 0$), then the solution of problem (1)–(4) tends to stationary regime $u^s(x)$, f^s (see (12), (13)) with respect to exponent law at $t \to \infty$. The second remark is concerned to question of stabilization. The results obtained can be interpreted as stability of stationary solution (12), (13) if conditions (15) are fulfilled.

The authors were surprised at research of some aspects of solution stabilization in problems on binary mixtures motion that the question on solutions solvability and stability in the problems close to (1)–(4) was not described anywhere in literature. And we were glad to fill this gap in the investigation of such problems.

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Обратная задача определения функции источника для параболического уравнения с краевыми условиями Неймана

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Аннотация. В работе изучается вторая начально-краевая задача для параболического уравнения, когда часть функции источника, зависящая только от времени, неизвестна. Показано, что в отличие от классической задачи Неймана для обратной задачи с интегральным условием переопределения возможна сходимость ее нестационарного решения к соответствующему стационарному при естественных ограничениях на входные данные.

Ключевые слова: параболическое уравнение, обратная задача, функция источника, априорная оценка, нелокальное условие переопределения.