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Two-layer Stationary Flow in a Cylindrical Capillary Taking into Account Changes in the Internal Energy of the Interface

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Abstract. The problem of two-dimensional stationary flow of two immiscible incompressible binary mixtures in a cylindrical capillary in the absence of mass forces is investigated. The mixtures are contacted through a common the interface on which the total energy condition is taken into account. The temperature and concentration in the mixtures are distributed according to a quadratic law, which is in good agreement with the velocity field of the type Hiemenz. The resulting conjugate boundary value problem is nonlinear and inverse with respect to the pressure gradients along the axis of the cylindrical capillary. The tau-method (a modification of the Galerkin method) was applied to this problem, which showed the possibility of the existence of two solutions. It is shown that the obtained solutions with a decrease in the Marangoni number converge to the solutions of the problem of the creeping flow of binary mixtures. When solving the model problem for small Marangoni numbers, it is found that the effect of the increments of the internal energy of the interfacial surface significantly affects the dynamics of flows of mixtures in layers.

Keywords: binary mixture, interface, internal energy, inverse problem, pressure gradient, thermal Marangoni number.

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Introduction

The specifics of the phenomena occurring at the interface of liquids are related to the existence of the energy and entropy of the surface phase, which are excessive in relation to the bulk phases in the transition layer [1]. However, the energy exchange between the bulk and surface phases has not been sufficiently studied. For ordinary liquids at room temperature, the effect of changes in the internal energy of the interfacial surface on the formation of heat fluxes, temperature fields, and velocities in its vicinity is insignificant in relation to viscous friction and heat transfer . However, at sufficiently high temperatures, when the viscosity and thermal conductivity of ordinary liquids are significantly reduced, as well as for liquids with reduced viscosity (for example, for some cryogenic liquids), the effect of the internal energy increments of the interfacial surface is significant [3].

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In this paper, we consider a mathematical model describing the two-dimensional stationary thermodiffusion motion of two immiscible incompressible binary mixtures in a cylindrical capillary in the absence of mass forces. The mixtures are contacted through a common interface on which the total energy condition is taken into account. In this geometry, the mechanism of influence of changes in the surface internal energy on the dynamics of binary mixtures is investigated. Without taking into account the effects of thermal diffusion, such a model was studied in the works [4, 5].

1. Statement of the problem

We consider a two-dimensional stationary axisymmetric flow of two immiscible incompressible binary mixtures in a cylindrical tube of radius R_2, the temperature of which is maintained constant. Binary mixture occupy the field: $\Omega_1 = \{0 \leqslant r \leqslant R_1, |z| < \infty\}$ and $\Omega_2 = \{R_1 \leqslant r \leqslant R_2, |z| < \infty\}$, where r, z are the radial and axial cylindrical coordinates. Here $r = R_1 = \text{const}$ is the total interface of binary mixtures, $r = R_2 = \text{const}$ is the solid wall. The values related to the regions Ω_1 and Ω_2 are denoted by indexes 1 and 2, respectively. The area of Ω_1 is called the core, and the area Ω_2 is an interlayer or film. It is assumed that its characteristic transverse size is small by compared to the radius of the core, $R_2 - R_1 \ll R_1$. Such a geometry corresponds, for example, to the case of displacement of the liquid that originally filled the capillary by another liquid.

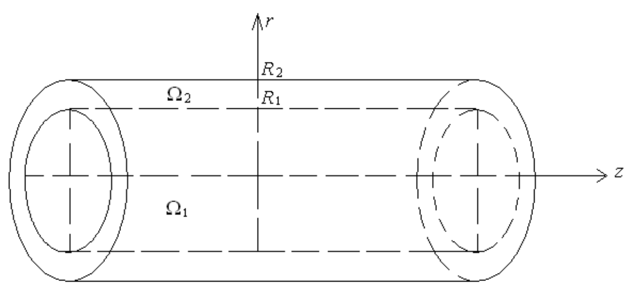

Fig. 1. The scheme of the flow region

Binary mixture is characterized by constant thermal conductivities k_j, specific heat capacities c_{pj}, dynamic viscosities μ_j, densities ρ_j; let $\chi_j = k_j/\rho_j c_{pj}$ is the thermal conductivity, $\nu_j = \mu_j/\rho_j$ is the kinematic viscosity (here and further, $j = 1, 2$). The influence of gravity is not taken into account, which may be justified, for example, if the tube it is quite narrow to the capillaries.

The system of equations of motion, continuity, internal energy balance and concentration transfer has the following form [6]:

$$
\begin{gathered}
u_j u_{jr} + w_j u_{jz} + \frac{1}{\rho_j} p_{jr} = \nu_j \left(\Delta u_j - \frac{u_j}{r^2} \right), \\
u_j w_{jr} + w_j w_{jz} + \frac{1}{\rho_j} p_{jz} = \nu_j \Delta w_j, \\
u_{jr} + \frac{u_j}{r} + w_{jz} = 0, \\
u_j \theta_{jr} + w_j \theta_{jz} = \chi_j \Delta \theta_j, \\
u_j c_{jr} + w_j c_{jz} = d_j \Delta c_j + \alpha_j d_j \Delta \theta_j,
\end{gathered}
\tag{1}
$$

where u_j, w_j are projections of the velocity vector on the r, z axis of the cylindrical coordinate system; p_j is the pressure in the layers; θ_j, c_j are deviations of temperature and concentration from their equilibrium values; d_j, α_j are the diffusion and thermal diffusion coefficients, respectively; $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2$ is the Laplace operator.

The linear dependence of the interfacial tension coefficient on temperature and concentration is assumed:

$$\sigma(\theta, c) = \sigma_0 - æ_1(\theta - \theta_0) - æ_2(c - c_0). \tag{2}$$

Here $æ_1 > 0$ is the temperature coefficient, $æ_2$ is the concentration coefficient of the surface tension (normally $æ_2 < 0$, because the surface tension increases with increasing concentration); θ_0, c_0 are the temperature and concentration on the interfacial surface in the as balance.

The solution to the problem is sought in a special form:

$$\begin{gathered} u_j = u_j(r), \quad w_j = z v_j(r), \quad p_j = p_j(r, z), \\ \theta_j = a_j(r) z^2 + b_j(r), \quad c_j = h_j(r) z^2 + g_j(r). \end{gathered} \tag{3}$$

A solution of the form (3) is called a solution of the type Hiemenz [7], in which the velocity field is linear with respect to the transverse coordinate. Thus, the temperature θ_j takes an extreme value at the point $z = 0$: the maximum at $a_j(r) < 0$ and the minimum at $a_j(r) > 0$. We get a similar interpretation for the concentration c_j, only instead of $a_j(r)$ the function $h_j(r)$ is considered.

After substituting the special form (3) into the equations of motion (1) we will have the following system with unknown functions $u_j(r)$, $v_j(r)$, $p_j(r)$, $a_j(r)$, $b_j(r)$, $h_j(r)$, $g_j(r)$:

$$u_j u_{jr} + \frac{1}{\rho_j} p_{jr} = \nu_j \Big(u_{jrr} + \frac{1}{r} u_{jr} - \frac{u_j}{r^2} \Big), \tag{4}$$

$$z(u_j v_{jr} + v_j^2) + \frac{1}{\rho_j} p_{jz} = \nu_j z \Big(v_{jrr} + \frac{1}{r} v_{jr} \Big), \tag{5}$$

$$u_{jr} + \frac{u_j}{r} + v_j = 0, \tag{6}$$

$$u_j a_{jr} + 2 v_j a_j = \chi_j \Big(a_{jrr} + \frac{1}{r} a_{jr} \Big), \tag{7}$$

$$u_j b_{jr} = \chi_j \Big(b_{jrr} + \frac{1}{r} b_{jr} + 2 a_j \Big), \tag{8}$$

$$u_j h_{jr} + 2 v_j h_j = d_j \Big(h_{jrr} + \frac{1}{r} h_{jr} \Big) + \alpha_j d_j \Big(a_{jrr} + \frac{1}{r} a_{jr} \Big), \tag{9}$$

$$u_j g_{jr} = d_j \Big(g_{jrr} + \frac{1}{r} g_{jr} + 2 h_j \Big) + \alpha_j \, d_j \Big(b_{jrr} + \frac{1}{r} b_{jr} + 2 a_j \Big). \tag{10}$$

From the equations (4), (5), we express the pressure gradients (p_{jr}, p_{jz}):

$$p_{jr} = \rho_j \nu_j \Big(u_{jrr} + \frac{1}{r} u_{jr} - \frac{u_j}{r^2} \Big) - \rho_j \, u_j \, u_{jr}, \tag{11}$$

$$p_{jz} = z \Big[\rho_j \nu_j \Big(v_{jrr} + \frac{1}{r} v_{jr} \Big) - \rho_j (u_j \, v_{jr} + v_j^2) \Big], \tag{12}$$

Conditions for the compatibility of the equations (11), (12) are satisfied identically: $p_{jrz} = p_{jzr} = 0$. It follows that the pressure in the layers will be restored by the formula:

$$p_j = -\rho_j f_j \frac{z^2}{2} + s_j(r), \tag{13}$$

where the derivative of the variable r from the functions $s_j(r)$ is exactly the right-hand side of the equation (11). Integrating this equation, we obtain for the functions $s_j(r)$ the following view:

$$s_j(r) = \rho_j \nu_j \Big(u_{jr} + \frac{1}{r} u_j \Big) - \frac{1}{2} \rho_j \, u_j^2 + s_{j0}, \quad s_{j0} \equiv \text{const}. \tag{14}$$

In turn, the functions $v_j(r)$ are defined from the equation:

$$u_j v_{jr} + v_j^2 = \nu_j \Big(v_{jrr} + \frac{1}{r}\, v_{jr} \Big) + f_j, \tag{15}$$

where $f_j \equiv$ const. The flow in the layers is induced by the longitudinal pressure gradients f_j. These are unknown constants that are subject to by definition. Therefore, the problem is reversed.

On a solid wall $r = R_2$, the boundary conditions are satisfied:

$$\begin{gathered} u_2(R_2) = 0, \quad v_2(R_2) = 0, \quad a_2(R_2) = a_{20}, \quad b_2(R_2) = b_{20}, \\ h_{2r}(R_2) + \alpha_2 a_{2r}(R_2) = 0, \quad g_{2r}(R_2) + \alpha_2 b_{2r}(R_2) = 0, \end{gathered} \tag{16}$$

with the given constants a_{20}, b_{20}. Note that when $a_{20} < 0$ the wall temperature has a maximum value at the point $z = 0$, and for $a_{20} > 0$ — minimal.

On the interface $r = R_1$, given the dependence (2), we will have the following conditions:

$$c u_1(R_1) = u_2(R_1), \quad v_1(R_1) = v_2(R_1), \tag{17}$$

$$\begin{gathered} a_1(R_1) = a_2(R_1), \quad b_1(R_1) = b_2(R_1), \\ h_1(R_1) = h_2(R_1), \quad g_1(R_1) = g_2(R_1), \end{gathered} \tag{18}$$

$$\mu_2 v_{2r}(R_1) - \mu_1 v_{1r}(R_1) = -2æ_1 a_1(R_1) - 2æ_2 h_1(R_1), \tag{19}$$

$$d_1[h_{1r}(R_1) + \alpha_1 a_{1r}(R_1)] = d_2[h_{2r}(R_1) + \alpha_2 a_{2r}(R_1)], \tag{20}$$

$$\begin{gathered} k_2 a_{2r}(R_1) - k_1 a_{1r}(R_1) = æ_1 a_1(R_1) v_1(R_1), \\ k_2 b_{2r}(R_1) - k_1 b_{1r}(R_1) = æ_1 b_1(R_1) v_1(R_1). \end{gathered} \tag{21}$$

The relation (21) is called the *energy condition on the interface* of two binary mixtures [8–10]. It means that the jump in the heat flow in the direction of the normal to the surface section $r = R_1$ is compensated by a change in the internal energy of this surface. In turn, this change is associated with both a change in temperature (and with it the specific internal energy) and a change in the area of the interface.

For a complete statement of the problem to the relations (17)–(21), it is necessary to add the boundedness of the functions on the axis of the cylindrical capillary at $r = 0$:

$$\begin{gathered} |u_1(0)| < \infty, \quad |v_1(0)| < \infty, \quad |s_1(0)| < \infty, \quad |a_1(0)| < \infty, \\ |b_1(0)| < \infty, \quad |h_1(0)| < \infty, \quad |g_1(0)| < \infty. \end{gathered} \tag{22}$$

2. Transformation to a problem in dimensionless variables

For what follows, it is essential that the equations (6), (7), (9), (15) are independent of the others and form a closed subsystem for defining the functions $v_j(r)$, $a_j(r)$, $h_j(r)$ and the constants f_j $(j = 1, 2)$. After solving it, the functions $b_j(r)$, $g_j(r)$ are found from the equations (8), (10), and $s_j(r)$ is uniquely restored by the formula (14). If we integrate the continuity equation (6) and exclude functions $u_j(r)$ in the equations (7), (9), (15) with given the conditions of boundedness (22) and sticking on a solid wall (16), the problem is reduced to the conjugate boundary value problem of finding only the functions $v_j(r)$, $a_j(r)$, $h_j(r)$ and the constants f_j. We introduce dimensionless variables and functions by equalities:

$$\begin{gathered} \xi = \frac{r}{R_1}\,, \quad R = \frac{R_2}{R_1} > 1\,, \quad V_j = \frac{R_1^2 v_j}{\mathrm{Ma}\,\nu_1}\,, \\ A_j = \frac{a_j}{a_{20}}\,, \quad H_j = \frac{h_j}{c_0}\,, \quad F_j = \frac{R_1^4 f_j}{\mathrm{Ma}\,\nu_1^2}\,, \end{gathered} \tag{23}$$

where a_{20}, c_0 are the characteristic temperature and concentration.

As the defining parameters of the problem under consideration, we choose the following:

$$\mathrm{Ma} = \frac{æ_1 a_{20} R_1^3}{\mu_2 \nu_1}, \quad \mathrm{Mc} = \frac{æ_2 c_0 R_1^3}{\mu_2 \nu_1}, \quad \mathrm{Pr}_j = \frac{\nu_j}{\chi_j}, \quad \mathrm{Sc}_j = \frac{\nu_j}{d_j}, \quad \mathrm{Sr}_j = \frac{\alpha_j a_{20}}{c_0},$$
$$\mu = \frac{\mu_1}{\mu_2}, \quad \nu = \frac{\nu_1}{\nu_2}, \quad k = \frac{k_1}{k_2}, \quad d = \frac{d_1}{d_2}, \quad \mathrm{M} = \frac{\mathrm{Mc}}{\mathrm{Ma}} = \frac{æ_2 c_0}{æ_1 a_{20}}. \tag{24}$$

Here Ma is the thermal Marangoni number, Mc is the concentration Marangoni number, Pr_j are the Prandtl numbers, Sc_j are the Schmidt numbers, Sr_j are the Soret numbers.

After de-dimensionalization, we obtain a nonlinear inverse boundary value problem in the domain with respect to the spatial variable ξ, which, for $j = 1$ varies between 0 and 1, and when $j = 2$ — in the range from 1 to R.

For $0 < \xi < 1$ we will have:

$$K_1(V_1, F_1) \equiv V_{1\xi\xi} + \frac{1}{\xi} V_{1\xi} + \frac{\mathrm{Ma}}{\xi} V_{1\xi} \int_0^\xi x V_1(x)\, dx - \mathrm{Ma} V_1^2 + F_1 = 0, \tag{25}$$

$$S_1(V_1, A_1) \equiv \frac{1}{\mathrm{Pr}_1} \left(A_{1\xi\xi} + \frac{1}{\xi} A_{1\xi} \right) + \frac{\mathrm{Ma}}{\xi} A_{1\xi} \int_0^\xi x V_1(x)\, dx - 2\mathrm{Ma} A_1 V_1 = 0; \tag{26}$$

$$T_1(V_1, A_1, H_1) \equiv \frac{1}{\mathrm{Sc}_1} \left(H_{1\xi\xi} + \frac{1}{\xi} H_{1\xi} \right) + \frac{\mathrm{Sr}_1}{\mathrm{Sc}_1} \left(A_{1\xi\xi} + \frac{1}{\xi} A_{1\xi} \right) + $$
$$+ \frac{\mathrm{Ma}}{\xi} H_{1\xi} \int_0^\xi x V_1(x)\, dx - 2\mathrm{Ma} H_1 V_1 = 0. \tag{27}$$

For $1 < \xi < R$, we have:

$$K_2(V_2, F_2) \equiv \frac{1}{\nu} \left(V_{2\xi\xi} + \frac{1}{\xi} V_{2\xi} \right) - \frac{\mathrm{Ma}}{\xi} V_{2\xi} \int_\xi^R x V_2(x)\, dx - \mathrm{Ma} V_2^2 + F_2 = 0, \tag{28}$$

$$S_2(V_2, A_2) \equiv \frac{1}{\mathrm{Pr}_2 \nu} \left(A_{2\xi\xi} + \frac{1}{\xi} A_{2\xi} \right) - \frac{\mathrm{Ma}}{\xi} A_{2\xi} \int_\xi^R x V_2(x)\, dx - 2\mathrm{Ma} A_2 V_2 = 0; \tag{29}$$

$$T_2(V_2, A_2, H_2) \equiv \frac{1}{\mathrm{Sc}_2 \nu} \left(H_{2\xi\xi} + \frac{1}{\xi} H_{2\xi} \right) + \frac{\mathrm{Sr}_2}{\mathrm{Sc}_2 \nu} \left(A_{2\xi\xi} + \frac{1}{\xi} A_{2\xi} \right) -$$
$$- \frac{\mathrm{Ma}}{\xi} H_{2\xi} \int_\xi^R x V_2(x)\, dx - 2\mathrm{Ma} H_2 V_2 = 0. \tag{30}$$

Then, on a solid wall $\xi = R$, the conditions are met:

$$V_2(R) = 0, \quad A_2(R) = 1, \quad H_{2\xi}(R) + \mathrm{Sr}_2 A_{2\xi}(R) = 0. \tag{31}$$

On the interface $\xi = 1$:

$$V_1(1) = V_1(1), \quad \int_0^1 x V_1(x)\, dx = 0, \quad \int_1^R x V_2(x)\, dx = 0, \tag{32}$$

$$A_1(1) = A_2(1), \quad H_1(1) = H_2(1), \tag{33}$$

$$V_{2\xi}(1) - \mu V_{1\xi}(1) = -2A_1(1) - 2\mathrm{M} H_1(1), \tag{34}$$

$$d(H_{1\xi}(1) + \mathrm{Sr}_1 A_{1\xi}(1)) = H_{2\xi}(1) + \mathrm{Sr}_2 A_{2\xi}(1), \tag{35}$$

$$A_{2\xi}(1) - kA_{1\xi}(1) = EA_1(1)V_1(1), \tag{36}$$

where $E = æ_1^2 a_{20} R_1^2/\mu_2 k_2$ is a parameter that determines the effect of the internal energy of the interface on the dynamics of the movement of liquids inside the layers.

On the axis of symmetry, the conditions of boundedness are set:

$$|V_1(0)| < \infty, \quad |A_1(0)| < \infty, \quad |H_1(0)| < \infty. \tag{37}$$

Remark. The integral redefinition conditions in (32), meaning the flow closure conditions, are necessary to find the unknown longitudinal pressure gradients F_j in the layers of binary mixtures, $j = 1, 2$.

3. Solving of the conjugate problem for small Marangoni numbers

We will assume that the thermal Marangoni number Ma $\ll 1$ (*a creeping motion*), and Ma $\sim$ Mc, that is, the thermal and concentration effects on the interface $\xi = 1$ of the same order. Formally decomposing the functions V_j, A_j, H_j in a series of Ma, we obtain for the first approximation the problem (25)–(27), (28)–(30) with Ma $= 0$. In the equations of momentum, energy, and concentration transport, the convective terms are discarded. As for the nonlinear boundary condition (36), it is remains unchanged. To do this, we must assume that $E = O(1)$.

Then the conjugate inverse boundary value problem for small Marangoni numbers becomes linear:

$$V_{1\xi\xi} + \frac{1}{\xi}V_{1\xi} = -F_1, \tag{38}$$

$$A_{1\xi\xi} + \frac{1}{\xi}A_{1\xi} = 0, \tag{39}$$

$$H_{1\xi\xi} + \frac{1}{\xi}H_{1\xi} = 0, \quad 0 < \xi < 1; \tag{40}$$

$$V_{2\xi\xi} + \frac{1}{\xi}V_{2\xi} = -F_2\nu, \tag{41}$$

$$A_{2\xi\xi} + \frac{1}{\xi}A_{2\xi} = 0, \tag{42}$$

$$H_{2\xi\xi} + \frac{1}{\xi}H_{2\xi} = 0, \quad 1 < \xi < R; \tag{43}$$

with the boundary conditions (31)–(37).

Common solutions of systems (38)–(43) are easily found (the boundedness conditions (37) are taken into account):

$$V_1(\xi) = C_1 - \frac{F_1}{4}\xi^2\,, \quad A_1(\xi) = C_2\,, \quad H_1(\xi) = C_3; \tag{44}$$

$$V_2(\xi) = C_4 + C_5\ln\xi - \frac{F_2\nu}{4}\xi^2\,, \quad A_2(\xi) = C_6 + C_7\ln\xi\,, \quad H_2(\xi) = C_8 + C_9\ln\xi, \tag{45}$$

with the constants $C_1, \ldots, C_9$, which are determined from the boundary conditions (31)–(36). Exactly,

$$C_1 = \frac{F_1}{8}\,, \quad C_2 = C_6 = \frac{8}{8 - EF_1\ln R}\,, \quad C_3 = C_8\,,$$
$$C_4 = \frac{2F_2\nu - F_1}{8}\,, \quad C_5 = \frac{2F_2\nu(R^2-1) + F_1}{8\ln R}\,, \quad C_7 = \frac{EF_1}{EF_1\ln R - 8}\,, \quad C_9 = -\mathrm{Sr}_2 C_7. \tag{46}$$

As for the constant C_3, from the boundary condition (34) it is defined as follows:

$$C_3 = \frac{F_2\nu - F_1\mu - 4C_2 - 2C_5}{4\mathrm{M}}. \tag{47}$$

But such a representation for C_3 makes it difficult to further search for the pressure gradients F_1, F_2 along the layers when solving the inverse boundary value problem. On the other hand, this constant can be found if you set the average concentration over the cross section $z = 0$, so $\int\limits_0^1 \xi H_1(\xi)\, d\xi = 0$. From where we get that $C_3 = 0$ and, therefore, $C_8 = 0$.

The pressure gradients F_1, F_2 are related by the relation $F_2 = F_1 N(R)$, where the function $N(R)$ is defined by the formula:

$$N(R) = \frac{R^2 - 2\ln R - 1}{2\nu(R^2-1)[(R^2+1)\ln R - R^2 + 1]}. \tag{48}$$

In addition, the functions $U_j(\xi)$are recovered from the continuity equation (6):

$$U_1(\xi) = \frac{F_1}{16}\xi(\xi-1)(\xi+1),$$
$$U_2(\xi) = \frac{F_1}{16\xi}[(R^2-\xi^2)(8C_4 - 4C_5 - F_2\nu(R^2+\xi^2)) + 8C_5(R^2\ln R - \xi^2\ln\xi)]. \tag{49}$$

If the expression for the constant C_3 from (47) vanishes, then after some calculations a quadratic equation arises with respect to the unknown pressure gradient F_1:

$$EL(R)\ln R\, F_1^2 - 8L(R)F_1 - 128\ln R = 0, \tag{50}$$

where $L(R)$ is defined by the formula:

$$L(R) = 4\nu\ln R(\rho - N(R)) + 2\nu N(R)(R^2-1) + 1. \tag{51}$$

Of interest are the cases related to the number of solutions of the equation (50).

1. If $E = 0$, we get the equation: $-8L(R)F_1 - 128\ln R = 0$, which has a unique solution $F_1 = -16\ln R/L(R)$. The pressure gradient F_2 is easily determined from the ratio (48).

2. If $R \to 1$, then we have the equation: $-8L(R)F_1 = 0$, which has the only solution $F_1 = 0$. Here it is taken into account that the function $L(R)$ takes positive values on the interval $(1, +\infty)$. Then it follows from (48) that $F_2 = 0$. The equality of the pressure gradients to zero means that there is no source of motion of the mixtures in both layers. Thus the mixtures are at rest.

Next, we find the discriminant of the quadratic equation (50):

$$D = 64L(R)(L(R) + 8E\ln^2 R), \tag{52}$$

depending on the sign of which the equation has a different number of roots.

3. If $D > 0$, we get: $E > -L(R)/8\ln^2 R$. In this case, the square equation has two roots:

$$F_1^{1,2} = \frac{4L(R) \pm 4\sqrt{L^2(R) + 8EL(R)\ln^2 R}}{EL(R)\ln R}. \tag{53}$$

4. The discriminant vanishes at $E = -L(R)/8\ln^2 R, (L(R) \neq 0)$. Then the equation will have a unique solution: $F_1 = -32\ln R/L(R)$. Note, what is the expression $L(R)/8\ln^2 R > 0$ when $R \in (1, +\infty)$. Therefore, the parameter E takes negative values. This is possible with $a_{20} < 0$, since E depends on this parameter.

5. The negative sign of the discriminant corresponds to the condition: $E < -L(R)/8\ln^2 R$, which is equivalent to the absence of real roots of the square equation.

Thus, the number of solutions to the equation (50) depends more on the parameter E. In other words, the energy of interfacial heat transfer has a significant effect on the processes occurring in the contacting liquids.

4. Model problem

We present the quantitative results of solving the problem for the model system formic acid (mixture 1) — transformer oil (mixture 2). According to the tabular data, the physical constants are as follows:

$$\mu_1 = 1.78 \cdot 10^{-3} \frac{\text{kg}}{\text{m} \cdot \text{s}}, \quad \mu_2 = 198.1 \cdot 10^{-4} \frac{\text{kg}}{\text{m} \cdot \text{s}}, \quad \nu_1 = 1.46 \cdot 10^{-6} \frac{\text{m}^2}{\text{s}}, \quad \nu_2 = 22.5 \cdot 10^{-6} \frac{\text{m}^2}{\text{s}},$$
$$\chi_1 = 1.057 \cdot 10^{-7} \frac{\text{m}^2}{\text{s}}, \quad \chi_2 = 7.55 \cdot 10^{-8} \frac{\text{m}^2}{\text{s}}, \quad k_1 = 0.267 \frac{\text{Wt}}{\text{m} \cdot \text{K}}, \quad k_2 = 0.1106 \frac{\text{Wt}}{\text{m} \cdot \text{K}},$$
$$\sigma_0 = 37.58 \cdot 10^{-3} \frac{\text{N}}{\text{m}}, \quad æ_1 = 1.2826 \cdot 10^{-4} \frac{\text{N}}{\text{m} \cdot \text{K}}.$$

The following parameter values were also used: $R = 1.5$, $R_1 = 10^{-9}$ m, $E = 0.7\,(a_{20} > 0)$. As a result of the calculations, two solutions were obtained for the longitudinal pressure gradients in the layers: $F_1^1 = -1.78305$, $F_2^1 = -71.22054$ and $F_1^2 = 29.96938$, $F_2^2 = 1197.06399$. It can be seen that for the second solution, the gradient values in both mixtures are too high, which is unphysical.

Fig. 2–4 demonstrates the function $V_j(\xi)$ and the velocity profile $U_j(\xi)$ depending on the various defining parameters of the model.

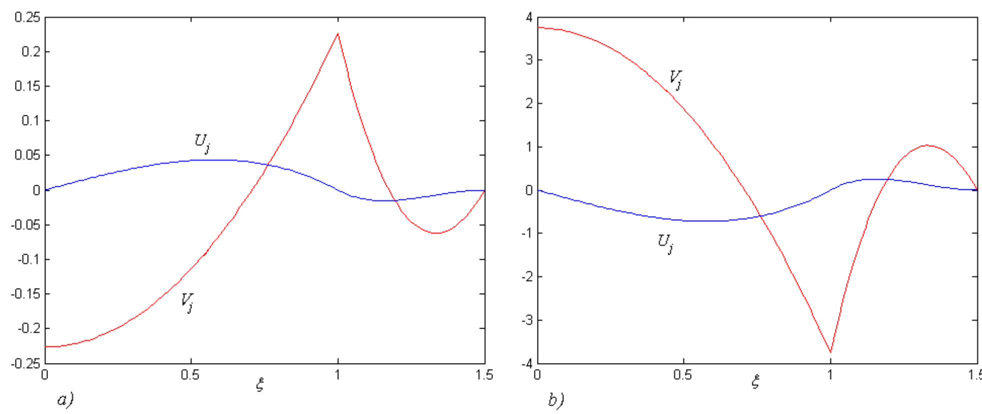

Fig. 2. The behavior of the function $V_j(\xi)$ and the velocity profile $U_j(\xi)$: a) for the first solution, b) for the second solution

Fig. 2 shows the functions $V_j(\xi)$, $U_j(\xi)$, corresponding to the two solutions $\{F_1^1, F_2^1\}$ and $\{F_1^2, F_2^2\}$.

Fig. 3 shows that as the parameter E increases, the values of the functions $V_j(\xi)$, $U_j(\xi)$ in absolute value decreases significantly. You can choose such values of E, at which the model problem will have a single solution. So, for $E = 0$ ($a_{20} = 0$) we get: $F_1 = -1.89641$, $F_2 = -76.27046$. By $E \approx -2.6$ ($a_{20} = -3.46 \cdot 10^{23}$) we have: $F_1 = -3.79282$, $F_2 = -152.54093$.

The increase of the parameter R is strongly influenced by the velocity profile $U_j(\xi)$ and the function $V_j(\xi)$. Fig. 4 shows that the absolute values of the functions increase. This is due to the fact that for a fixed R_1, the radius of the outer cylinder increases, since $R = R_2/R_1$. It is also important to trace how the change in the radius of the inner cylinder R_1 affects the flow pattern in the layers. It turned out that with the growth of R_1, the values of the functions $V_j(\xi)$, $U_j(\xi)$ in absolute value decreases. This is due to the fact that with an increase in the radius of the inner cylinder at fixed R and E, the influence of a constant temperature set on the surface of the outer cylindrical tube weakens.

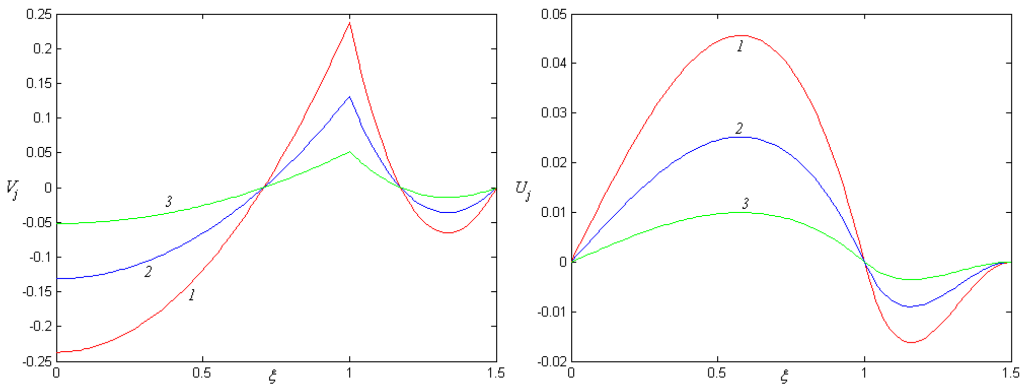

Fig. 3. The dependence of the functions $V_j(\xi)$, $U_j(\xi)$ on the parameter E: $1 - E = 0.05$, $2 - E = 0.2$, $3 - E = 0.7$

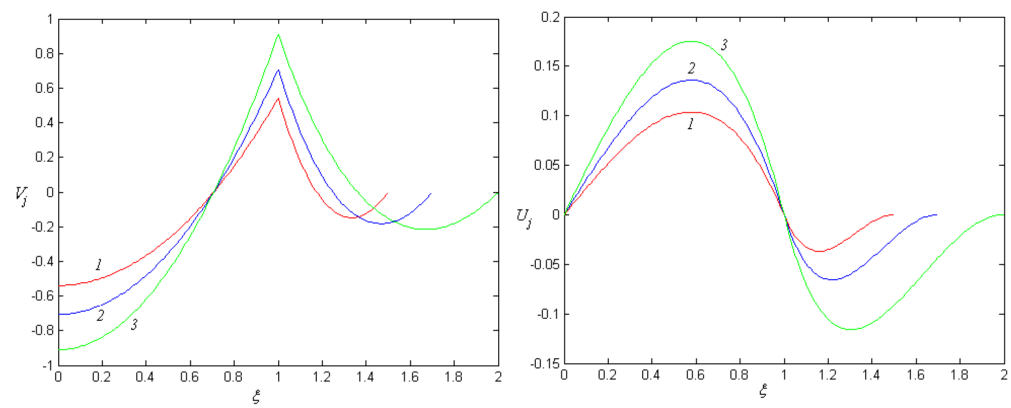

Fig. 4. The dependence of the functions $V_j(\xi)$, $U_j(\xi)$ on the parameter R: $1 - R = 1.5$, $2 - R = 1.7$, $3 - R = 2.0$

Fig. 5 shows the "temperature" and "concentration" functions $A_j(\xi)$, $H_j(\xi)$, corresponding to the first solution $\{F_1^1, F_2^1\}$. In the first layer, these functions are constant. In the second layer $A_j(\xi)$ increases and $H_j(\xi)$ decreases, which corresponds to the phenomenon of abnormal thermal diffusion.

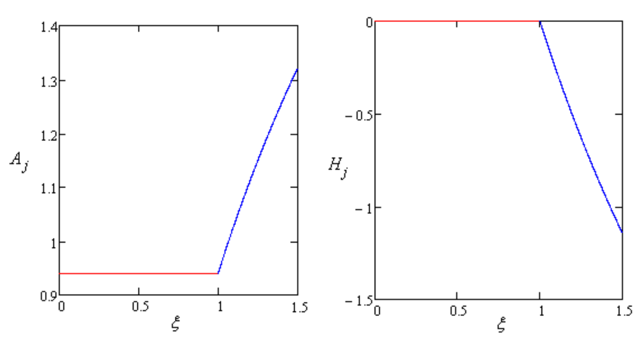

Fig. 5. The behavior of functions $A_j(\xi)$, $H_j(\xi)$ in the case of the first solution

Thus, the effect of changes in the internal energy of the interfacial surface on the two-layer flow of two immiscible binary mixtures in a cylindrical capillary is studied. It is found that with an increase in the parameter E, which is responsible for the influence of changes in the surface internal energy on the dynamics of liquids in layers, the absolute values of the functions $V_j(\xi)$, $U_j(\xi)$ decreases.

5. Derivation of a finite-dimensional system of nonlinear algebraic equations

To solve the nonlinear problem (25)–(37), the tau method is used, which is a modification of the Galerkin method [11]. For the future, it is essential to replace the variables: $\xi' = \xi$ with $j = 1$ and $\xi' = (\xi - R)/(1 - R)$ when $j = 2$ and re-assign $\xi' \leftrightarrow \xi$. An approximate solution is sought in the form of sums:

$$V_j^n(\xi) = \sum_{l=0}^{n} V_j^l R_k^{(0,1)}(\xi), \quad A_j^n(\xi) = \sum_{l=0}^{n} A_j^l R_k^{(0,1)}(\xi), \quad H_j^n(\xi) = \sum_{l=0}^{n} H_j^l R_k^{(0,1)}(\xi), \tag{54}$$

где $R_k^{(0,1)}(\xi)$ are the shifted Jacobi polynomials. In general, they are defined in terms of the Jacobi polynomials $P_k^{(\alpha,\beta)}(y)$ as follows ($\alpha > -1, \beta > -1$) [12]:

$$R_k^{(\alpha,\beta)}(y) = P_k^{(\alpha,\beta)}(2y - 1), \quad y \in [0, 1]. \tag{55}$$

Coefficients V_j^l, A_j^l, H_j^l and constants F_j are found from the Galerkin approximation system, namely:

$$\int_0^1 K_j(V_j^n, F_j) R_m^{(0,1)}(\xi)\, \xi\, d\xi = 0, \tag{56}$$

$$\int_0^1 S_j(V_j^n, A_j^n) R_m^{(0,1)}(\xi)\, \xi\, d\xi = 0, \tag{57}$$

$$\int_0^1 T_j(V_j^n, A_j^n, H_j^n) R_m^{(0,1)}(\xi)\, \xi\, d\xi = 0, \quad m = 0, \dots, n-2, \quad j = 1, 2. \tag{58}$$

It follows from the integral redefinition conditions of (32) that $V_1^0 = V_2^0 = 0$.

The boundary conditions are transformed as follows:

$$\sum_{l=0}^{n} (-1)^l V_2^l = 0, \quad \sum_{l=0}^{n} (-1)^l A_2^l = 1, \tag{59}$$

$$\sum_{l=1}^{n} (-1)^{l-1} l(l+1)(l+2)[H_2^l + \mathrm{Sr}_2 A_2^l] = 0. \tag{60}$$

$$\sum_{l=0}^{n} V_1^l = \sum_{l=0}^{n} V_2^l, \quad \sum_{l=0}^{n} A_1^l = \sum_{l=0}^{n} A_2^l, \quad \sum_{l=0}^{n} H_1^l = \sum_{l=0}^{n} H_2^l, \tag{61}$$

$$\sum_{l=1}^{n} l(l+2)(V_2^l - \mu V_1^l) = -2 \sum_{l=0}^{n} (A_1^l + \mathrm{M} H_1^l). \tag{62}$$

$$d \sum_{l=1}^{n} l(l+2)[H_1^l + \mathrm{Sr}_1 A_1^l] = \sum_{l=1}^{n} l(l+2)[H_2^l + \mathrm{Sr}_2 A_2^l], \tag{63}$$

$$\sum_{l=1}^{n} l(l+2)(A_2^l - k A_1^l) = -E \sum_{l=0}^{n} A_1^l \sum_{l=0}^{n} V_1^l. \tag{64}$$

Verbose output finite-dimensional system galerkins approximations for the coefficients V_j^l, A_j^l, H_j^l, $l = 0, \dots, n$, $j = 1, 2$, and also the calculation of definite integrals from different product of shifted Jacobi polynomials are present in the work [13].

As a result, the system of integro-differential equations are converted to a closed system of nonlinear algebraic equations unknown coefficients V_j^l, A_j^l, H_j^l and gradients of pressure F_j, where $l = 0, \dots, n$, $j = 1, 2$. Its solution was used Newton's method with a given accuracy $\varepsilon = 10^{-5}$. As an initial approximation, the results obtained in solving the model problem were taken.

Applied to a nonlinear inverse boundary value problem (25)–(37) the tau-method showed the possibility of existence of two solutions for the longitudinal pressure gradients and, accordingly, for the rest of the desired functions of the problem. Calculations were performed for $n = 10, 12$ in Galerkin approximations. As the number of n increases, a rapid increase in the accuracy of the solution is detected.

Fig. 6 shows the dependence of the functions $V_j(\xi)$, $U_j(\xi)$ on different values of the thermal Marangoni number, obtained for the first solution: $F_1 = -1.78355$, $F_2^1 = -71.73149$. We conclude that the solutions found with a decrease in the Marangoni number converge to solutions of the problem of the creeping flow of binary mixtures.

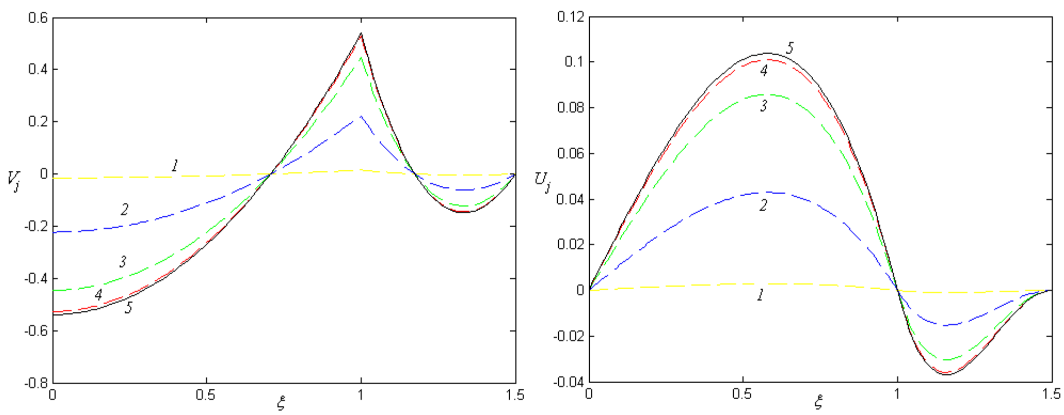

Fig. 6. The dependence of the functions $V_j(\xi)$, $U_j(\xi)$ of the thermal Marangoni number: 1 — Ma = 15, 2 — Ma = 3, 3 — Ma = 0.5, 4 — Ma = 0.28, 5 — a creeping current

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Двухслойное стационарное течение в цилиндрическом капилляре с учетом изменения внутренней энергии поверхности раздела

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Аннотация. Изучена задача о двумерном стационарном течении двух несмешивающихся несжимаемых бинарных смесей в цилиндрическом капилляре в отсутствие массовых сил. Смеси контактируют через общую поверхность раздела, на которой учитывается полное энергетическое условие. Температура и концентрация в смесях распределены по квадратичному закону, что хорошо согласуется с полем скоростей типа Хименца. Возникающая сопряженная краевая задача является нелинейной и обратной относительно градиентов давлений вдоль оси цилиндрического капилляра. К этой задаче применен тау-метод (модификация метода Галеркина), который показал возможность существования двух решений. Показано, что полученные решения с уменьшением числа Марангони сходятся к решениям задачи о ползущем течении бинарных смесей. При решении модельной задачи при малых числах Марангони установлено, что влияние приращений внутренней энергии межфазной поверхности существенно сказывается на динамике течения смесей в слоях.

Ключевые слова: бинарная смесь, поверхность раздела, внутренняя энергия, обратная задача, градиент давления, тепловое число Марангони.