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Analysis of the Boundary Value and Control Problems for Nonlinear Reaction–Diffusion–Convection Equation

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Abstract. The global solvability of the inhomogeneous mixed boundary value problem and control problems for the reaction–diffusion–convection equation are proved in the case when the reaction coefficient nonlinearly depends on the concentration. The maximum and minimum principles are established for the solution of the boundary value problem. The optimality systems are derived and the local stability estimates of optimal solutions are established for control problems with specific reaction coefficients.

Keywords: nonlinear reaction–diffusion–convection equation, mixed boundary conditions, maximum principle, control problems, optimality systems, local stability estimates.

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1. Introduction. Solvability of the boundary value problem

In recent years, there has been an increasing interest in the study of inverse and control problems for models of heat and mass transfer, electromagnetism and acoustics. A number of papers are devoted to the theoretical analysis of these problems, of which we note [1–16]. In these papers, the solvability of boundary value problems, inverse and extremum problems for the specified models was proved, and the questions of uniqueness and stability of their solutions were studied. Related problems for models of complex heat transfer were studied in [17, 18].

This paper which continues a series of papers by the authors [10–14] is devoted to the theoretical analysis of the boundary value and control problems for the nonlinear reaction–diffusion–convection equation, considered under inhomogeneous mixed boundary conditions on the boundary of the domain.

In bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ, consisting of two parts Γ_D and Γ_N, the following boundary value problem for nonlinear reaction–diffusion–convection equation is considered:

$$-\mathrm{div}(\lambda(\mathbf{x})\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi + k(\varphi, \mathbf{x})\varphi = f \text{ in } \Omega, \tag{1.1}$$

$$\varphi = \psi \text{ on } \Gamma_D,\ \lambda(\mathbf{x})(\partial\varphi/\partial n + \alpha(\mathbf{x})\varphi) = \chi \text{ on } \Gamma_N. \tag{1.2}$$

Here the function φ means the concentration of the substance, $\mathbf{u}$ is a given vector of velocity, f is a volume density of external sources of substance, $\lambda(\mathbf{x})$ is a diffusion coefficient, function $k(\varphi, \mathbf{x})$ is a reaction coefficient, $\mathbf{x} \in \Omega$. Below we will refer to the problem (1.1), (1.2) for the given functions $\lambda, k, f, \psi, \alpha$ and χ as Problem 1.

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In this paper, we first prove the global solvability of Problem 1 and the nonlocal uniqueness of its solution in the case, when the reaction coefficient $k(\varphi, \mathbf{x})$ is sufficiently arbitrarily depends on both the concentration φ and the spatial variable $\mathbf{x}$, and the nonlinearity $k(\varphi, \mathbf{x})\varphi$ is monotone. Under additional conditions on the functions $\lambda, f, \chi, \alpha, \psi$ and the reaction coefficient k the minimum and maximum principles are established for the concentration φ. Further, a control problem is formulated, in which the role of controls is played by the diffusion coefficient λ, the volume density of external sources of substance f and the density of boundary sources χ and its solvability is proved. For the mentioned problems, with specific reaction coefficients, an optimality system is derived and, based on its analysis a theorem on the local stability estimates of optimal solutions is formulated. This theorem can be proved according to the scheme described in detail in [11–16].

When analyzing the problems under study, we will use the Sobolev functional spaces $H^s(D)$, $s \in \mathbb{R}$. Here D means either a domain Ω, or some subset $Q \subset \Omega$, or part Γ_D of the boundary Γ. By $\|\cdot\|_{s,Q}, |\cdot|_{s,Q}$ and $(\cdot,\cdot)_{s,Q}$ we will denote the norm, seminorm and scalar product in $H^s(Q)$. The norms and scalar products in $L^2(Q)$, $L^2(\Omega)$ or in $L^2(\Gamma_N)$ will be denoted by $\|\cdot\|_Q$ and $(\cdot,\cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)$ or $\|\cdot\|_{\Gamma_N}$ and $(\cdot,\cdot)_{\Gamma_N}$, respectively. Let $L^p_+(D) = \{k \in L^p(D) : k \geqslant 0\}$, $p \geqslant 3/2$, $Z = \{\mathbf{v} \in L^4(\Omega)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}\cdot\mathbf{n}|_{\Gamma_N} = 0\}$, $H^s_{\lambda_0}(\Omega) = \{h \in H^s(\Omega) : h \geqslant \lambda_0 > 0 \text{ in } \Omega\}$, $s > 3/2$, $\mathcal{T} = \{\varphi \in H^1(\Omega) : \varphi|_{\Gamma_D} = 0\}$. Here and below $\varphi|_{\Gamma_0}$ denotes the trace of a function $\varphi \in H^1(\Omega)$ on the part Γ_0 of the boundary Γ. For any function $\varphi \in \mathcal{T}$ the Friedrichs–Poincaré inequality $\|\nabla\varphi\|^2_\Omega \geqslant \delta_0\|\varphi\|^2_{1,\Omega}$ holds, where positive constant δ_0 does not depend on φ.

Let the following conditions hold:

(i) Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma \in C^{0,1}$, consisting of closures of two non-intersecting open parts Γ_D and Γ_N ($\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$), and meas $\Gamma_D > 0$;

(ii) $\lambda \in H^s_{\lambda_0}(\Omega)$, $s > 3/2$, $f \in L^2(\Omega)$, $\chi \in L^2(\Gamma_N)$;

(iii) $\mathbf{u} \in Z$, $\psi \in H^{1/2}(\Gamma_D)$, $\alpha \in L^2_+(\Gamma_N)$.

(iv) The function $k : \mathbb{R} \times \Omega \to \mathbb{R}$ is nonnegative. In addition, for any function $v \in H^1(\Omega)$ the embedding $k(v,\cdot) \in L^p_+(\Omega)$ holds for some $p \geqslant 3/2$, independent of v, and on any ball $B_r = \{v \in H^1(\Omega) : \|v\|_{1,\Omega} \leqslant r\}$ of radius r the following inequality holds:

$$\|k(v_1,\cdot) - k(v_2,\cdot)\|_{L^p(\Omega)} \leqslant L_r\|v_1 - v_2\|_{L^4(\Omega)} \quad \forall v_1, v_2 \in B_r. \tag{1.3}$$

Here the constant L_r depends on r but does not depend on $v_1, v_2 \in B_r$;

(v) $(k(\varphi_1,\cdot)\varphi_1 - k(\varphi_2,\cdot)\varphi_2, \varphi_1 - \varphi_2) \geqslant 0$ for all $\varphi_1, \varphi_2 \in H^1(\Omega)$;

(vi) $\|k(\varphi,\cdot)\|_{L^p(\Omega)} \leqslant A\|\varphi\|^r_{1,\Omega} + B$ for all $\varphi \in H^1(\Omega)$, where number p is defined in (iii), $r \geqslant 0$ is a fixed number, A and B are nonnegative constants.

Let us note that the condition (iv) describes an operator acting from $H^1(\Omega)$ to $L^p(\Omega)$, $p \geqslant 3/2$, allowing to take into account the rather arbitrary dependence of the reaction coefficient k on both the concentration φ and the spatial variable $\mathbf{x}$. Condition (v) means that the nonlinearity $k(\varphi,\cdot)\varphi$ is monotone [19, p. 182], and condition (vi) restricts the growth in φ of the reaction coefficient by a power function with exponent r.

The specified conditions will provide a proof of the solvability of Problem 1 considered under the inhomogeneous Dirichlet condition on the part Γ_D of the boundary Γ. As an example of the function $k(\varphi,\cdot)$ satisfying (iv)–(vi) we give the function $k : \mathbb{R} \times \Omega \to \mathbb{R}$, such that $k(\varphi, \mathbf{x}) = \varphi^2$ for $\mathbf{x} \in Q$ where Q is a subdomain of domain Ω, $k(\varphi, \mathbf{x}) = k_0(\mathbf{x}) \in L^{3/2}_+(\Omega \setminus \overline{Q})$ for $\mathbf{x} \in \Omega \setminus \overline{Q}$.

Let us also remind that, by the Sobolev embedding theorem, the space $H^1(\Omega)$ is embedded into the space $L^s(\Omega)$ continuously at $s \leqslant 6$ and compactly at $s < 6$ and, with a certain constant C_s, depending on s and Ω, we have the estimate

$$\|\varphi\|_{L^s(\Omega)} \leqslant C_s\|\varphi\|_{1,\Omega} \quad \forall \varphi \in H^1(\Omega). \tag{1.4}$$

The following technical lemma holds (see details in [7]).

Lemma 1.1. *Let, in addition to condition (i)–(iii),* $\mathbf{u} \in Z$, $k_1(\cdot) \in L^p_+(\Omega)$, $p \geqslant 3/2$. *Then the following relations hold:*

$$|(\lambda\nabla\varphi, \nabla\eta)| \leqslant \gamma_s\|\lambda\|_{s,\Omega}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \forall\, \varphi, \eta \in H^1(\Omega),$$
$$(\lambda\nabla h, \nabla h) \geqslant \lambda_*\|h\|^2_{1,\Omega} \ \forall\, h \in \mathcal{T},\ \lambda_* \equiv \delta_0\lambda_0, \tag{1.5}$$

$$|(\mathbf{u}\cdot\nabla\varphi, \eta)| \leqslant \gamma_1\|\mathbf{u}\|_{L^4(\Omega)^3}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \forall\, \varphi, \eta \in H^1(\Omega),\ \ (\mathbf{u}\cdot\nabla h, h) = 0 \ \forall\, h \in \mathcal{T}, \tag{1.6}$$

$$|(\chi, \varphi)_{\Gamma_N}| \leqslant \gamma_2\|\chi\|_{\Gamma_N}\|\varphi\|_{1,\Omega} \ \ \forall\, \chi \in L^2(\Gamma_N),\ \varphi \in H^1(\Omega), \tag{1.7}$$

$$|(\lambda\alpha\varphi, \eta)_{\Gamma_N}| \leqslant \gamma^s_3\|\lambda\|_{s,\Omega}\|\alpha\|_{\Gamma_N}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \ \forall\, \varphi, \eta \in H^1(\Omega), \tag{1.8}$$

$$|(k_1\varphi, \eta)| \leqslant \gamma_p\|k_1\|_{L^p(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \ \forall\, \varphi, \eta \in H^1(\Omega). \tag{1.9}$$

Here $\lambda_* = \delta_0\lambda_0$, *constants* γ_1 *and* γ_2 *depend on* Ω, *constants* γ_s *and* γ^s_3 *depend on* Ω *and* s, γ_p *depends on* Ω *and* p.

Let us multiply the equation (1.1) by $h \in \mathcal{T}$ and integrate over Ω using Green's formulae. Taking into account (1.2), we obtain

$$(\lambda\nabla\varphi, \nabla h) + (k(\varphi, \cdot)\varphi, h) + (\mathbf{u}\cdot\nabla\varphi, h) + (\lambda\alpha\varphi, h)_{\Gamma_N} = (f, h) + (\chi, h)_{\Gamma_N} \ \forall h \in \mathcal{T},\ \varphi|_{\Gamma_D} = \psi. \tag{1.10}$$

Definition 1.1. *The function* $\varphi \in H^1(\Omega)$, *which satisfies (1.10), will be called a weak solution of Problem 1.*

To prove the solvability of Problem 1, we need the following lemma [12].

Lemma 1.2. *Let condition (i) holds. Then for any function* $\psi \in H^{1/2}(\Gamma_D)$ *there exists a function* $\varphi_0 \in H^1(\Omega)$, *such that* $\varphi_0 = \psi$ *on* Γ_D *and with some constant* C_Γ, *depending on* Ω *and* Γ_D, *the estimate* $\|\varphi_0\|_{1,\Omega} \leqslant C_\Gamma\|\psi\|_{1/2,\Gamma_D}$ *holds.*

We represent the solution to Problem 1 as the sum $\varphi = \tilde\varphi + \varphi_0$ where φ_0 is a given function from Lemma 1.2 and $\tilde\varphi \in \mathcal{T}$ is unknown function. Substituting $\varphi = \tilde\varphi + \varphi_0$ in (1.10) we will have

$$(\lambda\nabla\tilde\varphi, \nabla h) + (k(\tilde\varphi + \varphi_0, \cdot)(\tilde\varphi + \varphi_0), h) + (\mathbf{u}\cdot\nabla\tilde\varphi, h) + (\lambda\alpha\tilde\varphi, h)_{\Gamma_N} =$$
$$= (f, h) + (\chi, h)_{\Gamma_N} - (\lambda\nabla\varphi_0, \nabla h) - (\mathbf{u}\cdot\nabla\varphi_0, h) - (\lambda\alpha\varphi_0, h)_{\Gamma_N} \ \ \forall\, h \in \mathcal{T}. \tag{1.11}$$

Adding the term $-(k(\varphi_0, \cdot)\varphi_0, h)$ to both parts of (1.11), we obtain

$$(\lambda\nabla\tilde\varphi, \nabla h) + (k(\tilde\varphi + \varphi_0, \cdot)(\tilde\varphi + \varphi_0) - k(\varphi_0, \cdot)\varphi_0, h) + (\mathbf{u}\cdot\nabla\tilde\varphi, h) + (\lambda\alpha\tilde\varphi, h)_{\Gamma_N} =$$
$$\langle l, h\rangle \equiv (f, h) + (\chi, h)_{\Gamma_N} - (\lambda\nabla\varphi_0, \nabla h) - (\mathbf{u}\cdot\nabla\varphi_0, h) - (k(\varphi_0, \cdot)\varphi_0, h) - (\lambda\alpha\varphi_0, h)_{\Gamma_N} \ \forall\, h \in \mathcal{T}. \tag{1.12}$$

Using the Holder inequality, Lemmas 1.1, 1.2, estimate (1.4) and condition (vi), it is easy to show that $l \in \mathcal{T}^*$ and, moreover, the following estimate holds:

$$\|l\|_{\mathcal{T}^*} \leqslant M_l \equiv \|f\|_\Omega + \gamma_2\|\chi\|_{\Gamma_N} + C_\Gamma(\gamma_s\|\lambda\|_{s,\Omega} + \gamma_1\|\mathbf{u}\|_{L^4(\Omega)^3})\|\psi\|_{1/2,\Gamma_D} +$$
$$+ C_\Gamma[\gamma_p(AC^r_\Gamma\|\psi\|^r_{1/2,\Gamma_D} + B) + \gamma^s_3\|\lambda\|_{s,\Omega}\|\alpha\|_{\Gamma_N}]\|\psi\|_{1/2,\Gamma_D}. \tag{1.13}$$

Let us introduce the nonlinear operator $A : \mathcal{T} \to \mathcal{T}^*$ by

$$\langle A(\tilde\varphi), h\rangle \equiv (\lambda\nabla\tilde\varphi, \nabla h) + (k(\tilde\varphi + \varphi_0, \cdot)(\tilde\varphi + \varphi_0) - k(\varphi_0, \cdot)\varphi_0, h) + (\mathbf{u}\cdot\nabla\tilde\varphi, h) +$$
$$+ (\lambda\alpha\tilde\varphi, h)_{\Gamma_N} \ \ \forall\, \tilde\varphi,\ h \in \mathcal{T}. \tag{1.14}$$

It is clear that the problem (1.12) is equivalent to the operator equation $A(\tilde{\varphi}) = l$. According to [19, p. 182], to prove the existence of a solution $\tilde{\varphi} \in \mathcal{T}$ of problem (1.12) it suffices to show that: 1) the operator A is monotone on $\mathcal{T}$, that is $\langle A(u)-A(v), u-v\rangle \geqslant 0$ for all $u, v \in \mathcal{T}$; 2) the operator $A : \mathcal{T} \to \mathcal{T}^*$ is continuous and bounded; 3) the operator A is coercive on $\mathcal{T}$.

To prove the monotonicity of the operator A we subtract the relation (1.14) for $\tilde{\varphi} = \tilde{\varphi}_2$ from (1.14) for $\tilde{\varphi} = \tilde{\varphi}_1$ where $\tilde{\varphi}_1$ and $\tilde{\varphi}_2 \in \mathcal{T}$ are arbitrary elements. We obtain

$$\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), h\rangle = (\lambda \nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), \nabla h) + (k(\tilde{\varphi}_1 + \varphi_0, \cdot)(\tilde{\varphi}_1 + \varphi_0) - k(\tilde{\varphi}_2 + \varphi_0, \cdot)(\tilde{\varphi}_2 + \varphi_0), h) +$$

$$+ (\mathbf{u} \cdot \nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), h) + (\lambda\alpha(\tilde{\varphi}_1 - \tilde{\varphi}_2), h)_{\Gamma_N} \quad \forall\, h \in \mathcal{T}. \tag{1.15}$$

For $h = \tilde{\varphi}_1 - \tilde{\varphi}_2$ all terms in the right-hand side of (1.15) are nonnegative due to the properties of the functions $\lambda, \alpha, \mathbf{u}$ indicated in (ii), (iii) and monotonicity of nonlinearity $k(\varphi)\varphi$. Therefore

$$\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), \tilde{\varphi}_1 - \tilde{\varphi}_2\rangle \geqslant 0 \quad \forall \tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathcal{T}.$$

To prove the continuity and boundedness of the operator A we rewrite (1.15) in the form

$$\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), h\rangle = (\lambda \nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), \nabla h) + (k(\tilde{\varphi}_1 + \varphi_0, \cdot) - k(\tilde{\varphi}_2 + \varphi_0, \cdot), \tilde{\varphi}_1 + \varphi_0, h) +$$

$$+ (k(\tilde{\varphi}_2 + \varphi_0, \cdot)(\tilde{\varphi}_1 - \tilde{\varphi}_2), h) + (\mathbf{u} \cdot \nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), h) + (\lambda\alpha(\tilde{\varphi}_1 - \tilde{\varphi}_2), h)_{\Gamma_N} \quad \forall h \in \mathcal{T}. \tag{1.16}$$

Using the estimates of Lemma 1.1, the estimates (1.4), (1.9), and condition (iii), from (1.16) we deduce that

$$|\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), h\rangle| \leqslant (\gamma_s \|\lambda\|_{s,\Omega} + \gamma_p L C_4 \|\varphi_1\|_{1,\Omega}) \|\tilde{\varphi}_1 - \tilde{\varphi}_2\|_{1,\Omega} \|h\|_{1,\Omega} +$$

$$+ (\gamma_p \|k(\tilde{\varphi}_2 + \varphi_0, \cdot)\|_{L^p(\Omega)} + \gamma_1 \|\mathbf{u}\|_{L^4(\Omega)^3} + \gamma_3^s \|\lambda\|_{s,\Omega} \|\alpha\|_{\Gamma_N}) \|\tilde{\varphi}_1 - \tilde{\varphi}_2\|_{1,\Omega} \|h\|_{1,\Omega} \quad \forall h \in \mathcal{T}. \tag{1.17}$$

The inequality (1.17) implies the continuity and boundedness of the operator A. Finally, setting $h = \tilde{\varphi}$ in (1.14) and using conditions (ii), (iv), and (1.6), we arrive at the following inequality which implies the coercivity of the operator A:

$$\langle A(\tilde{\varphi}), \tilde{\varphi}\rangle = (\lambda \nabla \tilde{\varphi}, \nabla \tilde{\varphi}) + (k(\tilde{\varphi} + \varphi_0, \cdot)(\tilde{\varphi} + \varphi_0) - k(\varphi_0, \cdot)\varphi_0, \tilde{\varphi}) +$$

$$+ (\lambda\alpha\tilde{\varphi}, \tilde{\varphi})_{\Gamma_N} \geqslant \lambda_* \|\tilde{\varphi}\|^2_{1,\Omega} \quad \forall \tilde{\varphi} \in \mathcal{T}. \tag{1.18}$$

As a result we conclude that the solution $\tilde{\varphi} \in \mathcal{T}$ of the problem (1.11) exists and the estimate $\|\tilde{\varphi}\|_{1,\Omega} \leqslant C_* \|l\|_{\mathcal{T}^*}$, $C_* = \lambda_*^{-1}$ takes place. In this case, the function $\varphi = \varphi_0 + \tilde{\varphi}$ is the desired weak solution to Problem 1 and the following estimate holds:

$$\|\varphi\|_{1,\Omega} \leqslant M_\varphi \equiv C_* M_l + C_\Gamma \|\psi\|_{1/2,\Gamma_D} \quad (C_* = \lambda_*^{-1}). \tag{1.19}$$

Here the constant M_l was defined in (1.13) and C_Γ is the constant from Lemma 1.2.

Let us show that the solution to Problem 1 is unique. Let φ_1 and $\varphi_2 \in H^1(\Omega)$ be any two solutions to Problem 1. Then their difference $\varphi = \varphi_1 - \varphi_2 \in \mathcal{T}$ satisfies the identity

$$(\lambda \nabla \varphi, \nabla h) + (k(\varphi_1, \cdot)\varphi_1 - k(\varphi_2, \cdot)\varphi_2, h) + (\mathbf{u} \cdot \nabla \varphi, h) + (\lambda\alpha(\varphi_1 - \varphi_2), h)_{\Gamma_N} = 0 \; \forall\, h \in \mathcal{T}.$$

Setting here $h = \varphi$, by virtue of conditions (iii), (v) and (1.6) we arrive at the inequality $\lambda_* \|\varphi\|_{1,\Omega} \leqslant 0$, from which it follows that $\varphi_1 = \varphi_2$ in Ω. This proves the following theorem.

Theorem 1.1. *Let conditions (i)–(vi) hold. Then there exists a unique weak solution $\varphi \in H^1(\Omega)$ of Problem 1 and the estimate (1.19) holds.*

Within the framework of the approach of [20] we prove the maximum and minimum principles for a weak solution φ to Problem 1. To this end, we assume, in addition to (i)–(vi), that the following conditions are satisfied:

(vii) $\psi_{\min} \leqslant \psi \leqslant \psi_{\max}$ a.e. on Γ_D, $f_{\min} \leqslant f \leqslant f_{\max}$ and $\lambda_{\min} \leqslant \lambda \leqslant \lambda_{\max}$ a.e. in Ω, $\alpha_{\min} \leqslant \alpha \leqslant \alpha_{\max}$ and $\chi_{\min} \leqslant \chi \leqslant \chi_{\max}$ a.e. on Γ_N.
Here $\psi_{\min}$, $\psi_{\max}$, $f_{\min}$, $f_{\max}$, $\chi_{\min}$, $\chi_{\max}$ are nonnegative numbers, while $\alpha_{\min}$, $\alpha_{\max}$ and $\lambda_{\min}$, $\lambda_{\max}$ are positive numbers;

Besides, we will assume also that the reaction coefficient k satisfies the following conditions:

(viii) the reaction coefficient k has the form $k = k_1(\varphi)$ where $k_1(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous nonnegative function, satisfying conditions (iv)–(vi), in which one should set $k(\varphi, \cdot) = k_1(\varphi)$, and every of functional with respect to M_1 and m_1 equations

$$k_1(M_1)M_1 = f_{\max} \quad \text{and} \quad k_1(m_1)m_1 = f_{\min} \tag{1.20}$$

has at least one solution.

We set

$$M = \max\{\psi_{\max}, \chi_{\max}/\lambda_{\min}\alpha_{\min}, M_1\}, \quad m = \min\{\psi_{\min}, \chi_{\min}/\lambda_{\max}\alpha_{\max}, m_1\}. \tag{1.21}$$

Theorem 1.2. *Let conditions (i)–(iii), (vii), (viii) hold. Then for the solution $\varphi \in H^1(\Omega)$ of Problem 1 the following maximum and minimum principle holds:*

$$m \leqslant \varphi \leqslant M \ \text{a.e. in } \Omega. \tag{1.22}$$

Here the constants m and M are defined in (1.21) where M_1 is a minimum root of the first equation in (1.20) and m_1 is a maximum root of the second equation in (1.20).

Proof. Firstly, we prove the validity of the maximum principle in the form of the estimate $\varphi \leqslant M$ in Ω. For this purpose we introduce a nonnegative function $v = \max\{\varphi - M, 0\}$. From the definition of v it follows that the estimate $\varphi \leqslant M$ holds if and only if $v = 0$ in Ω. We denote by $\Omega_M \subset \Omega$ a measurable subset of Ω, at the points of which the inequality $\varphi > M$ holds, by Γ_M we denote the measurable subset of the part Γ_N, at the points of which the condition $v|_{\Gamma_M} > 0$ is satisfied. Set $\Omega_1 = \Omega \setminus \Omega_M$, $\Gamma_1 = \Gamma_N \setminus \Gamma_M$. From [21, p. 152] and [22] it follows by the definition of the constant M in (1.21) that $v \in \mathcal{T}$, and the following relations hold:

$$v = \varphi - M > 0 \text{ and } \nabla v = \nabla\varphi \text{ in } \Omega_M; \quad v = 0 \text{ and } \nabla v = \mathbf{0} \text{ in } \Omega_1; v|_{\Gamma_1} = 0,$$

$$(\lambda\nabla\varphi, \nabla v) = (\lambda\nabla v, \nabla v)_{\Omega_M} = (\lambda\nabla v, \nabla v), \quad (\mathbf{u}\cdot\nabla\varphi, v) = (\mathbf{u}\cdot\nabla\varphi, v)_{\Omega_M} = (\mathbf{u}\cdot\nabla v, v) = 0.$$

We set $h = v$ in (1.10) at $k(\varphi) = k_1(\varphi)$ and add to both sides of the resulting equality the term $-(k_1(M)M, v)_{Q_M} - (\lambda\alpha M, v)_{\Gamma_M}$. Taking into account the properties of v we obtain

$$(\lambda\nabla v, \nabla v) + (k_1(v+M)(v+M) - k_1(M)M, v)_{Q_M} + (\lambda\alpha v, v)_{\Gamma_M} =$$

$$= (f - k_1(M)M, v)_{Q_M} + (\chi - \lambda\alpha M, v)_{\Gamma_M}. \tag{1.23}$$

From the definition of the constant M in (1.21), relations (1.20) and conditions (ii),(iii), (vi) and (vii) it follows that the right-hand side in (1.23) is non-positive while the second and third terms in the left-hand side are nonnegative. Taking into account this fact and the second inequality in (1.5) from (1.23) we arrive at the estimate $\|v\|^2_{1,\Omega} \leqslant 0$, from which it follows that $v = 0$. This means the validity of the estimate of $\varphi \leqslant M$ in Ω.

To prove the minimum principle in the form of the estimate $\varphi \geqslant m$ in Ω we introduce a non-positive function $w = \min\{\varphi - m, 0\}$ and note that the validity of the minimum principle is equivalent to the condition $w = 0$ in Ω. Let us denote by Ω_m a measurable subset of Ω, at the points of which $\varphi < m$. By Γ_m we denote a measurable subset of the part Γ_N, at the points of which $\varphi|_{\Gamma_m} < m$. Set $\Omega_2 = \Omega \setminus \Omega_m$, $\Gamma_2 = \Gamma_N \setminus \Gamma_m$. By definition of Ω_m and Γ_m we have

$$w = \varphi - m < 0 \text{ and } \nabla w = \nabla\varphi \text{ in } \Omega_m; \quad w = 0 \text{ and } \nabla w = \mathbf{0} \text{ in } \Omega_2, \ w = 0 \text{ on } \Gamma_2.$$

Setting $h = w$ in (1.10) at $k(\varphi) = k_1(\varphi)$ we add to both sides of the resulting relation the term $-(k_1(m)m, w)_{Q_m} - (\lambda\alpha m, w)_{\Gamma_m}$. Taking into account the properties of the function w we obtain

$$(\lambda\nabla w, \nabla w) + (k_1(w+m)(w+m) - k_1(m)m, w)_{Q_m} + (\lambda\alpha w, w)_{\Gamma_m} =$$
$$= (f - k_1(m)m, w)_{Q_m} + (\chi - \lambda\alpha m, w)_{\Gamma_m}. \tag{1.24}$$

From the definition of the constant m in (1.21), (1.20) and conditions (ii), (iii) (v), (vii) it follows that the right-hand side in (1.24) is non-positive while the second and third terms in the left-hand side are nonnegative. Taking into account this fact, from (1.24) we derive that $w = 0$. $\square$

Remark 1.2. For power-law reaction coefficients, the parameters M_1 and m_1 are easily calculated. For example, for $k_1(\varphi) = \varphi^2$, we easily deduce that $M_1 = f_{\max}^{1/3}$, $m_1 = f_{\min}^{1/3}$.

2. Formulation and solvability of control problem

To formulate the control problem we divide the set of initial data of Problem 1 into two groups: a group of fixed data, to which we assign the functions $\mathbf{u}, k(\varphi,\cdot), \alpha$ and ψ, and the control group, to which we assign the functions λ, f and χ, assuming that they can change in some sets K_1, K_2 and K_3 satisfying the condition

(j) $K_1 \subset H^s_{\lambda_0}(\Omega)$, $K_2 \subset L^2(\Omega)$ and $K_3 \subset L^2(\Gamma_N)$ are nonempty convex closed sets.

Define the space $Y = \mathcal{T}^* \times H^{1/2}(\Gamma_D)$. Setting $u = (\lambda, f, \chi)$, $K = K_1 \times K_2 \times K_3$ we introduce the operator $F = (F_1, F_2) : H^1(\Omega) \times K \to Y$ by formulae: $F_2(\varphi) = \varphi|_{\Gamma_D} - \psi$ and

$$\langle F_1(\varphi, u), h\rangle = (\lambda\nabla\varphi, \nabla h) + (k(\varphi,\cdot)\varphi, h) + (\mathbf{u}\cdot\nabla\varphi, h) + (\lambda\alpha\varphi, h)_{\Gamma_N} - (f, h) - (\chi, h)_{\Gamma_N}$$

and rewrite (1.10) in the form $F(\varphi, u) = 0$. Considering this equality as a conditional restriction on the state $\varphi \in H^1(\Omega)$ and control $u \in K$, we introduce the cost functional I and formulate the following conditional minimization problem:

$$J(\varphi, u) \equiv \frac{\mu_0}{2} I(\varphi) + \frac{\mu_1}{2}\|\lambda\|^2_{s,\Omega} + \frac{\mu_2}{2}\|f\|^2_\Omega + \frac{\mu_3}{2}\|\chi\|^2_{\Gamma_N} \to \inf,$$
$$F(\varphi, u) = 0, \quad (\varphi, u) \in H^1(\Omega) \times K. \tag{2.1}$$

We denote by $Z_{ad} = \{(\varphi, u) \in H^1(\Omega) \times K : F(\varphi, u) = 0, J(\varphi, u) < \infty\}$ the set of admissible pairs for the problem (2.1) and suppose that the following condition is satisfied:

(jj) $\mu_0 > 0$, $\mu_i \geqslant 0$, $i = 1, 2, 3$, and K is a bounded set or $\mu_i > 0$, $i = 0, 1, 2, 3$ and functional I is bounded below.

We use the following cost functionals:

$$I_1(\varphi) = \|\varphi - \varphi^d\|^2_Q = \int_Q |\varphi - \varphi^d|^2 d\mathbf{x}, \quad I_2(\varphi) = \|\varphi - \varphi^d\|^2_{1,Q}. \tag{2.2}$$

Here $\varphi^d \in L^2(Q)$ (or $\varphi^d \in H^1(Q)$) is a given function in some subdomain $Q \subset \Omega$.

Theorem 2.1. *Let, in addition to conditions (i), (iii)–(vi), and (j), (jj), $I : H^1(\Omega) \to \mathbb{R}$ be a weakly semicontinuous below functional and let $Z_{ad} \neq \emptyset$. Then there exists at least one solution $(\varphi, u) \in H^1(\Omega) \times K$ of the control problem (2.1).*

Proof. Let $(\varphi_m, u_m) \in Z_{ad}$ be a minimizing sequence for which the following is true

$$\lim_{m\to\infty} J(\varphi_m, u_m) = \inf_{(\varphi,u)\in Z_{ad}} J(\varphi, u) \equiv J^*.$$

Condition (jj) and Theorem 1.1 yield the following estimates:

$$\|\lambda_m\|_{s,\Omega} \leqslant c_1,\ \|f_m\|_\Omega \leqslant c_2,\ \|\chi_m\|_{\Gamma_N} \leqslant c_3,\ \ \|\varphi_m\|_{1,\Omega} \leqslant c_4 \tag{2.3}$$

where the constants c_i, $i = 1, 2, 3, 4$ don't depend on m.

From the estimates (2.3) and from the condition (j) it follows that there exist weak limits $\lambda^* \in K_1$, $f^* \in K_2$, $\chi^* \in K_3$ and $\varphi^* \in H^1(\Omega)$ of some subsequences of sequences $\{\lambda_m\}$, $\{f_m\}$, $\{\chi_m\}$ and $\{\varphi_m\}$, respectively. Corresponding subsequences will be also denoted by $\{\lambda_m\}$, $\{f_m\}$, $\{\chi_m\}$ and $\{\varphi_m\}$. Moreover, due to the compactness of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $p < 6$, $H^{1/2}(\Gamma_N) \subset L^q(\Gamma_N)$ for $q < 4$, $H^s(\Omega) \subset L^\infty(\Omega)$ and $H^{s-1/2}(\Gamma_N) \subset L^\infty(\Gamma_N)$ for $s > 3/2$ we can assume for $m \to \infty$, that

$$\begin{gathered}
\varphi_m \to \varphi^* \text{ weakly in } H^1(\Omega),\ \text{weakly in } L^6(\Omega) \text{ and strongly in } L^s(\Omega),\ s < 6,\\
\varphi_m|_{\Gamma_N} \to \varphi^*|_{\Gamma_N} \text{ weakly in } H^{1/2}(\Gamma_N),\ \text{weakly in } L^4(\Gamma_N) \text{ and strongly in } L^q(\Gamma_N),\ q < 4,\\
f_m \to f^* \text{ weakly in } L^2(\Omega),\ \ \chi_m \to \chi^* \text{ weakly in } L^2(\Gamma_N),\\
\lambda_m \to \lambda^* \text{ weakly in } H^s(\Omega) \text{ and strongly in } L^\infty(\Omega),\\
\lambda_m|_{\Gamma_N} \to \lambda^*|_{\Gamma_N} \text{ weakly in } H^{s-1/2}(\Gamma_N) \text{ and strongly in } L^\infty(\Gamma_N),\ \ s > 3/2.
\end{gathered} \tag{2.4}$$

It is clear, that $F_2(\varphi^*) = 0$. Let us show that $F_1(\varphi^*, u^*) = 0$, that is, that

$$(\lambda^*\nabla\varphi^*, \nabla h) + (k(\varphi^*, \cdot)\varphi^*, h) + (\mathbf{u}\cdot\nabla\varphi^*, h) + (\lambda^*\alpha\varphi^*, h)_{\Gamma_N} = (f^*, h) + (\chi^*, h)_{\Gamma_N}\ \forall\, h \in \mathcal{T}. \tag{2.5}$$

To this end we note that the pair (φ_m, u_m) satisfies the identity

$$(\lambda_m\nabla\varphi_m, \nabla h) + (k(\varphi_m, \cdot)\varphi_m, h) + (\mathbf{u}\cdot\nabla\varphi_m, h) + (\lambda_m\alpha\varphi_m, h)_{\Gamma_N} =$$

$$= (f_m, h) + (\chi_m, h)_{\Gamma_N}\ \forall\, h \in \mathcal{T}. \tag{2.6}$$

Let us pass to the limit in (2.6) as $m \to \infty$. From (2.4) it follows that all linear terms in (2.6) pass into corresponding ones in (2.5).

Let us study the behaviour of nonlinear terms for $m \to \infty$ starting with $(k(\varphi_m, \cdot)\varphi_m, h)$. To prove the convergence

$$(k(\varphi_m, \cdot)\varphi_m, h) \to (k(\varphi^*, \cdot)\varphi^*, h)\ \text{ as } m \to \infty\ \forall\, h \in \mathcal{T} \tag{2.7}$$

it is enough to show that $k(\varphi_m, \cdot)\varphi_m \to k(\varphi^*, \cdot)\varphi^*$ weakly in $L^{6/5}(\Omega)$ as $m \to \infty$. From (1.3) it follows that $k(\varphi_m, \cdot) \to k(\varphi^*, \cdot)$ strongly in $L^{3/2}(\Omega)$, and from (2.4) it follows that $\varphi_m \to \varphi^*$ weakly in $L^6(\Omega)$ as $m \to \infty$. We derive from these properties that $k(\varphi_m, \cdot)\varphi_m \to k(\varphi^*, \cdot)\varphi^*$ weakly in $L^{6/5}(\Omega)$ and therefore (2.7) also holds.

For the term $(\lambda_m\nabla\varphi_m, \nabla h)$ the following equality holds:

$$(\lambda_m\nabla\varphi_m, \nabla h) - (\lambda^*\nabla\varphi^*, \nabla h) = ((\lambda_m - \lambda^*)\nabla\varphi_m, \nabla h) + (\nabla(\varphi_m - \varphi^*), \lambda^*\nabla h). \tag{2.8}$$

Since $\lambda^*\nabla h \in L^2(\Omega)^3$, then from (2.4) it follows that $(\nabla(\varphi_m - \varphi^*), \lambda^*\nabla h) \to 0$ as $m \to \infty$ for all $h \in \mathcal{T}$. Using Holder's inequality, (2.3) and (2.4) we easily deduce for the first term in the right-hand side of (2.8) that

$$|((\lambda_m - \lambda^*)\nabla\varphi_m, \nabla h)| \leqslant \|\lambda_m - \lambda^*\|_{L^\infty(\Omega)}\|\nabla\varphi_m\|_\Omega\|\nabla h\|_\Omega \to 0\ \text{ as } m \to \infty\ \ \forall\, h \in \mathcal{T}.$$

Then from (2.8) we obtain that $(\lambda_m \nabla\varphi_m, \nabla h) \to (\lambda^* \nabla\varphi^*, \nabla h)$ as $m \to \infty\ \forall\, h \in \mathcal{T}$.

Similarly, for the nonlinear term $(\lambda_m \alpha\varphi_m, h)_{\Gamma_N}$ we have that

$$(\lambda_m \alpha\varphi_m, h)_{\Gamma_N} - (\lambda^* \alpha\varphi^*, h)_{\Gamma_N} = ((\lambda_m - \lambda^*)\alpha\varphi_m, h)_{\Gamma_N} + (\lambda^*\alpha(\varphi_m - \varphi^*), h)_{\Gamma_N}. \tag{2.9}$$

Since $\lambda^*\alpha h \in L^{4/3}(\Gamma_N)$ then by virtue of (2.4) $(\varphi_m - \varphi^*, \lambda^*\alpha h)_{\Gamma_N} \to 0$ for all $h \in \mathcal{T}$ as $m \to \infty$. Using Holder inequality, (2.4) and the uniform boundedness of the quantity $\|\varphi_m\|_{L^4(\Gamma_N)}$ for any m, we deduce for the first term in the right-hand side of (2.9), that

$$|((\lambda_m - \lambda^*)\alpha\varphi_m, h)_{\Gamma_N}| \leqslant \|\lambda_m - \lambda^*\|_{L^\infty(\Gamma_N)} \|\alpha\|_{\Gamma_N} \|\varphi_m\|_{L^4(\Gamma_N)} \|h\|_{L^4(\Gamma_N)} \to 0 \ \text{ as } \ m \to \infty.$$

To complete the proof notice that the fact $J(\varphi^*, u^*) = J^*$ follows from aforesaid and from the weakly continuity below on $H^1(\Omega) \times H^s(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N)$ of the functional J. □

Remark 2.1. The functionals defined in (2.2) satisfy the conditions of Theorem 2.1.

3. Derivation of the optimality system and stability estimates

The next stage in the study of the control problem (2.1) is the derivation of the optimality system. It provides valuable information about additional properties of optimal solutions for specific reaction coefficients, for example, in the case when $k(\varphi, \cdot) = \varphi^2|\varphi|$. Based on its analysis, one can establish, in particular, the uniqueness and stability of the optimal solutions More details about the method for deriving estimates of local stability of optimal solutions can be found in [11–16].

Based on the theory developed in [11–16] we introduce the space $Y^* = \mathcal{T} \times H^{1/2}(\Gamma_D)^*$ dual of the space Y. It is easy to show that for the case $k(\varphi, \cdot) = \varphi^2|\varphi|$ the Fréchet derivative of the operator $F = (F_1, F_2) : H^1(\Omega) \times K \to Y$ with respect to φ at any point $(\hat\varphi, \hat u) = (\hat\varphi, \hat\lambda, \hat f, \hat\chi)$ is a linear continuous operator $F'_\varphi(\hat\varphi, \hat u) : H^1(\Omega) \to Y$ that maps each element $\tau \in H^1(\Omega)$ into an element $F'_\varphi(\hat\varphi, \hat u)(\tau) = (\hat y_1, \hat y_2) \in Y$. Here the elements $\hat y_1 \in \mathcal{T}^*$ and $\hat y_2 \in H^{1/2}(\Gamma_D)$ are defined by $\hat\varphi, \hat\lambda$ and τ with the help of the following relations:

$$\langle \hat y_1, h\rangle = (\hat\lambda \nabla\tau, \nabla h) + 4(\hat\varphi^2|\hat\varphi|\tau, h) + (\hat\lambda\alpha\tau, h)_{\Gamma_N} + (\mathbf{u}\cdot\nabla\tau, h) \ \ \forall\, h \in \mathcal{T},\ y_2 = \tau|_{\Gamma_D}. \tag{3.1}$$

By $F'_\varphi(\hat\varphi, \hat u)^* : Y^* \to H^1(\Omega)^*$ we denote an operator adjoint of $F'_\varphi(\hat\varphi, \hat u)$.

According to the general theory of smooth-convex extremum problems [23], we introduce an element $\mathbf{y}^* = (\theta, \zeta) \in Y^*$, to which we will refer as to an adjoint state and we will define the Lagrangian $\mathcal{L} : H^1(\Omega) \times K \times Y^* \to \mathbb{R}$ by

$$\mathcal{L}(\varphi, u, \mathbf{y}^*) = J(\varphi, u) + \langle \mathbf{y}^*, F(\varphi, u)\rangle_{Y^*\times Y} \equiv J(\varphi, u) + \langle F_1(\varphi, u), \theta\rangle_{\mathcal{T}^*\times\mathcal{T}} + \langle \zeta, F_2(\varphi, u)\rangle_{\Gamma_D},$$

where $\langle \zeta, \cdot\rangle_{\Gamma_D} = \langle \zeta, \cdot\rangle_{H^{1/2}(\Gamma_D)^*\times H^{1/2}(\Gamma_D)}$.

Since $\hat\varphi^2|\hat\varphi| \in L^2_+(\Omega)$ then from [12] it follows that for any $f \in \mathcal{T}^*$ and $\psi \in H^{1/2}(\Gamma_D)$ there exists a unique solution $\tau \in H^1(\Omega)$ of the linear problem

$$(\hat\lambda \nabla\tau, \nabla h) + 4(\hat\varphi^2|\hat\varphi|\tau, h) + (\hat\lambda\alpha\tau, h)_{\Gamma_N} + (\mathbf{u}\cdot\nabla\tau, h) = \langle f, h\rangle\ \forall\, h \in \mathcal{T},\ \tau|_{\Gamma_D} = \psi. \tag{3.2}$$

Therefore the operator $F'_\varphi(\hat\varphi, \hat u) : H^1(\Omega) \to Y$ is an isomorphism and from [23] the following assertion follows.

Theorem 3.1. *Let, under conditions (i), (iii)–(vi) and (j), (jj), $k(\varphi, \cdot) = \varphi^2|\varphi|$, the functional $I : H^1(\Omega) \to \mathbb{R}$ is continuously differentiable with respect to φ at the point $\hat\varphi$ and let an element*

$(\hat\varphi,\hat u)\in H^1(\Omega)\times K$ be a local minimizer for the problem (2.1). Then there exists a unique Lagrange multiplier (adjoint state) $\mathbf{y}^*=(\theta,\zeta)\in Y^*$, such that the Euler–Lagrange equation $F'_\varphi(\hat\varphi,\hat u)^*\mathbf{y}^*=-J'_\varphi(\hat\varphi,\hat u)$ in $H^1(\Omega)^*$ takes place which is equivalent to the relation

$$(\hat\lambda\nabla\tau,\nabla\theta)+4(\hat\varphi^2|\hat\varphi|\tau,\theta)+(\hat\lambda\alpha\tau,\theta)_{\Gamma_N}+(\mathbf{u}\cdot\nabla\tau,\theta)+\langle\zeta,\tau\rangle_{\Gamma_D}=$$

$$=-(\mu_0/2)\langle I'_\varphi(\hat\varphi),\tau\rangle\ \ \forall\tau\in H^1(\Omega),\tag{3.3}$$

and the minimum principle $\mathcal{L}(\hat\varphi,\hat u,\mathbf{y}^*)\leqslant\mathcal{L}(\hat\varphi,u,\mathbf{y}^*)$ $\forall u\in K$ holds which is equivalent to the inequalities

$$\mu_1(\hat\lambda,\lambda-\hat\lambda)_{s,\Omega}+((\lambda-\hat\lambda)\nabla\hat\varphi,\nabla\theta)+((\lambda-\hat\lambda)\alpha\hat\varphi,\theta)_{\Gamma_N}\geqslant 0\ \forall\lambda\in K_1,\tag{3.4}$$

$$\mu_2(\hat f,f-\hat f)_\Omega-(f-\hat f,\theta)\geqslant 0\ \ \forall f\in K_2,\tag{3.5}$$

$$\mu_3(\hat\chi,\chi-\hat\chi)_{\Gamma_N}-(\chi-\hat\chi,\theta)_{\Gamma_N}\geqslant 0\ \ \forall\chi\in K_3.\tag{3.6}$$

The relations (3.3)–(3.6) together with the operator restriction $F(\hat\varphi,\hat u)=0$ comprise an *optimality system* for problem (2.1). It plays an important role in the study of uniqueness and stability of its solutions.

In conclusion, we formulate a theorem on the local stability of optimal solutions of problem (2.1) for $I(\varphi)=\|\varphi-\varphi^d\|_Q^2$, which is proved according to the scheme proposed in [11].

Theorem 3.2. *Assume that the conditions (i), (iii)–(vi) and (j), (jj) take place and $k(\varphi,\cdot)==\varphi^2|\varphi|$. Let the quadruple $(\varphi_i,\lambda_i,f_i,\chi_i)\in X\times K$ be a solution of the problem (2.1) at $I(\varphi)==\|\varphi-\varphi_i^d\|_Q^2$, which corresponds to a specified function $\varphi_i^d\in L^2(\Omega)$, $i=1,2$. Let the data of the problem (2.1) or parameters μ_0,μ_1,μ_2 and μ_3 be such that the following condition hold:*

$$\eta_1^2\mu_0\leqslant(1-\varepsilon)\mu_1,\ \ \eta_2^2\mu_0\leqslant(1-\varepsilon)\mu_2,\ \ \eta_3^2\mu_0\leqslant(1-\varepsilon)\mu_3,\tag{3.7}$$

where $\varepsilon\in(0,1)$ is an arbitrary number, the parameters η_k, $k=1,2,3,4$, monotonically depend on the norms of the initial data of the problem (2.1). Then the following local stability estimates hold:

$$\|\lambda_1-\lambda_2\|_{s,\Omega}\leqslant\sqrt{\mu_0/(\varepsilon\mu_1)}(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q,\tag{3.8}$$

$$\|f_1-f_2\|_\Omega\leqslant\sqrt{\mu_0/(\varepsilon\mu_2)}(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q,\tag{3.9}$$

$$\|\chi_1-\chi_2\|_{\Gamma_N}\leqslant\sqrt{\mu_0/(\varepsilon\mu_3)}(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q,\tag{3.10}$$

$$\|\varphi_1-\varphi_2\|_{1,\Omega}\leqslant(\omega_1\sqrt{\mu_0/(\varepsilon\mu_1)}+\omega_2\sqrt{\mu_0/(\varepsilon\mu_2)}+\omega_3\sqrt{\mu_0/(\varepsilon\mu_3)})(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q.\tag{3.11}$$

Here $\omega_1=C_*(\gamma_3^s\|\alpha\|_{\Gamma_N}M_\varphi+\gamma_sM_\varphi)$, $\omega_2=C_*$, $\omega_3=\gamma_2C_*$, where $\lambda_*,\gamma_2,\gamma_3^s,\gamma_s,C_*=1/\lambda_*$ are the constants from Lemma 1.1 and M_φ is defined in (1.19).

A similar theorem can be formulated and proved for the functional $I_2(\varphi)$ in (2.2). The authors plan to devote a separate paper to a more detailed study of the issues of uniqueness and stability of optimal solutions.

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Анализ краевых задач и задач управления для нелинейного уравнения реакции-диффузии-конвекции

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Аннотация. Доказывается глобальная разрешимость неоднородной смешанной краевой задачи и задач управления для уравнения реакции-диффузии-конвекции в случае, когда коэффициент реакции нелинейно зависит от концентрации. Для решения краевой задачи устанавливаются принципы максимума и минимума. Для задач управления с конкретными коэффициентами реакции выводятся системы оптимальности и устанавливаются оценки локальной устойчивости оптимальных решений.

Ключевые слова: нелинейное уравнение реакции-диффузии-конвекции, смешанные граничные условия, принцип максимума, задачи управления, системы оптимальности, оценки локальной устойчивости.