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Delta-extremal Functions in $\mathbb{C}^n$

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Abstract. The article is devoted to properties of a weighted Green function. We study the (δ,ψ)-extremal Green function $V_\delta^*(z,K,\psi)$ defined by the class $\mathcal{L}_\delta = \{u(z) \in psh(\mathbb{C}^n) : \ u(z) \leqslant C_u + \delta \ln^+|z|, \ z \in \mathbb{C}^n\}, \ \delta > 0$. We see that the notion of regularity of points with respect to different numbers δ differ from each other. Nevertheless, we prove that if a compact set $K \subset \mathbb{C}^n$ is regular, then δ-extremal function is continuous in the whole space $\mathbb{C}^n$.

Keywords: plurisubharmonic function, Green function, weighted Green function, δ-extremal function.

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1. Introduction and preliminaries

The Green function in the multidimensional complex space $\mathbb{C}^n$ is one of the main objects for the study of analytic and plurisubharmonic (psh) functions. The Green function was introduced and applied in the works of P. Lelong, J. Sichak, V. Zaharyuta, A. Zeriahi, A. Sadullaev and others (see [1–7]). Recall that a function $u(z) \in psh(\mathbb{C}^n)$ is said to be of logarithmic growth if there is a constant C_u such that

$$u(z) \leqslant C_u + \ln^+|z|, \quad z \in \mathbb{C}^n,$$

where $\ln^+|z| = \max\{\ln|z|, 0\}$. The family of all such functions is called the Lelong class and denoted by $\mathcal{L}$. We also introduce a class $\mathcal{L}^+$ as follows:

$$\mathcal{L}^+ := \{u(z) \in psh(\mathbb{C}^n), \quad c_u + \ln^+|z| \leqslant u(z) \leqslant C_u + \ln^+|z|\}.$$

For a fixed compact set $K \subset \mathbb{C}^n$ we put

$$V(z,K) = \sup\{u(z) : u(z) \in \mathcal{L}, u(z)|_K \leqslant 0\}.$$

Then the regularization of

$$V^*(z,K) = \overline{\lim_{w\to z}} V(w,K)$$

is called *the Green function* of the compact set K. For a non-pluripolar compact set K, the function $V^*(z,K)$ exists ($V^*(z,K) \not\equiv +\infty$) and belongs to the class $\mathcal{L}^+$. The Green function $V^*(z,K) \equiv +\infty$ if and only if K is pluripolar.

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Definition 1. *A compact set $K \subset \mathbb{C}^n$ is called globally pluri-regular at a point z_0 if $V^*(z^0, K) = 0$. It is called locally pluri-regular at a point z_0 if $V^*(z^0, K \cap B(z^0, r)) = 0$ for any ball $B(z^0, r)$), $r > 0$. A compact set K is globally pluri-regular if it is globally pluri-regular at every point of itself. A compact set K is locally pluri-regular if it is locally pluri-regular at every point of itself.*

Theorem 1.1 (see for example, J. Siciak [4], V. Zakharyuta [3])**.** *If a compact set K is globally pluriregular, then the function $V^*(z, K)$ is continuous in $\mathbb{C}^n$, and $V^*(z, K) = V(z, K)$.*

2. Weighted Green functions in $\mathbb{C}^n$

Let $\psi(z)$ be a bounded function on a compact set $K \subset \mathbb{C}^n$. Consider the class of functions

$$\mathcal{L}(K, \psi) := \{u(z) \in \mathcal{L},\ u(z)|_K \leqslant \psi(z)\}$$

and

$$V(z, K, \psi) := \sup\{u(z) : u(z) \in \mathcal{L}(K, \psi)\},\ z \in \mathbb{C}^n.$$

Then $V^*(z, K, \psi) = \overline{\lim_{w \to z}} V(w, K, \psi)$ is said to be a weighted Green function of K with respect to $\psi(z)$. Note that in the case $\psi(z) \equiv 0$ the function $V^*(z, K, \psi)$ coincides with the Green function $V^*(z, K)$, i.e., $V^*(z, K, 0) \equiv V^*(z, K, \psi)$. Extremal weighted Green functions are the subject of study by many authors (see [7, 10–13]). They are successfully applied in multidimensional complex analysis, in the approximation theory of functions, in multidimensional complex dynamical systems etc.

It is clear that for any compact set $K \subset \mathbb{C}^n$ we have the inequality

$$V^*(z, K) + \min_K \psi(z) \leqslant V^*(z, K, \psi) \leqslant V^*(z, K) + \max_K \psi(z). \tag{1}$$

If a function $\psi(z)$ extends to the space $\mathbb{C}^n$ as a function from the class $\mathcal{L}$, i.e. if there is a function

$$\Psi \in \mathcal{L} :\ \Psi|_K \equiv \psi, \tag{2}$$

then it is obvious $V(z, K, \psi) \geqslant \Psi(z)$ and

$$V(z, K, \psi) = \psi(z)\ \ \forall z \in K. \tag{3}$$

However, if the condition (2) is not met, then generally speaking, the equality (3) is not true.

Example 1. Let $K = \{|z| \leqslant 1\} \subset \mathbb{C}$ and $\psi(z) = 1 - |z|^2$. Then by the maximum principle

$$V(z, K, \psi) = V(z, K) = V(z, K) = \ln^+ |z|.$$

Therefore, $V(z, K, \psi) = 0 < \psi(z)\ \ \forall \mid z \mid < 1$.

According to this example, in order to introduce the concept of regularity, below we assume that the Green function satisfies the condition (3).

Definition 2. *We say that a compact set K is globally ψ-regular at z^0 if $V^*(z^0, K, \psi) = \psi(z^0)$. We say that a compact set K is locally ψ-regular at z^0 if $V^*(z^0, K \cap B(z^0, r), \psi) = \psi(z^0)$ for every ball $B(z^0, r)$, $r > 0$.*

A. Sadullaev [7] proved the following theorem.

Theorem 2.1. *Let K be a compact set, and $\psi(z)$ is a weight on K such that there exists a strictly plurisubharmonic function*

$$\Psi \in \mathcal{L} \cap C^2(\mathbb{C}^n): \quad dd^c\Psi > 0, \ \Psi|_K = \psi. \tag{4}$$

Then K is locally ψ-regular at $z^0 \in K$ if and only if K is globally ψ-regular at z^0.

Note that Theorem 2.1, generally speaking, is not true if Ψ is not a strictly plurisubharmonic function. For the weight function $\psi(z) \equiv 0$ and for the compact set $K = \{|z| = 1\} \cup \{z = 0\} \subset \mathbb{C}$ the point $z = 0$ is globally regular, but it is not locally regular. In this example K is not polynomially convex $\hat{K} \neq K$. In the work [5] A. Sadullaev constructed the following interesting example.

Example 2. The compact set $K = K_1 \cup K_2 \subset \mathbb{C}^2(z_1, z_2)$, where $K_1 = \{|z_1| < 1, z_2 = 0\}$, $K_2 = \{z_1 = e^{i\varphi}, Rez_2 = 0,\ 0 \leqslant Imz_2 \leqslant e^{\frac{1}{\cos\varphi - 1}}, -\pi \leqslant \varphi \leqslant \pi\}$, has the following properties:

a) K is polynomially convex, i.e., $\hat{K} = K$;

b) K is globally pluri-regular, i.e., $V^*(z, K) = 0, \forall z \in K$;

c) K is not locally pluri-regular at the points $z \in K_1$.

In connection with this example and with Theorem 2.1, the following problem arises (see [7]).

Problem 1. *Let K be a compact set in $\mathbb{C}^n$. Under a weaker condition that the weight function $\psi(z)$ continues only to a neighbourhood $U \supset K$ as a strictly plurisubharmonic function, prove that K is locally ψ-regular at $z_0 \in K$ if and only if K is globally ψ-regular at $z_0 \in K$.*

The following theorem relates to local regularity for different weight functions.

Theorem 2.2. *Let K be a compact set, and $\psi(z)$ is a weight on $K : \psi(z) \in C(K)$. Then K is locally ψ-regular at $z^0 \in K$ if and only if K is locally regular (case $\psi \equiv 0$) at z^0.*

Proof. Indeed, we use the inequality (1). If the point $z^0 \in K$ is not locally pluri-regular, i.e., if $V^*(z^0, K \cap \overline{B}) = \sigma > 0$ for some neighborhood $B: \quad z^0 \in B \subset \mathbb{C}^n$, then $V^*(z^0, K \cap \overline{B}_1) \geqslant \sigma$ for any $z^0 \in B_1 \subset B$. Therefore, by (1)

$$V^*(z^0, K \cap B_1, \psi) \geqslant V^*(z^0, K \cap B_1) + \min_{K \cap B_1} \psi(z) \geqslant \sigma + \min_{K \cap B_1} \psi(z). \tag{5}$$

Since $\psi(z)$ is continuous, choosing the neighborhood B_1 small enough we can make the right part of (5) to be greater than $\psi(z^0)$ i.e., $V^*(z, K \cap B_1, \psi) > \psi(z^0)$ and the point z^0 is not locally ψ-regular.

Reversing the roles of $V^*(z, K \cap B_1, \psi)$ and $V^*(z, K \cap B_1)$ from (1) we can prove the second part of the theorem: if the point $z^0 \in K$ is not locally ψ-regular, then it is not locally pluri-regular. □

It should be noted here that the conditions of continuity of the function $\psi(z)$ in Theorem 2.2 is essential. An example is given in [15], when the function $\psi(z)$ is discontinuous, Theorem 2.2 is false, i.e., some point $z^0 \in K \subset \mathbb{C}$ is a ψ-regular point, but it is not pluri-regular.

3. δ-extremal functions

Let $K \subset \mathbb{C}^n$ be a compact set and $\psi(z)$ be some bounded function on K. Consider the following generalization of the Lelong class

$$\mathcal{L}_\delta := \{u(z) \in psh(\mathbb{C}^n)\colon u(z) \leqslant C_u + \delta \ln^+ |z|,\ z \in \mathbb{C}^n\},\ \delta > 0.$$

It is clear that if $v(z) \in \mathcal{L}$, then $c \cdot v(z) \in \mathcal{L}_\delta$, where $0 < c \leqslant \delta$. Put

$$\mathcal{L}_\delta(K, \psi) := \{u(z) \in \mathcal{L}_\delta,\ u(z)|_K \leqslant \psi(z)\}.$$

Definition 3. *The function $V^*_\delta(z, K, \psi) = \overline{\lim_{w\to z}} V_\delta(w, K, \psi)$ is called a δ-extremal function of K with respect to $\psi(z)$, where*

$$V_\delta(z, K, \psi) := \sup\{u(z) : u(z) \in \mathcal{L}_\delta(K, \psi)\},\quad z \in \mathbb{C}^n.$$

We list simple properties of δ-extremal functions:

1°. If $\delta_1 \leqslant \delta_2$, then $V_{\delta_1}(z, K, \psi) \leqslant V_{\delta_2}(z, K, \psi)$.

2°. If $\psi_1 \leqslant \psi_2, \forall z \in K$, then $V_\delta(z, K, \psi_1) \leqslant V_\delta(z, K, \psi_2)$.

3°. $V_\delta(z, K, \psi) = \delta V\left(z, K, \frac{\psi}{\delta}\right)$, in particular $V_\delta(z, K) = \delta V(z, K)$.

4°. $V_\delta(z, K, \psi + c) = c + V_\delta(z, K, \psi)$, $\forall c \in \mathbb{R}$.

If a function $\psi(z)$ extends to the space $\mathbb{C}^n$ as a function from the class $\mathcal{L}_\delta$, i.e. if there is a function

$$\Psi \in \mathcal{L}_\delta :\ \Psi|_K \equiv \psi, \tag{6}$$

then it is obvious $V_\delta(z, K, \psi) \geqslant \Psi(z)$ and

$$V_\delta(z, K, \psi) = \psi(z)\ \ \forall z \in K. \tag{7}$$

However, if the condition (6) is not met, then generally speaking, the equality (7) is not true. In this section, as above we assume that the Green function $V_\delta(z, K, \psi)$ satisfies the condition (7). For such a function ψ we can introduce the concept of (δ, ψ)-regularity.

Definition 4. *We say that a compact set K is globally (δ, ψ)-regular at z^0 if $V^*_\delta(z^0, K, \psi) = \psi(z^0)$. We say that a compact set K is locally (δ, ψ)-regular at z^0 if $V^*_\delta(z^0, K \cap B(z^0, r), \psi) = \psi(z^0)$ for any ball $B(z^0, r), r > 0$.*

The following theorem is proved similarly to the proof of Theorem 2.2 and we omit it.

Theorem 3.1. *Let K be a compact set and $\psi(z)$ is a weight on $K : \psi(z) \in C(K)$, $V_\delta(z, K, \psi) = \psi(z)$ $\forall z \in K$. Then K is locally (δ, ψ)-regular at $z^0 \in K$ if and only if K is locally $(\delta, 0)$-regular at z^0.*

Similarly to Theorem 1.1 the continuity of the δ-extremal function takes place.

Theorem 3.2. *Let $\psi(z)$ be continuous on K. If K is globally (δ, ψ)-regular i.e. if K is globally (δ, ψ)-regular at a point $z^0 \in K$, then $V^*_\delta(z, K, \psi) = V_\delta(z, K, \psi)$ and $V^*_\delta(z, K, \psi)$ is continuous in $\mathbb{C}^n$.*

Proof. Let $\psi(z)$ be a function defined and continuous on K. It is well known that $\psi(z)$ can be extended continuously to K, i.e., there is a function $\Psi(z) \in C(\mathbb{C}^n)$ such that $\Psi(z)|_K = \psi(z)$ (see Whitney H. [8]). We use the standard approximation $u_j \downarrow V^*_\delta(z, K, \psi)$, where $u_j \in \mathcal{L}_\delta \cap C^\infty(\mathbb{C}^n)$. Since $V^*_\delta(z, K, \psi) \equiv \Psi(z),\ z \in K$, for any $\varepsilon > 0$ there is an open set $\{z \in \mathbb{C}^n,\ V^*_\delta(z, K, \psi) < \Psi(z) + \varepsilon\}$ contained K. Therefore, by the Hartogs lemma, there exists $j_0 \in \mathbb{N}$ such that $u_j(z) < \Psi(z) + 2\varepsilon = \psi(z) + 2\varepsilon,\ \forall z \in K,\ j > j_0$. From here, $u_j - 2\varepsilon \in \mathcal{L}_\delta(\psi, K)$ and

$$u_j - 2\varepsilon \leqslant V_\delta(z, K, \psi) \leqslant V^*_\delta(z, K, \psi) \leqslant u_j, \quad j > j_0, \quad z \in \mathbb{C}^n.$$

This means that the sequence u_j converges to $V^*_\delta(z, K, \psi)$ uniformly and $V^*_\delta(z, K, \psi) = V_\delta(z, K, \psi) \in C(\mathbb{C}^n)$. □

In the case when $\delta = 1$ and $\psi(z)$ continues throughout $\mathbb{C}^n$ as a continuous function of the class $\mathcal{L}$, Theorem 3.2 was proved by A. Sadullaev.

4. δ-extremal functions for different δ

Note that in the general case $V_\delta(z, K, \psi)$ and the weight function ψ do not have to be equal on K for all δ. In other words, the condition (7) may not be satisfied.

Example 3 (see Alan [10]). Let $K = \overline{B(0,1)}$ and $\psi(z) = |z|^2$. Then one can prove that

$$V_\delta(z, K, \psi) = \begin{cases} |z|^2, & |z| \leqslant \sqrt{\dfrac{\delta}{2}}, \\ \delta \ln|z| + \dfrac{\delta}{2} - \dfrac{\delta}{2}\ln\left|\dfrac{\delta}{2}\right|, & |z| > \sqrt{\dfrac{\delta}{2}}. \end{cases}$$

We see $V_\delta(z, K, \psi) = |z|^2,\ \forall z \in \left\{|z| \leqslant \sqrt{\dfrac{\delta}{2}}\right\}$ and $V_\delta(z, K, \psi) < |z|^2,\ \forall z \in \left\{\sqrt{\dfrac{\delta}{2}} < |z| \leqslant 1\right\}$.

We denote by $\Lambda = \Lambda(K, \psi)$ the set of numbers δ for which the equality of type (7) holds, i.e.

$$\Lambda = \Lambda(K, \psi) = \{\delta > 0 : V_\delta(z, K, \psi)|_K \equiv \psi(z)\}.$$

For Alan's example, $\Lambda = [2, +\infty)$. In fact,

$$V_2(z, K, \psi) = \begin{cases} |z|^2, & |z| \leqslant 1, \\ 2\ln|z| + 1, & |z| > 1. \end{cases}$$

So, $V_2(z, K, \psi)|_K \equiv \psi(z)$ and by property 1° from Section 3 $V_\delta(z, K, \psi) \geqslant V_2(z, K, \psi)$ for all $\delta \in [2, +\infty)$. If $\delta \in (0, 2)$ then there is a point $z^0 \in K$ such that $V_\delta(z^0, K, \psi) < \psi(z^0)$, that is $(0, 2) \cap \Lambda = \emptyset$.

The sets Λ may be empty. For example, for $K = \{|z| \leqslant 1\} \subset \mathbb{C}$ and $\psi(z) = 1 - |z|^2$, by property 3° we have

$$V_\delta(z, K, \psi) = V_\delta(z, K) = \delta V(z, K) = \delta \ln^+ |z|.$$

Therefore, for any $\delta > 0$, $V_\delta(z, K, \psi) < \psi(z)$, $\forall |z| < 1$. That is, in this case $\Lambda = \emptyset$.

If $\psi(z) \equiv c$, where c is a constant, then $V_\delta(z, K, c) = c + V_\delta(z, K) = c + \delta V(z, K)$. Since the Green function $V(z, K) \geqslant 0,$, for any $\delta > 0$ and $z \in K$ the equality $V_\delta(z, K, c) = c$ holds. This means that $\Lambda = (0, +\infty)$.

Let $\Lambda \neq \emptyset$. If $\delta \in \Lambda$, then from property 1° we easily get $\delta_1 \in \Lambda$ for $\delta_1 > \delta$. On the other hand

Proposition 1. *If* $\delta_j \in \Lambda,\ \forall j \in \mathbb{N}$ *and* $\delta_j \downarrow \delta_0 \neq 0$ *as* $j \to \infty$ *then* $\delta_0 \in \Lambda$.

Proof. Indeed, by the hypothesis we have $V_{\delta_j}(z, K, \psi) = \psi(z), z \in K$. Using properties 2° and 3°, we get

$$V_{\delta_j}(z, K, \psi) = \delta_j V\big(z, K, \tfrac{\psi}{\delta_j}\big) \leqslant \delta_j V\big(z, K, \tfrac{\psi}{\delta_0}\big).$$

Consequently, $\forall j \in \mathbb{N}$ we have $\psi(z) = V_{\delta_j}(z, K, \psi) \leqslant \delta_j V\big(z, K, \frac{\psi}{\delta_0}\big),\ z \in K$. As j tends to infinity, we get

$$\psi(z) \leqslant \delta_0 V\big(z, K, \tfrac{\psi}{\delta_0}\big) = V_{\delta_0}(z, K, \psi), \quad z \in K,$$

i.e. $\psi(z) = \delta_0 V\big(z, K, \frac{\psi}{\delta_0}\big) = V_{\delta_0}(z, K, \psi),\ z \in K$ and $\delta_0 \in \Lambda$. □

Proposition 1 follows, if $\Lambda \neq \emptyset$ then $\Lambda = (0, \infty)$ or $\Lambda = [\delta_0, +\infty), \delta_0 > 0$. Note that if $\delta \in \Lambda(K, \psi)$, then $V_\delta(z, K, \psi) = \psi(z),\ z \in K$. Therefore, by monotonicity $V_\delta(z, K \cap \bar{B}, \psi) = \psi(z),\ z \in K \cap \bar{B}$, for any ball $B \cap K \neq \emptyset$. It follows that if $\delta \in \Lambda(K, \psi)$, then $\delta \in \Lambda(K \cap B, \psi)$.

Definition 5. *Let* $\delta \in \Lambda(K)$. *A compact set* K *is called globally* (δ, ψ)*-regular at a point* $z^0 \in K$ *if* $V_\delta^*(z^0, K, \psi) = \psi(z^0)$. *It is called locally* (δ, ψ)*-regular at a point* $z^0 \in K$ *if for every nonempty ball* $B(z^0, r):\ V_\delta^*(z^0, K \cap \bar{B}(z^0, r), \psi) = \psi(z^0)$. *A compact set* K *is globally* (δ, ψ)*-regular if it is globally* (δ, ψ)*-regular at every point of itself. A compact* K *is locally* (δ, ψ)*-regular if it is locally* (δ, ψ)*-regular at every point of itself.*

Note that global or local (δ, ψ)-regularity can only be defined for $\delta \in \Lambda$. It is easy to see that any locally (δ, ψ)-regular point is globally (δ, ψ)-regular. We denote by $\Lambda_{reg} = \Lambda_{reg}(K, \psi)$ the set of numbers $\delta \subset \Lambda$, for which K is globally regular, we denote by $\Lambda_{reg}^{loc} = \Lambda_{reg}^{loc}(K, \psi)$ the set of numbers $\delta \subset \Lambda$, for which K is locally regular. We see, $\Lambda_{reg}^{loc} \subset \Lambda_{reg} \subset \Lambda$.

Proposition 2. *Let* $\delta_1, \delta_2 \in \Lambda$ *and* $\delta_1 \leqslant \delta_2$. *If a point* z^0 *is* (δ_2, ψ)*-regular, then it is* (δ_1, ψ)*-regular.*

The proof follows from property 1° of Section 3. For a continuous function ψ there holds

Theorem 4.1. *Let* $\delta \in \Lambda$, *and a function* $\psi(z)$ *be continuous on* K. *Then a fixed point* $z^0 \in K \subset \mathbb{C}^n$ *is locally* (δ, ψ)*-regular if and only if it is locally pluri-regular.*

Proof. We show that for any compact set $K \subset \mathbb{C}^n$ the following is true:

$$\delta V^*(z, K) + \min_K \psi(z) \leqslant V_\delta^*(z, K, \psi) \leqslant \delta V^*(z, K) + \max_K \psi(z). \tag{8}$$

In fact, if $u \in \mathcal{L}_\delta(K, \psi)$, i.e., $u \in \mathcal{L}_\delta,\ \ u|_K \leqslant \psi$, then

$$u(z) - \max_K \psi(z) \in \mathcal{L}_\delta(K).$$

Therefore

$$u(z) - \max_K \psi(z) \leqslant V_\delta^*(z, K)$$

and

$$V_\delta^*(z, K, \psi) - \max_K \psi(z) \leqslant V_\delta^*(z, K) = \delta V^*(z, K), \quad \forall z \in \mathbb{C}^n.$$

Conversely, if $u \in \mathcal{L}_\delta(K)$, then $u(z) + \min_K \psi(z) \in \mathcal{L}_\delta(K, \psi)$. Therefore,

$$V_\delta^*(z, K) + \min_K \psi(z) = \delta V^*(z, K) + \min_K \psi(z) \leqslant V_\delta^*(z, K, \psi),$$

so that (8) holds.

Using (8) we can now prove the theorem. If a fixed point $z^0 \in K$ is not locally pluri-regular, i.e., if $V^*(z^0, K \cap \overline{B}) = \sigma > 0$ for some neighborhood $B: \ z^0 \in B \subset \mathbb{C}^n$, then $V^*(z^0, K \cap \overline{B}_1) \geqslant \sigma$ for any $z^0 \in B_1 \subset B$. Therefore, by (8)

$$V_\delta^*(z^0, K \cap B_1, \psi) \geqslant \delta V^*(z^0, K \cap B_1) + \min_{K \cap B_1} \psi(z) \geqslant \delta\sigma + \min_{K \cap B_1} \psi(z). \tag{9}$$

Since $\psi(z)$ is continuous, choosing a neighborhood B_1 small enough we can make the right part of (9) to be greater than $\psi(z^0)$ i.e., $V_\delta^*(z, K \cap B_1, \psi) > \psi(z^0)$. This means that the point z^0 is not locally (δ, ψ)-regular.

Reversing the roles of $V_\delta^*(z, K \cap B_1, \psi)$ and $V^*(z, K \cap B_1)$ from (8) we can prove the second part of the theorem: if a point $z^0 \in K$ is not locally (δ, ψ)-regular, then it is not locally pluri-regular. □

Corollary 1. *Let $\delta_1, \delta_2 \in \Lambda$ and a function $\psi(z)$ be continuous on K. Then a fixed point $z^0 \in K \subset \mathbb{C}^n$ is locally (δ_1, ψ)-regular if and only if it is locally (δ_2, ψ)-regular.*

Proposition 3. *If $\delta_j \in \Lambda_{reg}, \ \forall j \in \mathbb{N}$ and $\delta_j \uparrow \delta$ as $j \to \infty$, then $\delta \in \Lambda_{reg}$.*

Proof. In fact, since $\psi(z) = V_{\delta_j}^*(z, K, \psi), z \in K$, we get

$$\psi(z) = V_{\delta_j}^*(z, K, \psi) = \delta_j V^*\big(z, K, \tfrac{\psi}{\delta_j}\big) \geqslant \delta_j V^*\big(z, K, \tfrac{\psi}{\delta}\big).$$

Therefore, $\forall j \in \mathbb{N}$ we have $\psi(z) \geqslant \delta_j V^*\big(z, K, \frac{\psi}{\delta}\big), \ z \in K$. As j tends to infinity, we get

$$\psi(z) \geqslant \delta V^*\big(z, K, \tfrac{\psi}{\delta}\big) = V_\delta^*(z, K, \psi), \ z \in K.$$

This means that $\delta \in \Lambda_{reg}$. □

Corollary 2. *If $\Lambda = [\delta_0, \infty)$, then* $\Lambda_{reg} = \begin{cases} or & [\delta_0, \delta_1] \\ or & [\delta_0, \infty). \end{cases}$

Corollary 3. *If $\Lambda = (0, \infty)$, then* $\Lambda_{reg} = \begin{cases} or & (0, \delta_1] \\ or & (0, \infty). \end{cases}$

In the paper [10] M. Alan studied the concepts of (δ, ψ)-regularity and posed the following problem

Problem 2 ([10]). *Let K be a compact set in $\mathbb{C}^n$, $\psi(z)$ extends to $\mathcal{L}_{\delta_1}^+$ (see (6)) and $0 < \delta_1 < \delta_2$. If K is (δ_1, ψ)-regular at $z_0 \in K$, then K is (δ_2, ψ)-regular at z_0.*

5. The property of (δ, ψ)-regularity

Further properties of δ-extremal function are associated with pluri-thin sets.

Definition 6. *Let $E \subset \mathbb{C}^n$ and let E' be its limit point set. Then E is said to be pluri-thin at z^0 if either $z^0 \notin E'$ or $z^0 \in E'$ but there exists a neighbourhood U of z^0 and a function $u(z) \in psh(U)$ such that*

$$\overline{\lim_{\substack{z \to z^0 \\ z \in E \setminus \{z^0\}}}} u(z) < u(z^0).$$

So, if the set E is not thin at the point z^0, then for any plurisubharmonic function $u(z)$ in the neighborhood of z^0

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}} u(z) = \overline{\lim_{\substack{z\to z^0\\ z\in E}}} u(z) = u(z^0).$$

Proposition 4 ([16])**.** *If $E \subset \mathbb{C}^n$ is pluri-thin at a limit point z^0 of E, then there exists a plurisubharmonic function $u \in \mathcal{L}^+$ such that*

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}} u(z) = -\infty < u(z^0).$$

Theorem 5.1. *If z^0 is a pluri-thin point of K, then z^0 is locally (δ, ψ)-irregular point of K. Here the function $\psi \in L^\infty(K)$ and $\delta \in \Lambda$.*

Proof. Let K be pluri-thin at the point $z^0 \in K$. Then, according to Proposition 4, there exists a function $u(z) \in \mathcal{L}_\delta$ such that

$$\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}} u(z) = -\infty < u(z^0).$$

Without loss of generality, we can assume $u(z^0) > 0$ and find a ball $B(z^0, r)$ such that

$$\begin{cases} u(z) \leqslant \inf\limits_{z\in K} \psi(z) - \psi(z^0) \text{ for } z \in K \cap B \setminus \{z^0\}, \\ u(z^0) > 0. \end{cases}$$

Put $w(z) = u(z) + \psi(z^0)$. It is easy to see that $w(z) \in L_\delta(\psi, K \cap B \setminus \{z^0\})$, because for $z \in K \cap B \setminus \{z^0\}$

$$w(z) = u(z) + \psi(z^0) \leqslant \inf_{z\in K} \psi(z) - \psi(z^0) + \psi(z^0) = \inf_{z\in K} \psi(z) \leqslant \psi(z).$$

Consequently,

$$w(z) \leqslant V_\delta^*(z, K \cap B \setminus \{z^0\}, \psi) = V_\delta^*(z, K \cap B, \psi), \ \forall z \in \mathbb{C}^n.$$

From here

$$w(z^0) \leqslant V_\delta^*(z^0, K \cap B, \psi).$$

On the other hand

$$w(z^0) = u(z^0) + \psi(z^0) > \psi(z^0).$$

Therefore

$$\psi(z^0) < w(z^0) \leqslant V_\delta^*(z^0, K \cap B, \psi).$$

Hence, the point z^0 is a locally (δ, ψ) irregular point of the compact set K. □

Note that if $n > 1$, the necessary condition of Theorem 5.1, generally speaking, is not true.

Example 4. Let $(\delta, \psi) = (1, 0)$ and $K = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z| \leqslant 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 \colon z_2 = 0, |z_1| \leqslant 2\}$.

The compact set K is a union of the unit ball in $\mathbb{C}^2$ and a pluripolar set. We have

$$V(z, K) = \begin{cases} \ln^+ |z| & \text{for } z_2 \neq 2 \\ \ln^+ \left|\dfrac{z_1}{2}\right| & \text{for } z_2 = 0 \end{cases}$$

and

$$V^*(z, K) = \ln^+ |z|.$$

A point $(2, 0) \in K$ is an irregular point, but it is not pluri-thin.

References

[1] P.Lelong, Plurisubharmonic functions and positive differential forms, Gordon and Breach, New York, 1968.

[2] V.P.Zakharyuta, Bernstein's theorem on optimal polynomial approximations to holomorphic and harmonic functions of several variables, All-Union Conf. Theory of functions, Khar'kov, 1971, 80–81.

[3] V.P.Zakharyuta, Extremal plurisubharmonic functions, orthogonal polynomials, and the Bernstein–Walsh theorem for analytic functions of several complex variables, *Ann. Polon. Math.*, **33**(1976), 137–148.

[4] J.Siciak, Extremal plurisubharmonic functions in $\mathbb{C}^n$, Proceedings of the First Finnish – Polish Summer School in complex Analysis at Podlesice, Lods: University of Lods, 1977, 115–152.

[5] A.Sadullaev, Plurisubharmonic measures and capacities on complex manifolds, *Russian Math. Surveys*, **36**(1981), no. 4, 53–105.

[6] A.Sadullaev, Pluripotential theory. Applications, Palmarium Academic Publishing, 2012 (in Russian)

[7] A.Sadullaev, Pluriregular compacts in $\mathbb{P}^n$, *Contemporary mathematics. AMS*, **662**(2016), 145–156.

[8] H.Whitney, Analytic extensions of differentiable functions defined in closed sets, *Trans. Amer. Math. Soc.*, **36**(1934), 63–89.

[9] A.Zeriahi, Fonction de Green pluricomplexe a pole a l'infini sur un espace de Stein parabolique, *Math. Scand.*, **69**(1991), 89–126.

[10] M.A.Alan, Weighted regularity and two problems of Sadullaev on weighted regularity, *Complex Analysis and its Synergies. Springer*, **8**(2019), no. 5.
DOI: 10.1007/s40627-019-0026-4

[11] M.A.Alan, N. Gökhan Göğüş, Supports of weighted equilibrium measures: complete characterization, Potential Analysis, **39**(2013), no. 4, 411–415 (SCI).

[12] T.Bloom, N.Levenberg, Weighted pluripotential theory in $\mathbb{C}^N$, *Amer. J. Math.*, **125**(2003), no. 1, 57–103.

[13] T.Bloom, Weighted polynomials and weighted pluripotential theory, *Trans. Amer. Math. Soc.*, **361**(2009), no. 4, 2163–2179.

[14] M. Klimek, Pluripotential theory, Oxford University Press, New York, 1991.

[15] N.X.Narzillaev, About ψ-regular points, *Tashkent. Acta NUUz*, **2**(2017), no. 2, 173–175 (in Russian).

[16] E.Bedford, B.A.Taylor, A new capacity for plurisubharmonic functions, *Acta Mathematica*, **149**(1982), 1–41.

Дельта-экстремальная функция в пространстве $\mathbb{C}^n$

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Аннотация. В этой статье мы изучаем (δ,ψ)-экстремальную функцию Грина $V^*_\delta(z,K,\psi)$, которая определяется при помощи класса $\mathcal{L}_\delta = \left\{u(z) \in psh(\mathbb{C}^n) : \ u(z) \leqslant C_u + \delta \ln^+ |z|, \ z \in \mathbb{C}^n\right\}, \ \delta > 0$. Покажем, что понятие регулярности точек для разных δ не совпадают. Тем не менее мы доказываем, что если компакт $K \subset \mathbb{C}^n$ регулярен, то δ-экстремальная функция Грина непрерывна во всем пространстве $\mathbb{C}^n$.

Ключевые слова: плюрисубгармонические функции, экстремальная функция Грина, функция Грина с весом, δ-экстремальная функция.