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Determination of a Multidimensional Kernel in Some Parabolic Integro–differential Equation

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Abstract. A multidimensional parabolic integro-differential equation with the time-convolution integral on the right side is considered. The direct problem is represented by the Cauchy problem for this equation. The inverse problem is studied in this paper. The problem consists in finding the time and spatial dependent kernel of the equation from the solution of direct problem in a hyperplane $x_n = 0$ for $t > 0$. This problem is reduced to the more convenient inverse problem with the use of the resolvent kernel. The last problem is replaced by the equivalent system of integral equations with respect to unknown functions. The unique solvability of the direct and inverse problems is proved with use of the principle of contraction mapping.

Keywords: integro-differential equation, inverse problem, Hölder space, kernel, resolvent.

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1. Introduction. Formulation of problem

Integro–differential equations of convolution type arise in mathematical models of physical, biological, engineering systems and in other areas where it is necessary to take into consideration the prehistory of processes. Constitutive relations in the linear non-homogeneous diffusion and wave propagation processes with memory contain time- and space-dependent memory kernel. Often in practice these kernels are unknown functions. Problems of memory kernels identification in parabolic and hyperbolic integro–differential equations have been intensively studied (see e.g., [1–4]).

In many cases equations that describe propagation of electrodynamic and elastic waves with integral convolution are reduced to one second-order hyperbolic integro–differential equation. Various problems of recovering the kernel of convolution integral in these equations were investigated [1–12]. Determination of time- and space-dependent kernels in parabolic integro-differential equations with several additional conditions was considered by many authors (see e.g., [13–23]). Existence, uniqueness and stability theorems were proved. The linear inverse source and nonlinear inverse coefficient problems for parabolic integro-differential equations were discussed [18–23]. A numerical approach for solving this kind of problems was also applied.

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We consider an inverse problem of determining functions $u(x,t)$, $k(x',t)$, $x = (x_1, x_2, \ldots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n$, $t > 0$ that satisfy the following equations

$$u_t - Lu = -\int_0^t k(x',\tau)Lu(x,t-\tau)d\tau, \quad (x,t) \in \mathbb{R}^n_T, \tag{1.1}$$

$$u\Big|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

$$u\Big|_{x_n=0} = f(x',t),\ 0 \leqslant t \leqslant T,\ f(x',0) = \varphi(x',0), \tag{1.3}$$

where $Lu = \triangle u + c(x)u$, $\triangle$ is the Laplace operator with respect to spatial variables $x = (x_1, \ldots, x_n)$ and $\mathbb{R}^n_T = \{(x,t) | x = (x', x_n) \in \mathbb{R}^n,\ 0 < t < T\}$ is a strip with thickness T, $T > 0$ is an arbitrary fixed number. There are a heat conduction operator on the left side of equation (1.1)that acts on function $u(x;t)$ and a convolution type integral on the right side of the equation. In fact, if kernel k of the integral in equation (1.1) is known then the problem of finding function u with condition (1.2) is called the direct problem. Note that direct problem in this case is the Cauchy problem for equation (1.1).

Since after finding kernel k the solution of the direct problem becomes known then inverse problem (1.1)–(1.3) is said to be the problem of finding functions u and k.

Problems that are close to problem (1.1)–(1.3) were considered in [15]–[17]. The uniqueness theorem for the solution of kernel determination problem for one-dimensional heat conduction equation was proven [15]. Inverse problems with kernel depending on time and $(n-1)$-dimensional spatial variable x' were considered in [16, 17]. Determination of the kernel which is convolution of elliptic operator and the solution of the direct problem is of great interest in applications. One of these integro-differential equations for which the inverse problem is posed is considered in this paper.

Let us consider the inverse problem. It is required to find kernel k in the integral of equation (1.1) if condition (1.3) is known for the solution of the direct problem. Function φ in condition (1.2) and function f in condition (1.3) are called the data of direct and inverse problems, respectively. Condition (1.3) is the condition of compatibility for given functions.

In what follows, Hölder space $H^l(Q)$ with exponent l is used for functions that depend on spatial variables and Hölder space $H^{l,l/2}(Q_T)$ with exponents l and $l/2 is used for functions that depend on spatial and time variables.$

Everywhere in this paper, we assume that $\varphi(x) \in H^{l+8}(\mathbb{R}^n)$, $\varphi(x) \geqslant \varphi_0 = const > 0$, $c(x) \in H^{l+4}(\mathbb{R}^n)$, $f(x',t) \in H^{l+6,(l+6)/2}\left(\bar{\mathbb{R}}^{n-1}_T\right)$,

$$\bar{\mathbb{R}}^{n-1}_T = \left\{(x',t)\Big| x' \in \mathbb{R}^{n-1},\ 0 \leqslant t \leqslant T\right\},\ l \in (0,1),$$

spaces $H^l(Q)$, $H^{l,l/2}(Q_T)$ and norms in them are defined in [24, pp. 18–27]. In what follows, for norm of functions in space $H^{l,l/2}(Q_T)$ ($Q_T = \mathbb{R}^n_T$ or $Q_T = \mathbb{R}^{n-1}_T$) that depend on spatial and time variables notation $|\cdot|^{l,l/2}_T$ is used, and for functions that depend only on spatial variables the notation $|\cdot|^l$ is used (in this case $Q = \mathbb{R}^n$ or $Q = \mathbb{R}^{n-1}$).

To begin with we prove the following lemma:

Lemma 1.1. *Let $k(x',t) \in H^{l+2,(l+2)/2}\left(\bar{\mathbb{R}}^{n-1}_T\right)$. Then problem (1.1)–(1.3) is equivalent to the problem of finding functions $u(x,t)$, $r(x',t)$ that satisfy equation*

$$u_t(x,t) = Lu - \int_0^t r(x',t-\tau)u_\tau(x,\tau)d\tau, \tag{1.4}$$

and conditions (1.2), (1,3), where $r(x',t)$ is resolvent of kernel $k(x',t)$.

Proof. Let $u(x,t)$ be the solution of Cauchy problem (1.1), (1.2). Let us note that equation (1.1) at fixed x can be considered as Volterra integral equation of the second kind with kernel $k(x',t)$ with respect to the operator Lu:

$$Lu = \int_0^t k(x', t-\tau)Lu(x,\tau)d\tau + u_t.$$

It follows from the general theory of integral equations (see e.g. [25, pp. 39–44]) that the solution of this equation has the form

$$Lu = u_t(x,t) + \int_0^t r(x', t-\tau)u_\tau(x,\tau)d\tau.$$

This equality leads to equation (1.4). In equation (1.4) kernels $k(x',t)$ and $r(x',t)$ are related by the formula

$$k(x',t) = r(x',t) - \int_0^t r(x', t-\tau)k(x',\tau)d\tau. \tag{1.5}$$

To verify this formula we substitute (1.5) into (1.1). Then we have

$$u_t - Lu = -\int_0^t \left[r(x', t-\tau) - \int_0^{t-\tau} r(x', t-\tau-\alpha)k(x',\alpha) \right] Lu(x,\tau)d\alpha d\tau =$$

$$= -\int_0^t r(x', t-\tau)Lu(x,\tau)d\tau + \int_0^t \int_0^{t-\tau} r(x', t-\tau-\alpha)k(x',\alpha)Lu(x,\tau)d\alpha d\tau =: I_1 + I_2,$$

where

$$I_1 := -\int_0^t r(x', t-\tau)Lu(x,\tau)d\tau, \quad I_2 := \int_0^t \int_0^{t-\tau} r(x', t-\tau-\alpha)k(x',\alpha)Lu(x,\tau)d\alpha d\tau.$$

Integral I_1 can be rewritten in the form

$$I_1 := -\int_0^t r(x', t-\tau)Lu(x,\tau)d\tau = -\int_0^t r(x',\beta)Lu(x, t-\beta)d\beta.$$

To reduce I_2 to more convenient form we perform the following substitution $\alpha = t-\tau-\beta$ in the inner integral of I_2 and change the order of integration in resulting twice repeated integral

$$I_2 = \int_0^t \int_0^{t-\tau} r(x',\beta)k(x', t-\tau-\beta)Lu(x,\tau)d\beta d\tau = d\int_0^t r(x',\beta)\int_0^{t-\beta} k(x', t-\tau-\beta)Lu(x,\tau)d\tau d\beta.$$

Then we have

$$u_t(t) - Lu(x,t) = -I_1 + I_2 = -\int_0^t r(x',\beta)\left[Lu(x, t-\beta) - \int_0^{t-\beta} k(x', t-\beta-\tau)Lu(x,\tau)d\tau \right] d\beta.$$

Since function $u(x,t)$ satisfies equation (1.1) then the expression in the squared brackets of the last equality is equal to $u_t(x, t-\beta)$. It means that the last equation coincides with (1.4). Let us write integral equation (1.5) with respect to $r(x',t)$ as

$$r(x',t) = k(x',t) + \int_0^t k(x', t-\tau)r(x',\tau)d\tau$$

and substitute this relation instead of $r(x',t)$ in (1.4). Then we obtain

$$u_t - Lu(x,t) = -\int_0^t k(x',\beta)\left[u_\beta(x, t-\beta) + \int_0^{t-\beta} r(x', t-\tau-\beta)u_\tau(x,\tau)d\tau \right] d\beta.$$

Taking into account that $u(x,t)$ is a solution of (1.4), we conclude that expression standing in the squared brackets of this equality is equal to $Lu(x, t-\beta)$. This leads to equation (1.1).

The lemma is proved. □

Thus, if $u(x,t)$, $r(x',t)$ is the solution of problem (1.4), (1.2), (1.3) then function $k(x',t)$ is the solution of integral equation (1.5).

2. Auxiliary problem

In what follows, we denote function $r_t(x',t)$ by $h(x',t)$, i.e., $h(x',t) = r_t(x',t)$.

Lemma 2.1. *Problem* (1.1) – (1.5) *is equivalent to the following auxiliary problem for functions* $\vartheta(x,t),\ h(x',t)$

$$\begin{aligned}&\vartheta_t - L\vartheta - 2c_{x_n}\vartheta^{(2)}_{x_n} - c_{x_nx_n}\vartheta^{(2)} + r(x',0)\vartheta + \\ &+h(x',t)\Big[L\varphi_{x_nx_n}(x) + 2c_{x_n}\varphi_{x_n}(x) + c_{x_nx_n}\varphi(x)\Big] + \int_0^t h(x',t-\tau)\vartheta(x,\tau)d\tau = 0\end{aligned} \tag{2.1}$$

$$\vartheta\Big|_{t=0} = Y_{x_nx_n}(x) \tag{2.2}$$

$$\begin{aligned}\vartheta(x',0,t) = f_{ttt}(x',t) - \Delta_{x'}f_{tt}(x',t) - c(x',0)f_{tt}(x',t) + r(x',0)f_{tt}(x',t)+ \\ +h(x',t)L\varphi\Big|_{x_n=0} + \int_0^t f_{tt}(x',t-\tau)h(x',\tau)d\tau.\end{aligned} \tag{2.3}$$

where $\vartheta(x,t) = u_{ttx_nx_n}(x,t)$, $\vartheta^{(2)} = u_{tt}$, $\triangle_{x'} = \sum\limits_{i=1}^{n-1}\dfrac{\partial^2}{\partial x_i^2}$, $Y_{x_nx_n}(x) = \dfrac{\partial^2}{\partial x_n^2}(L^2\varphi(x) - r(x',0)L\varphi(x))$

$$r(x',0) = \frac{L^2\varphi(x',0) - f_{tt}(x'0)}{L\varphi(x',0)}. \tag{2.4}$$

Proof. Let us introduce new function $\vartheta^{(1)}(x,t) = u_t(x,t)$ and differentiate (1.4) and (1.3) with respect to t. Then we obtain the following problem for functions $\vartheta^{(1)}(x,t),\ h(x',t)$ from

$$\vartheta^{(1)}_t - L\vartheta^{(1)} + r(x',0)\vartheta^{(1)} + \int_0^t h(x',\tau)\vartheta^{(1)}(x,t-\tau)d\tau = 0, \tag{2.5}$$

$$\vartheta^{(1)}(x,0) = L\varphi(x),\ x\in\mathbb{R}^n, \tag{2.6}$$

$$\vartheta^{(1)}|_{x_n=0} = f_t(x',t),\ (x',t)\in\mathbb{R}^{n-1}_T,\ L\varphi(x',0) = f_t(x',0). \tag{2.7}$$

Here, initial condition (2.6) is obtained from (1.4) by setting $t = 0$. The problem for functions $\vartheta^{(2)}(x,t) = \vartheta^{(1)}_t(x,t)$, $h(x',t)$ is obtained from (2.5)–(2.7) in an analogous way:

$$\vartheta^{(2)}_t - L\vartheta^{(2)} + r(x',0)\vartheta^{(2)} + h(x',t)L\varphi(x) - \int_0^t h(x',\tau)\vartheta^{(2)}(x,t-\tau)d\tau = 0, \tag{2.8}$$

$$\vartheta^{(2)}\Big|_{t=0} = Y(x),\ x\in\mathbb{R}^n, \tag{2.9}$$

$$\vartheta^{(2)}\Big|_{x_n=0} = f_{tt}(x',t),\ (x',t)\in\mathbb{R}^{n-1}_T, \tag{2.10}$$

where $Y(x) := L^2\varphi(x) - r(x',0)L\varphi(x)$. Requiring that equalities (2.9) and (2.10) be equal at $t = 0$ and $x_n = 0$, we obtain some relation and equality (2.4) follows from this relation. Further, function $r(x',0)$ is assumed to be known.

Now let us denote function $\vartheta^{(2)}_{x_nx_n}(x,t)$ by $\vartheta(x,t)$. Differentiating (2.8) and (2.9) twice with respect to x_n, we obtain equations (2.1) and (2.2). To derive the additional condition for $\vartheta(x,t)$ at $x_n = 0$ we substitute function $\vartheta^{(2)}_{x_nx_n}(x,t)$ into $\triangle\vartheta^{(2)}(x,t)$ in (2.8), i.e., $L\vartheta^{(2)}(x,t) = = \vartheta^{(2)}_{x_nx_n}(x,t) + \Delta_{x'}\vartheta^{(2)}(x,t) + c(x)\vartheta^{(2)}(x,t)$. Taking into account this relation and taking $x_n = 0$ in (2.8), after some mathematical treatment we obtain (2.3). Thus, problem (1.4), (1.2), (1.3) is reduced to problem (2.1)–(2.3). It is not difficult to show that inverse transformations take place [15].

The lemma is proved. □

3. Existence and uniqueness

In this section existence and uniqueness for problem (2.1)–(2.3) are proved with the use of the principle of contraction mapping [25, pp. 87–97]. The idea is to write the integral equations for unknown functions $\vartheta(x,t),\ h(x',t)$ as a system with a non-linear operator, and to prove that this operator is a contraction mapping operator. Then existence and uniqueness immediately follow

We recall the definition of a contraction mapping operator.

Definition. *Let F be an operator defined on a closed set Ω which is a subset of a Banach space. F is called a contraction mapping operator in Ω if it satisfies the following two properties:*

1) if $y \in \Omega$ then $Fx \in \Omega$ (i.e. F maps Ω into itself);

2) if $y, z \in \Omega$ then $\|Fy - Fz\| \leqslant \rho\,\|y - z\|$ with $\rho < 1$ (constant ρ is independent of y and z).

Lemma (principle of contraction mapping)**.** *If F is a contraction mapping operator from Ω to Ω then equation*

$$y = Fy$$

has a unique solution $y_0 \in \Omega$.

Now we write Cauchy problem (2.1) and (2.2) as integral equation with respect to function $\vartheta(x,t)$. Using Poisson's formula, we obtain

$$\begin{aligned}\vartheta(x,t) = \int_{R^n} Y_{\xi_n\xi_n}(\xi) G(x-\xi,t)d\xi + \int_0^t \int_{R^n} \Big(c\vartheta + 2c_{\xi_n}\vartheta^{(2)}_{\xi_n} + c_{\xi_n\xi_n}\vartheta^{(2)} - \\ -r(\xi',0)\vartheta - h(\xi',\tau)\Big[L\varphi_{\xi_n\xi_n}(\xi) + 2c_{\xi_n}\varphi_{\xi_n}(\xi) + c_{\xi_n\xi_n}\varphi(\xi)\Big] - \\ -\int_0^\tau h(\xi',\tau-\alpha)\vartheta(\xi,\alpha)d\alpha\Big) G(x-\xi,t-\tau)d\xi d\tau,\end{aligned} \tag{3.1}$$

where $G(x;t) = \dfrac{1}{(2\sqrt{\pi t})^n} e^{\frac{-|x|^2}{4t}}$ is the fundamental solution of the heat operator $\dfrac{\partial}{\partial t} - \triangle,\ \xi = = (\xi_1,\dots,\xi_n),\ \xi' = (\xi_1,\dots,\xi_{n-1}),\ d\xi = d\xi_1\dots d\xi_n,\ |x|^2 = x_1^2 + \dots + x_n^2.$

The integral equation for $h(x',t)$ is obtained from (3.1) by considering it at $x_n = 0$ and using equality (2.3) :

$$\begin{aligned}h(x',t) = \frac{1}{L\varphi(x',0)}\Big[- f_{ttt}(x',t) + \Delta_{x'} f_{tt}(x',t) + c(x',0) f_{tt}(x',t) - r(x',0) f_{tt}(x',t) + \\ + \int_{R^n} Y_{\xi_n\xi_n}(\xi) G(x'-\xi',\xi_n,t)d\xi\Big] + \frac{1}{L\varphi(x',0)}\Big[- \int_0^t f_{tt}(x',t-\tau)h(x',\tau)d\tau + \\ + \int_0^t\int_{R^n}\Big(c\vartheta + 2c_{\xi_n}\vartheta^{(2)}_{\xi_n} + c_{\xi_n\xi_n}\vartheta^{(2)} - r(\xi',0)\vartheta - h(\xi',\tau)\Big[L\varphi_{\xi_n\xi_n}(\xi) + 2c_{\xi_n}\varphi_{\xi_n}(\xi) + \\ + c_{\xi_n\xi_n}\varphi(\xi)\Big] - \int_0^\tau h(\xi',\tau-\alpha)\vartheta(\xi,\alpha)d\alpha\Big) G(x'-\xi',\xi_n,t-\tau)d\xi d\tau\Big]\end{aligned} \tag{3.2}$$

where $G(x'-\xi',\xi_n;t-\tau) = G(x-\xi;t-\tau)\Big|_{x_n=0}$.

Note that integral equations (3.1) and (3.2) contain unknown functions $\vartheta^{(2)}$ and $\vartheta^{(2)}_{x_n}$. To close system of equations (3.1) and (3.2), we write the integral equations for these functions. The equation for $\vartheta^{(2)}$ follows from equalities (2.8) and (2.9):

$$\begin{aligned}\vartheta^{(2)}(x,t) = \int_{R^n} Y(\xi)G(x-\xi,t)d\xi + \int_0^t\int_{R^n}\Big(c\vartheta^{(2)} - r(\xi',0)\vartheta^{(2)} - h(\xi',\eta)L\varphi(\xi) - \\ -\int_0^\tau h(\xi',\tau-\alpha)\vartheta^{(2)}(\xi,\alpha)d\alpha\Big) G(x-\xi,t-\tau)d\xi d\tau.\end{aligned} \tag{3.3}$$

Differentiating equalities (2.8), (2.9) with respect to x_n and then applying Poisson's formula, we obtain equation for $\vartheta^{(2)}_{x_n}$

$$\vartheta^{(2)}_{x_n}(x,t)=\int_{R^n}Y_{\xi_n}(\xi)G(x-\xi,t)d\xi+ \\ +\int_0^t\int_{R^n}\Big(c\vartheta^{(2)}_{\xi_n}+c_{\xi_n}\vartheta^{(2)}-r(\xi',0)\vartheta^{(2)}_{\xi_n}-h(\xi',\tau)\Big(L\varphi_{\xi_n}(\xi)+c_{\xi_n}\varphi(\xi)\Big)- \\ -\int_0^\tau h(\xi',\tau-\alpha)\vartheta^{(2)}_{\xi_n}(\xi,\alpha)d\alpha\Big)G(x-\xi,t-\tau)d\xi d\tau. \tag{3.4}$$

Theorem (existence and uniqueness)**.** *If conditions $\varphi(x)\in H^{l+8}(\mathbb{R}^n)$, $|L\varphi(x',0)|^l\geqslant const>0$, $c(x)\in H^{l+4}(\mathbb{R}^n)$, $f(x',t)\in H^{l+6,(l+6)/2}\left(\bar{\mathbb{R}}^{n-1}_T\right)$, $l\in(0,1)$ and equalities $f(x',0)==\varphi(x',0)$, $f_t(x',0)=L\varphi(x',0)$ are satisfied then for sufficiently small number $T>0$ the unique solution of integral equations (3.1)–(3.4) in the class of functions $\left\{\vartheta(x,t),\vartheta^{(2)}(x,t),\vartheta^{(2)}_{x_n}(x,t)\right\}\in H^{l+2,(l+2)/2}\left(\overline{\mathbb{R}}^n_T\right)$, $h(x',t)\in H^{l,l/2}\left(\overline{\mathbb{R}}^n_T\right)$ exists. Thus, there is the unique classical solution of problem (2.1)–(2.3).*

Proof. System of equations (3.1)–(3.4) is a closed system for the unknown functions $\vartheta(x,t)$, $h(x',t)$, $\vartheta^2(x,t)$, $\vartheta^2_{x_n}(x,t)$ in the domain $\mathbb{R}^n_T$. It can be rewritten in the form of a non-linear operator equation

$$\psi=A\psi, \tag{3.5}$$

where $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^*=\left(\vartheta(x,t),\ h(x',t),\vartheta^{(2)},\vartheta^{(2)}_{x_n}\right)^*$, symbol $*$ means transposition. According to equations (3.1)-(3.4), operator $A\psi=[(A\psi)_1,\ (A\psi)_2,\ (A\psi)_3,\ (A\psi)_4]$ has the form

$$(A\psi)_1=\psi_{01}(x,t)+\int_0^t\int_{R^n}\Big((c(\xi)-r(\xi',0))\psi_1(\xi,\tau)+2c_{\xi_n}(\xi)\psi_4(\xi,\tau)+ \\ +c_{\xi_n\xi_n}(\xi)\psi_3(\xi,\tau)-\psi_2(\xi',\tau)\Big[L\varphi_{\xi_n\xi_n}(\xi)+2c_{\xi_n}\varphi_{\xi_n}(\xi)+c_{\xi_n\xi_n}\varphi(\xi)\Big]- \\ -\int_0^\tau\psi_2(\xi',\tau-\alpha)\psi_1(\xi,\alpha)d\alpha\Big)G(x-\xi,t-\tau)d\xi d\tau, \tag{3.6}$$

$$(A\psi)_2=\psi_{02}(x',t)+\frac{1}{L\varphi(x',0)}\Big[-\int_0^t f_{tt}(x',t-\tau)\psi_2(x',\tau)d\tau+ \\ +\int_0^t\int_{R^n}\Big((c(\xi)-r(\xi',0))\psi_1(\xi,\tau)+2c_{\xi_n}(\xi)\psi_4(\xi,\tau)+c_{\xi_n\xi_n}(\xi)\psi_3(\xi,\tau)- \\ -\psi_2(\xi',\tau)\Big[L\varphi_{\xi_n\xi_n}(\xi)+2c_{\xi_n}\varphi_{\xi_n}(\xi)+c_{\xi_n\xi_n}\varphi(\xi)\Big]- \\ \int_0^\tau+\psi_2(\xi',\tau-\alpha)\psi_1(\xi,\alpha)d\alpha\Big)G(x'-\xi',\xi_n,t-\tau)d\xi d\tau\Big], \tag{3.7}$$

$$(A\psi)_3=\psi_{03}(x,t)+\int_0^t\int_{R^n}\Big((c(\xi)-r(\xi',0))\psi_3(\xi,\tau)-\psi_2(\xi',\eta)L\varphi(\xi)- \\ -\int_0^\tau\psi_2(\xi',\tau-\alpha)\psi_3(\xi,\alpha)d\alpha\Big)G(x-\xi,t-\tau)d\xi d\tau, \tag{3.8}$$

$$(A\psi)_4=\psi_{04}(x,t)+\int_0^t\int_{R^n}\Big((c(\xi)-r(\xi',0))\psi_4(\xi,\tau)+c_{\xi_n}(\xi)\psi_3(\xi,\tau)- \\ -\psi_2(\xi',\tau)\Big(L\varphi_{\xi_n}(\xi)+c_{\xi_n}\varphi(\xi)\Big)-\int_0^\tau\psi_2(\xi',\tau-\alpha)\psi_4(\xi,\alpha)d\alpha\Big)G(x-\xi,t-\tau)d\xi d\tau. \tag{3.9}$$

The following notations are introduced in (3.6)–(3.9):

$$\psi_{01}(x,t)=\int_{R^n} Y_{\xi_n\xi_n}(\xi)G(x-\xi,t)d\xi,$$

$$\psi_{02}(x',t)=\frac{1}{L\varphi(x',0)}\Big[-f_{ttt}(x',t)+\Delta_{x'}f_{tt}(x',t)+c(x',0)f_{tt}(x',t)-$$

$$-r(x',0)f_{tt}(x',t)+\int\ {}_{R^n}Y_{\xi_n\xi_n}(\xi)G(x'-\xi',\xi_n,t)d\xi\Big],$$

$$\psi_{03}(x,t)=\int_{R^n} Y(\xi)G(x-\xi,t)d\xi,$$

$$\psi_{04}(x,t)=\int_{R^n} Y_{\xi_n}(\xi)G(x-\xi,t)d\xi.$$

** Let us introduce the following designation: $|\psi|^l_T=\max\Big(|\psi_1|^l_T\,,\ |\psi_2|^l_T\,,\ |\psi_3|^l_T\,,\ |\psi_4|^l_T\Big)$ and consider in the space $H^{l,l/2}\left(\mathbb{R}^n_T\right)$ the set $S(T)$ of functions $\psi(x,t)$ that satisfies the inequality

$$|\psi-\psi_0|^l_T\leqslant|\psi_0|^l_{T_0}\,,\ T<T_0, \tag{3.10}$$

where $\psi_0=(\psi_{01},\,\psi_{02},\,\psi_{03},\,\psi_{04})$ and $|\psi_0|^l_{T_0}=\max\Big(|\psi_{01}|^l_{T_0}\,,|\psi_{02}|^l_{T_0}\,,|\psi_{03}|^l_{T_0}\,,|\psi_{04}|^l_{T_0}\Big)$.

It can be shown that when T is sufficiently small the operator A is a contraction mapping operator in $S(T)$. Then theorem of existence and uniqueness immediately follows from the contraction mapping principle.

First we show that A has the first property of a contraction mapping operator. Let $\psi\in S(T)$, $T<T_0$. Then we have from inequality (3.10) that

$$|\psi_i|^l_T\leqslant 2\,|\psi_0|^l_{T_0}\,,\ i=1,2,3,4.$$

It is easy to see that

$$|(A\psi)_1-\psi_{01}|^l_T=\Big|\int_0^t\!\!\int_{R^n}\Big((c(\xi)-r(\xi',0))\psi_1(\xi,\tau)+2c_{\xi_n}(\xi)\psi_4(\xi,\tau)+$$

$$+c_{\xi_n\xi_n}(\xi)\psi_3(\xi,\tau)-\psi_2(\xi',\tau)\Big[L\varphi_{\xi_n\xi_n}(\xi)+2c_{\xi_n}\varphi_{\xi_n}(\xi)+c_{\xi_n\xi_n}\varphi(\xi)\Big]-$$

$$-\int_0^\tau\psi_2(\xi',\tau-\alpha)\psi_1(\xi,\alpha)d\alpha\Big)G(x-\xi,t-\tau)d\xi d\tau\Big|^l_T\leqslant$$

$$\leqslant\int_0^t\!\!\int_{R^n}\Big((|c(\xi)|^l+|r(\xi',0)|^l_T)|\psi_1(\xi,\tau)|^l_T+2|c_{\xi_n}(\xi)|^l|\psi_4(\xi,\tau)|^l_T+$$

$$|c_{\xi_n\xi_n}(\xi)|^l|\psi_3(\xi,\tau)|^l_T+|\psi_2(\xi',\tau)|^l_T\Big[|L\varphi_{\xi_n\xi_n}(\xi)|^l+2|c_{\xi_n}|^l|\varphi_{\xi_n}(\xi)|^l+|c_{\xi_n\xi_n}(\xi)|^l|\varphi(\xi)|^l+$$

$$+|c_{\xi_n\xi_n}(\xi)|^l|\psi_3(\xi,\tau)|^l_T\Big]+\int_0^\tau|\psi_2(\xi',\tau-\alpha)|^l_T|\psi_1(\xi,\alpha)|^l_Td\alpha\Big)|G(x-\xi,t-\tau)|^l_Td\xi d\tau\leqslant$$

$$\leqslant|\psi_0|^l_{T_0}\cdot 2T\Big(c_0+r_0+2c_1(1+\varphi_5)+c_2(1+\varphi_4)+\varphi_3+2|\psi_0|^l_{T_0}T\Big):=|\psi_0|^l_{T_0}\beta_1.$$

In a similar way we obtain

$$|(A\psi)_2-\psi_{02}|^l_T\leqslant|\psi_0|^l_{T_0}\cdot 2T\varphi_0\Big(f_1+c_0+r_0+2c_1(1+\varphi_5)+c_2(1+\varphi_4)+$$

$$+\varphi_3+2T|\psi_0|^l_{T_0}\Big):=|\psi_0|^l_{T_0}\beta_2,$$

$$|(A\psi)_3-\psi_{03}|_T^l|\psi_0|_{T_0}^l \leqslant |\psi_0|_{T_0}^l\cdot 2T\Big(c_0+r_0+\varphi_1+2T|\psi_0|_{T_0}^l\Big):=|\psi_0|_{T_0}^l\beta_3,$$

$$|(A\psi)_4-\psi_{04}|_T^l \leqslant |\psi_0|_{T_0}^l\cdot 2T\Big(c_0+r_0+c_1(1+2\varphi_5)+c_2\varphi_4+2T|\psi_0|_{T_0}^l\Big):=|\psi_0|_{T_0}^l\beta_4.$$

where

$$c_0:=|c(x)|^l,\quad c_1:=|c_{x_n}(x)|^l,\quad c_2:=|c_{x_nx_n}(x)|^l$$

$$\varphi_0:=|(L\varphi(x',0))^{-1}|_T^l,\quad \varphi_1:=|L\varphi(x)|^l,\quad \varphi_2:=|L\varphi_{x_n}(x)|^l,\quad \varphi_3:=|L\varphi_{x_nx_n}(x)|^l,$$

$$\varphi_4:=|\varphi(x)|^l,\quad \varphi_5:=|\varphi_{x_n}(x)|^l,\quad L\varphi(x',0)=(\Delta\varphi(x)+c(x))|_{x_n=0}.$$

$$f_1:=|f_{tt}(x',t)|_T^l,\quad r_0:=|r(x',0)|_T^l.$$

We note that $\beta_i(T)\to 0$ when $T\to 0$, $i=1,2,3,4$. Therefore, if we choose T $(T<T_0)$ so that inequality $\beta_0:=\max\limits_{1\leqslant i\leqslant 4}\beta_i<1$, is satisfied then operator A has the first property of a contraction mapping operator, i.e., $A\psi\in S(T)$.

Next we consider the second property of a contraction mapping operator for A. Let $\psi^{(1)}=\\=\left(\psi_1^{(1)},\,\psi_2^{(1)},\,\psi_3^{(1)},\,\psi_4^{(1)}\right)\in S(T)$, $\psi^{(2)}=\left(\psi_1^{(2)},\,\psi_2^{(2)},\,\psi_3^{(2)},\,\psi_4^{(2)}\right)\in S(T)$. Then the simple mathematical treatment shows that the following relations are true:

$$\left|\psi_2^{(1)}\psi_1^{(1)}-\psi_2^{(2)}\psi_1^{(2)}\right|_T^l=\left|\left(\psi_2^{(1)}-\psi_2^{(2)}\right)\psi_1^{(1)}+\psi_2^{(2)}\left(\psi_1^{(1)}-\psi_1^{(2)}\right)\right|_T^l\leqslant$$
$$\leqslant 2\left|\psi^{(1)}-\psi^{(2)}\right|_T^l\max\left(\left|\psi_1^{(1)}\right|_T^l,\ \left|\psi_2^{(2)}\right|_T^l\right)\leqslant 4\,|\varphi_0|_T^l\left|\psi^{(1)}-\psi^{(2)}\right|_T^l.$$

Taking this into account, we carry out the estimations as follows

$$|((A\psi)^{(1)}-A\psi)^{(2)})_1|_T^l=\Big|\int_0^t\int_{R^n}\Big((c(\xi)-r(\xi',0))[\psi_1^{(1)}(\xi,\tau)-\psi_1^{(2)}(\xi,\tau)]+$$
$$+2c_{\xi_n}(\xi)[\psi_4^{(1)}(\xi,\tau)-\psi_4^{(2)}(\xi,\tau)]+c_{\xi_n\xi_n}(\xi)[\psi_3^{(1)}(\xi,\tau)-\psi_3^{(2)}(\xi,\tau)]+$$
$$+[\psi_2^{(1)}(\xi,\tau)-\psi_2^{(2)}(\xi,\tau)]\Big[L\varphi_{\xi_n\xi_n}(\xi)+2c_{\xi_n}\varphi_{\xi_n}(\xi)+c_{\xi_n\xi_n}\varphi(\xi)\Big]+$$
$$+\int_0^\tau[\psi_2^{(1)}(\xi',\tau-\alpha)\psi_1^{(1)}(\xi,\alpha)-\psi_2^{(2)}(\xi',\tau-\alpha)\psi_1^{(2)}(\xi,\alpha)]d\alpha\Big)G(x-\xi,t-\tau)d\xi d\tau\Big|_T^l\leqslant$$
$$\leqslant\int_0^t\int_{R^n}\Big((|c(\xi)|^l+|r(\xi',0)|_T^l)|[\psi_1^{(1)}(\xi,\tau)-\psi_1^{(2)}(\xi,\tau)]|_T^l+$$
$$+2|c_{\xi_n}(\xi)|^l|[\psi_4^{(1)}(\xi,\tau)-\psi_4^{(2)}(\xi,\tau)]|_T^l+|c_{\xi_n\xi_n}(\xi)|^l|[\psi_3^{(1)}(\xi,\tau)-\psi_3^{(2)}(\xi,\tau)]|_T^l+$$
$$+|[\psi_2^{(1)}(\xi,\tau)-\psi_2^{(2)}(\xi,\tau)]|_T^l\Big[|L\varphi_{\xi_n\xi_n}(\xi)|^l+|2c_{\xi_n}|^l|\varphi_{\xi_n}(\xi)|^l+|c_{\xi_n\xi_n}|^l|\varphi(\xi)|^l\Big]+$$
$$\int_0^\tau|[\psi_2^{(1)}(\xi',\tau-\alpha)\psi_1^{(1)}(\xi,\tau)-\psi_2^{(2)}(\xi',\tau-\alpha)\psi_1^{(2)}(\xi,\tau)]|_T^l d\alpha\Big)|G(x-\xi,t-\tau)|_T^l d\xi d\tau\leqslant$$
$$\leqslant|\psi^{(1)}-\psi^{(2)}|_{T_0}^l\cdot T\Big(c_0+r_0+2c_1(1+\varphi_5)+c_2(1+\varphi_4)+\varphi_3+4T|\psi_0|_{T_0}^l\Big):=$$
$$:=|\psi^{(1)}-\psi^{(2)}|_{T_0}^l\mu_1.$$

The similar argument gives the following relations

$$|((A\psi)^{(1)}-A\psi)^{(2)})_2|_T^l\leqslant|\psi^{(1)}-\psi^{(2)}|_{T_0}^l\cdot T\varphi_0\Big(f_1+c_0+r_0+2c_1(1+\varphi_5)+$$

$$+c_2(1+\varphi_4)+\varphi_3+4T|\psi_0|^l_{T_0}\Big) := \psi^{(1)}-\psi^{(2)}|^l_{T_0}\mu_2,$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_3|^l_T \leqslant |\psi^{(1)}-\psi^{(2)}|^l_{T_0}\cdot T\Big(c_0+r_0+\varphi_1+4T|\psi_0|^l_{T_0}\Big) := |\psi^{(1)}-\psi^{(2)}|^l_{T_0}\mu_3,$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_4|^l_T \leqslant |\psi^{(1)}-\psi^{(2)}|^l_{T_0}\cdot T\Big((c_0+r_0)+$$

$$+c_1(1+2\varphi_5)+c_2\varphi_4+\varphi_3+4T|\psi_0|^l_{T_0}\Big) \leqslant |\psi^{(1)}-\psi^{(2)}|^l_{T_0}\mu_4.$$

Hence, $\left|\left(A\psi^{(1)}-A\psi^{(2)}\right)\right|^l_T < \mu\left|\psi^{(1)}-\psi^{(2)}\right|^l_T$ if T satisfies the condition $\mu_0 := \max\limits_{1\leqslant i\leqslant 4}\mu_i < 1$.

It is not difficult to see that if T is chosen from the conditions $\max\{\beta_0,\ \mu_0\}$ and $T < T_0$ then operator A satisfies both properties of a contraction mapping operator, i.e., $A\psi \in S(T)$ for $\psi \in S(T)$. Then, according to Banach theorem (see, for instance, [26, pp. 87–97]) there exists in the set $S(T)$ only one fixed point of operator A, i.e., there exists only one solution of problem (3.5). Hence, solving system (3.1)–(3.4), for example, by the method of successive approximations, we uniquely find functions $\vartheta(x,t)$, $h(x',t)$ which belong to $H^{l,l/2}\left(\mathbb{R}^n_T\right)$ and $H^{l,l/2}\left(\mathbb{R}^{n-1}_T\right)$, respectively. Moreover, it follows from the general theory of parabolic equations [27, pp. 380–384] that if conditions of the theorem are fulfilled then solution of integral equation (3.1) $\vartheta(x,t)$ belongs to $H^{l+2,(l+2)/2}\left(\mathbb{R}^n_T\right)$.

The theorem is proved. □

Since $h(x',t) = r_t(x',t)$, the obtained function $h(x',t)$ can be used to determine the function $r(x',t)$ with the use of relation

$$r(x',t) = r(x',0) + \int_0^t h(x',\tau)d\tau, \quad (x',t) \in \mathbb{R}^{n-1}_T,$$

where $r(x',0)$ is the known function determined by (2.4). Then solving integral equation (1.5) at every fixed x', we uniquely find $k(x',t)$. Due to the proved theorem and the given above considerations, we conclude that problem (1.1)–(1.3) has a unique solution such that $\vartheta(x,t) \in H^{l+6,(l+6)/2}\left(\mathbb{R}^n_T\right)$, $k(x',t) \in H^{l+2,(l+2)/2}\left(\mathbb{R}^{n-1}_T\right)$.

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Проблема определения многомерного ядра в одном параболическом интегро-дифференциальном уравнении

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Аннотация. Рассматривается многомерное параболическое интегро-дифференциальное уравнение с интегралом временной свертки в правой части. Прямая задача представлена задачей Коши для этого уравнения. В данной статье исследуется обратная задача, заключающаяся в нахождении зависимого от времени и пространства ядра интегрируемого члена на известном в гиперплоскости $x_n = 0$ for $t > 0$ решению прямой задачи. С использованием резольвенты ядра эта задача сводится к исследованию более удобной обратной задачи. Последняя задача заменена эквивалентной системой интегральных уравнений относительно неизвестных функций, и на основе принципа сжимающего отображения доказана однозначная разрешимость прямой и обратной задач.

Ключевые слова: интегро-дифференциальное уравнение, обратная задача, пространство Гёльдера, ядро, резольвента.