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On a Creeping 3D Convective Motion of Fluids with an Isothermal Interface

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Abstract. In the work the 3D two-layer motion of liquids, the velocity field of which has a special form, is considered. The arising conjugate initial boundary value problem for the Oberbek–Boussinesq model is reduced to a system of ten integrodifferential equations with full conditions on a flat interface. It is shown that for small Marangoni numbers the stationary problem can have up to two solutions. The case when the stationary flow arises due to a change in the internal interphase energy is analyzed separately.

Keywords: Oberbek-Boussinesq model, interphase energy, creeping flow, inverse problem.

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1. Statement of the problem and basic equations

Suppose that two viscous heat-conducting fluids with a common interface $z = l_1 < l_2$ move in a layer $|x| < \infty$, $|y| < \infty$, $0 < z < l_2$, l_j are constants. The fluid "1" occupies the region $0 < z < l_1$ and fluid "2" occupies the region $l_1 < z < l_2$. The planes $z = 0$ and $z = l_2$ are solid fixed walls, the force of gravity is directed perpendicular to the layers. Oberbeck-Boussinesq equations are used as a mathematical model of fluid motion. Solutions are sought in a special way

$$u_j(x, y) = (f_j(z, t) + h_j(z, t))x, \quad v_j(x, y) = (f_j(z, t) - h_j(z, t))y, \quad w_j = -2\int_{z_0}^{z} f_j(\xi, t)\, d\xi, \quad (1)$$

$$\frac{1}{\rho_j} p_j = b_j(z, t)x^2 + d_j(z, t)y^2 + q_j(z, t), \quad (2)$$

$$T_j = a_j(z, t)x^2 + c_j(z, t)y^2 + \theta_j(z, t), \quad (3)$$

where $u_j(x, y, z, t)$, $v_j(x, y, z, t)$, $w_j(x, y, z, t)$ are projections of velocity vectors on the x, y, z axis, respectively; $p_j(x, y, z, t)$ are pressures; ρ_j are constants of density; $T_j(x, y, z, t)$ are absolute temperatures, $j = 1, 2$. The functions f_j, h_j, b_j, d_j, q_j, a_j, c_j, θ_j are new unknown function.

Substitution of the formulas (1)–(3) in the systems of Oberbeck-Boussinesq equations leads to the following systems

$$f_{jt} + f_j^2 + h_j^2 - 2f_{jz}\int_{z_0}^{z} f_j(\xi, t)\, d\xi + g\beta_j \int_{z_0}^{z} (a_j(\xi, t) + c_j(\xi, t))\, d\xi = \nu_j f_{jzz} + n_{j1}(t), \quad (4)$$

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$$h_{jt} + 2f_j h_j - 2h_{jz} \int_{z_0}^{z} f_j(\xi,t)\,d\xi + g\beta_j \int_{z_0}^{z} (a_j(\xi,t) - c_j(\xi,t))\,d\xi = \nu_j h_{jzz} + n_{j2}(t), \tag{5}$$

$$a_{jt} + 2a_j(f_j + h_j) - 2a_{jz} \int_{z_0}^{z} f_j(\xi,t)\,d\xi = \chi_j a_{jzz}, \tag{6}$$

$$c_{jt} + 2c_j(f_j - h_j) - 2c_{jz} \int_{z_0}^{z} f_j(\xi,t)\,d\xi = \chi_j c_{jzz}, \tag{7}$$

$$\theta_{jt} - 2\theta_{jz} \int_{z_0}^{z} f_j(\xi,t)\,d\xi = \chi_j \theta_{jzz} + 2\chi_j(a_j + c_j). \tag{8}$$

Here $\nu_j > 0$, $\chi_j > 0$, $\beta_j > 0$ are constants of kinematic viscosities, thermal diffusivities and thermal expansion coefficients of liquids; $n_{j1}(t)$, $n_{j2}(t)$ are arbitrary functions of time. By the known functions a_j, c_j the functions b_j, d_j are determined by quadratures

$$b_j(z,t) = g\beta_j \int_{z_0}^{z} a_j(\xi,t)\,d\xi - n_{j1}(t), \quad d_j(z,t) = g\beta_j \int_{z_0}^{z} c_j(\xi,t)\,d\xi - n_{j2}(t). \tag{9}$$

In the integral terms, the constant z_0 is equal to "0" for the first fluid ($j = 1$) and l_1 for the second fluid ($j = 2$). It can be verified that pressures in liquids are determined as follows

$$\begin{aligned} \frac{1}{\rho_j} p_j = &\left[g\beta_j \int_{z_0}^{z} a_j(\xi,t)\,d\xi - n_{j1}(t) \right] x^2 + \left[g\beta_j \int_{z_0}^{z} c_j(\xi,t)\,d\xi - n_{j2}(t) \right] y^2 - 2\nu_j f_j - gz + \\ &+ g\beta_j \int_{z_0}^{z} \theta_j(\xi,t)\,d\xi + 2\int_{z_0}^{z} (z-\xi) f_{jt}(\xi,t)\,d\xi + 2\left(\int_{z_0}^{z} f(\xi,t)\,d\xi \right)^2 + q_{j0}(t), \end{aligned} \tag{10}$$

with arbitrary functions $q_{j0}(t)$.

Remark 1. The velocity field (1), proposed in [1] is a special case of the velocity field for the Navier-Stokes equations [2].

2. Boundary and initial conditions

On solid boundaries, the sticking conditions for the velocities are satisfied, which implies equalities

$$f_1(0,t) = h_1(0,t) = 0, \quad f_2(l_2,t) = h_2(l_2,t) = \int_{l_1}^{l_2} f_2(\xi,t)\,d\xi = 0 \tag{11}$$

And the temperature is set

$$\begin{aligned} a_1(0,t) = a_0(t), \quad c_1(0,t) = c_0(t), \quad \theta_1(0,t) = \theta_1(t), \\ a_2(l_2,t) = a_2(t), \quad c_2(l_2,t) = c_2(t), \quad \theta_2(l_2,t) = \theta_2(t). \end{aligned} \tag{12}$$

The top wall can also be thermally insulated

$$a_{2z}(l_2,t) = c_{2z}(l_2,t) = \theta_{2z}(l_2,t) = 0, \tag{13}$$

To formulate the conditions on the undeformed interface $z = l_1$, we assume that the surface tension depends linearly on temperature

$$\sigma(T) = \sigma_0 - æ(T - T_0), \tag{14}$$

whereσ_0, æ, T_0 are given positive constants, $T(x,y,l_1,t)$ is temperature on this border.

On the interface $z = l_1$ there are equalities of velocities and temperatures. Taking into account the representation (1), (3) we get [3]

$$\begin{gathered} f_1(l_1,t) = f_2(l_1,t), \quad h_1(l_1,t) = h_2(l_1,t), \quad a_1(l_1,t) = a_2(l_1,t), \\ c_1(l_1,t) = c_2(l_1,t), \quad \theta_1(l_1,t) = \theta_2(l_1,t). \end{gathered} \tag{15}$$

Tangential stresses are reduced to two relations

$$\begin{aligned} \mu_2 f_{2z}(l_1,t) - \mu_1 f_{1z}(l_1,t) &= -æ(a_1(l_1,t) + c_1(l_1,t)), \\ \mu_2 h_{2z}(l_1,t) - \mu_1 h_{1z}(l_1,t) &= -æ(a_1(l_1,t) - c_1(l_1,t)), \end{aligned} \tag{16}$$

where $\mu_j = \rho_j \nu_j$ are dynamic viscosity of liquids.

The kinematic condition for a fixed and non-deformable interface ($w_1(l_1,t) = w_2(l_1,t) = 0$) is equivalent to the integral equality

$$\int_0^{l_1} f_1(\xi,t)\, d\xi = 0. \tag{17}$$

The energy condition [3], taking into account the assumptions (8), can be written as

$$k_2 T_{2z}(x,y,l_1,t) - k_1 T_{1z}(x,y,l_1,t) = æT(x,y,l_1,t)\mathrm{div}_\Gamma \mathbf{u}. \tag{18}$$

where k_j are constant coefficients of thermal conductivity of liquids; $\mathrm{div}_\Gamma \mathbf{u}$ is surface divergence of the velocity vector; $T(x,y,l_1,t) = T_1(x,y,l_1,t) = T_2(x,y,l_1,t)$. Since in our case $\mathrm{div}_\Gamma \mathbf{u} = u_x + v_y$, then using the formulas (1), (3) from (18) we derive the relations

$$\begin{aligned} k_2 a_{2z}(l_1,t) - k_1 a_{1z}(l_1,t) &= 2æa_1(l_1,t) f_1(l_1,t), \\ k_2 c_{2z}(l_1,t) - k_1 c_{1z}(l_1,t) &= 2æc_1(l_1,t) f_1(l_1,t), \\ k_2 \theta_{2z}(l_1,t) - k_1 \theta_{1z}(l_1,t) &= 2æ\theta_1(l_1,t) f_1(l_1,t). \end{aligned} \tag{19}$$

The relation order of equation right-hand side (18) to the first terms of its left-hand side is estimated by the parameter $E = æ^2\theta^*/\mu_2 k_2$ (for the second term $\mu_1 k_1$), where θ^* is the characteristic temperature on the interface [3]. These parameters for ordinary liquid media are small and instead of (18) the equality of heat fluxes is used. However, for low-viscosity liquids and small k_j the right-hand side in (18) (right-hand sides in (19)) must be taken into account, for example, for cryogenic media [3].

At the initial moment of time, all functions are set

$$f_j(z,0) = f_{j0(z)}, \quad h_j(z,0) = h_{j0}(z), \quad a_j(z,0) = a_{j0}(z), \quad c_j(z,0) = c_{j0}(z), \quad \theta_j(z,0) = \theta_j 0(z), \tag{20}$$

that satisfy the conditions of agreement with (12), (13), (15)–(17), (19). For example, $f_{10}(l_1) = f_{20}(l_1)$ etc.

Remark 2. The formulated initial-boundary value problem (4)–(9), (11)–(17), (19), (20) is the inverse, since the functions $n_{j1}(t)$, $n_{j2}(t)$ must be found along with its solution. For a complete statement of this problem, two more conditions must be set

$$\int_0^{l_1} h_1(\xi,t)\, d\xi = 0, \quad \int_{l_1}^{l_2} h_2(\xi,t)\, d\xi = 0, \tag{21}$$

which together with the integral equalities (11), (17) mean closedness of motion.

3. Dimensionless variables

We introduce dimensionless variables and parameters

$$\tau = \frac{\chi_1}{l_1^2}t, \quad \xi = \frac{z}{l_2}, \quad \chi = \frac{\chi_1}{\chi_2}, \quad \mathrm{P}_j = \frac{\nu_j}{\chi_j}, \quad \mu = \frac{\mu_1}{\mu_2}, \quad k = \frac{k_1}{k_2}, \quad l = \frac{l_1}{l_2} < 1,$$
$$\mathrm{G}_j = \frac{a^* l_2 l_1^4 g\beta_j}{\chi_1^2}, \quad \varepsilon_j = \frac{\chi_j}{\chi_1}, \quad \mathrm{M} = \frac{æ a^* l_1^2 l_2}{\mu_2\chi_1}, \quad F_j(\xi,\tau) = \frac{l_1^2}{\chi_1 \mathrm{M}} f_j(z,t),$$
$$H_j(\xi,\tau) = \frac{l_1^2}{\chi_1 \mathrm{M}} h_j(z,t), \quad A_j(\xi,\tau) = \frac{a_j(z,t)}{a^*\mathrm{M}}, \quad C_j(\xi,\tau) = \frac{c_j(z,t)}{a^*\mathrm{M}},$$
$$N_j(\tau) = \frac{l_1^4 n_j(t)}{\chi_1^2 \mathrm{M}}, \quad Q_j(\xi,\tau) = \frac{\theta_j(z,t)}{a^* l_1^2 \mathrm{M}}. \tag{22}$$

Here P_j are Prandtl numbers, G_j are Grashof numbers, M is Marangoni number. It is further believed that $a^* = \max\limits_{t\geqslant 0} |a_1(t)| > 0$ and the characteristic temperature at the interface is $\theta^* = a^* l_1^2$.

In the new variables, the system (4)–(8) will be rewritten as follows

$$F_{j\tau} + \mathrm{M}\Big[F_j^2 + H_j^2 - 2F_{j\xi}\int_{z_0/l_2}^{\xi} F_j(\zeta,\tau)\,d\zeta\Big] + \mathrm{G}_j\int_{z_0/l_2}^{\xi} (A_j(\zeta,\tau) + C_j(\zeta,\tau))\,d\zeta = \\ = \mathrm{P}_j l^2 \varepsilon_j F_{j\xi\xi} + N_{j1}(\tau), \tag{23}$$

$$H_{j\tau} - 2\mathrm{M}\Big[F_j H_j - 2H_{j\xi}\int_{z_0/l_2}^{\xi} F_j(\zeta,\tau)\,d\zeta\Big] + \mathrm{G}_j\int_{z_0/l_2}^{\xi} (A_j(\zeta,\tau) - C_j(\zeta,\tau))\,d\zeta = \\ = \mathrm{P}_j l^2 \varepsilon_j H_{j\xi\xi} + N_{j2}(\tau), \tag{24}$$

$$A_{j\tau} + 2\mathrm{M}A_j(F_j + H_j) - 2\mathrm{M}A_{j\xi}\int_{z_0/l_2}^{\xi} F_j(\zeta,\tau)\,d\zeta = l^2\varepsilon_j A_{j\xi\xi}, \tag{25}$$

$$C_{j\tau} + 2\mathrm{M}C_j(F_j - H_j) - 2\mathrm{M}C_{j\xi}\int_{z_0/l_2}^{\xi} F_j(\zeta,\tau)\,d\zeta = l^2\varepsilon_j C_{j\xi\xi}, \tag{26}$$

$$Q_{j\tau} - 2\mathrm{M}Q_{j\xi}\int_{z_0/l_2}^{\xi} F_j(\zeta,\tau)\,d\zeta = l^2\varepsilon_j Q_{j\xi\xi} + 2\varepsilon_j(A_j + C_j). \tag{27}$$

In integral expressions for $j = 1$ the $z_0 = 0$ and at $j = 2$ we have $z_0 = l_1$, so that $0 < \xi < l$ in the first layer and $l < \xi < 1$ in the second layer.

The boundary conditions (11)–(13), (15)–(17), (19), (21) are rewritten as

$$F_1(0,\tau) = H_1(0,\tau) = 0, \quad F_2(1,\tau) = H_2(1,\tau) = \int_l^1 F_2(\xi,\tau)\,d\xi = 0, \tag{28}$$

$$A_1(0,\tau) = A_1(\tau), \quad C_1(0,\tau) = C_1(\tau), \quad Q_1(0,\tau) = Q_1(\tau), \\ A_2(1,\tau) = A_2(\tau), \quad C_2(1,\tau) = C_2(\tau), \quad Q_2(1,\tau) = Q_2(\tau), \tag{29}$$

$$A_{2\xi}(1,\tau) = C_{2\xi}(1,\tau) = Q_{2\xi}(1,\tau) = 0, \tag{30}$$

$$F_1(l,\tau) = F_2(l,\tau), \quad H_1(l,\tau) = H_2(l,\tau), \quad A_1(l,\tau) = A_2(l,\tau), \\ C_1(l,\tau) = C_2(l,\tau), \quad Q_1(l,\tau) = Q_2(l,\tau), \tag{31}$$

$$F_{2\xi}(l,\tau) - \mu F_{1\xi}(l,\tau) = -\mathrm{M}(A_1(l,\tau) + C_1(l,\tau)), \\ H_{2\xi}(l,\tau) - \mu H_{1\xi}(l,\tau) = -\mathrm{M}(A_1(l,\tau) - C_1(l,\tau)), \tag{32}$$

$$\int_0^l F_1(\xi,\tau)\,d\xi = 0, \tag{33}$$

$$\begin{gathered} A_{2\xi}(l,\tau) - kA_{1\xi}(l,\tau) = 2\mathrm{E}A_1(l,\tau)F_1(l,\tau), \\ C_{2\xi}(l,\tau) - kC_{1\xi}(l,\tau) = 2\mathrm{E}C_1(l,\tau)F_1(l,\tau), \\ Q_{2\xi}(l,\tau) - kQ_{1\xi}(l,\tau) = 2\mathrm{E}Q_1(l,\tau)F_1(l,\tau), \end{gathered} \tag{34}$$

$$\int_0^l H_1(\xi,\tau)\,d\xi = 0, \quad \int_l^1 H_2(\xi,\tau)\,d\xi = 0. \tag{35}$$

The initial data (20) will be of the form

$$\begin{gathered} F_j(\xi,0) = F_{j0}(\xi), \quad H_j(\xi,0) = H_{j0}(\xi), \quad A_j(\xi,0) = A_{j0}(\xi), \\ C_j(\xi,0) = C_{j0}(\xi), \quad Q_j(\xi,0) = Q_{j0}(\xi). \end{gathered} \tag{36}$$

4. Stationary creeping flow with an isothermal interface

In this case, the right-hand sides (32) must be zero. It means that $A_1(l,\tau) = C_1(l,\tau) = 0$ and the task set above will be redefined. Here we consider the creeping motion ($\mathrm{M} \ll 1$). It is necessary to assume that the initial initial data are of the order M. Let $\mathrm{M} \to 0$, then the equations (23)–(27) will be linear and the right-hand sides of the boundary conditions are equal to zero. However, the relations (34) remain nonlinear.

Remark 3. If, assume that $A_j(\xi,\tau) = 0$, $C_j(\xi,\tau) = 0$, then the interface will be isothermal: $T_1(x,y,l,\tau) = T_2(x,y,l,\tau) = \theta_1(l,\tau) = \theta_2(l,\tau) = 0$.

In this paragraph, we assume that the upper plane is thermally insulated and conditions (30) are satisfied on it; initial data (36) are omitted. Let A_1^s, C_1^s, Q_1^s are specified stationary values of boundary conditions (29). Not complicated, but rather long calculations lead to representations

$$\begin{gathered} A_1(\xi) = \alpha_1\xi + A_1^s, \quad A_2(\xi) = \alpha_2 \equiv \alpha_1 l + A_1^s, \\ C_1(\xi) = \gamma_1\xi + C_1^s, \quad C_2(\xi) = \gamma_2 \equiv \gamma_1 l + C_1^s, \\ F_1(\xi) = \frac{1}{\mathrm{P}_1 l^2}\left[\mathrm{G}_1\left(\frac{\alpha_1+\gamma_1}{24}\xi^4 + \frac{A_1^s + C_1^s}{6}\xi^3\right) - \frac{N_{11}\xi^2}{2}\right] + D_1\xi, \\ F_2(\xi) = \frac{\chi}{\mathrm{P}_2 l^2}\left[\mathrm{G}_2(\alpha_2+\gamma_2)\left(\frac{\xi^3-1}{6} - \frac{l}{2}(\xi^2-1)\right) - \frac{N_{21}}{2}(\xi^2-1)\right] + D_2(\xi-1), \end{gathered} \tag{37}$$

$$\begin{gathered} H_1(\xi) = \frac{1}{\mathrm{P}_1 l^2}\left[\mathrm{G}_1\left(\frac{\alpha_1-\gamma_1}{24}\xi^4 + \frac{A_1^s - C_1^s}{6}\xi^3\right) - \frac{N_{12}\xi^2}{2}\right] + D_3\xi, \\ H_2(\xi) = \frac{\chi}{\mathrm{P}_2 l^2}\left[\mathrm{G}_2(\alpha_2-\gamma_2)\left(\frac{\xi^3-1}{6} - \frac{l}{2}(\xi^2-1)\right) - \frac{N_{22}}{2}(\xi^2-1)\right] + D_4(\xi-1). \end{gathered} \tag{38}$$

The constants $D_1,\ldots,D_4$ are found from the integral equalities (28), (33), (35):

$$\begin{gathered} D_1 = \frac{1}{3\mathrm{P}_1 l}\left[N_{11} - \mathrm{G}_1\left(\frac{(\alpha_1+\gamma_1)l^2}{20} + \frac{(A_1^s + C_1^s)l}{4}\right)\right], \\ D_2 = \frac{\chi}{\mathrm{P}_2 l^2}\left[\frac{N_{21}(l+2)}{3} + \frac{\mathrm{G}_2(\alpha_2+\gamma_2)(l^2+2l-1)}{4}\right], \\ D_3 = \frac{1}{3\mathrm{P}_1 l}\left[N_{12} - \mathrm{G}_1\left(\frac{(\alpha_1-\gamma_1)l^2}{20} + \frac{(A_1^s - C_1^s)l}{4}\right)\right], \\ D_4 = \frac{\chi}{\mathrm{P}_2 l^2}\left[\frac{N_{22}(l+2)}{3} + \frac{\mathrm{G}_2(\alpha_2-\gamma_2)(l^2+2l-1)}{4}\right]. \end{gathered} \tag{39}$$

By virtue of (37),

$$\alpha_2+\gamma_2=(\alpha_1+\gamma_1)l+A_1^s+C_1^s,\quad \alpha_2-\gamma_2=(\alpha_1-\gamma_1)l+A_1^s-C_2^s. \tag{40}$$

To determine the remaining unknowns $\alpha_1,\gamma_1,N_{11},N_{21},N_{12},N_{22}$, there are relations

$$\begin{gathered}F_1(l)=F_2(l),\quad F_{2\xi}(l)=\mu F_{1\xi}(l),\quad H_1(l)=H_2(l),\quad H_{2\xi}=\mu H_{1\xi},\\ A_{2\xi}(l)-kA_{1\xi}(l)=2l^{-2}\mathrm{E}A_1(l)F_1(l),\quad C_{2\xi}(l)-kC_{1\xi}(l)=2l^{-2}\mathrm{E}C_1(l)F_1(l),\end{gathered} \tag{41}$$

where

$$\begin{gathered}F_1(l)=\frac{1}{\mathrm{P}_1}\left[-\frac{1}{6}N_{11}+\mathrm{G}_1\left(\frac{(\alpha_1+\gamma_1)l^2}{40}+\frac{(A_1^s+C_1^s)l}{12}\right)\right],\\ A_1(l)=\alpha_1 l+A_1^s,\quad C_1(l)=\gamma_1 l+C_1^s.\end{gathered} \tag{42}$$

Further,

$$\begin{aligned}F_2(l)&=-\frac{\chi(l-1)^2}{6\mathrm{P}_2l^2}\left[\frac{\mathrm{G}_2(\alpha_2+\gamma_2)(l-1)}{2}+N_{21}\right],\\ H_1(l)&=\frac{1}{2\mathrm{P}_1}\left[-\frac{\mathrm{G}_1(\alpha_1-\gamma_1)l^2}{20}+\frac{\mathrm{G}_1(A_1^s-C_1^s)l}{6}-\frac{N_{12}}{3}\right],\\ H_2(l)&=-\frac{\chi(l-1)^2}{6\mathrm{P}_2l^2}\left[\frac{\mathrm{G}_2(\alpha_2-\gamma_2)(l-1)}{2}+N_{22}\right],\\ F_{1\xi}(l)&=\frac{1}{\mathrm{P}_1l}\left[-\frac{2}{3}N_{11}+\frac{3}{20}\mathrm{G}_1(\alpha_1+\gamma_1)l^2+\frac{5\mathrm{G}_1}{12}(A_1^s+C_1^s)l\right],\\ F_{2\xi}(l)&=-\frac{\chi(l-1)}{\mathrm{P}_2l^2}\left[\frac{2}{3}N_{21}+\frac{\mathrm{G}_2(\alpha_2+\gamma_2)(l-1)}{4}\right],\\ H_{1\xi}(l)&=\frac{1}{\mathrm{P}_1l}\left[-\frac{2}{3}N_{12}+\frac{3}{20}\mathrm{G}_1(\alpha_1-\gamma_1)l^2+\frac{5\mathrm{G}_1}{12}(A_1^s-C_1^s)l\right],\\ H_{2\xi}(l)&=-\frac{\chi(l-1)}{\mathrm{P}_2l^2}\left[\frac{2}{3}N_{22}+\frac{\mathrm{G}_2(\alpha_2-\gamma_2)(l-1)}{4}\right].\end{aligned} \tag{43}$$

Now from the first two equalities (41) we find N_{11} and N_{21}; from the last two equalities (41) we find N_{12} and N_{22}; from the last two equalities, taking into account the formulas (40), we define $\alpha_1+\gamma_1$, $\alpha_1-\gamma_1$, and therefore α_1, γ_1. Below we find the indicated values for $A_1^s=C_1^s$. This is the case of radial heating of the substrate. Here $\alpha_2+\gamma_2=(\alpha_1+\gamma_1)l+2A_1^s$, $\alpha_2-\gamma_2=(\alpha_1-\gamma_1)l$. Let's consider the simplest option: $\alpha_1=\gamma_1$ $(A_1(\xi)=C_1(\xi))$. Then $\alpha_2+\gamma_2=2(\alpha_1+A_1^s)$, $\alpha_2=\gamma_2$ and the formulas (37)–(43) are greatly simplified. Unknown will be α_1, N_{11}, N_{21}, N_{12}, N_{22}. Calculations show that in the general case

$$N_{12}=N_{22}=0,\quad N_{11}=K_1\alpha_1+K_2A_1^s,\quad N_{21}=K_3\alpha_1+K_4A_1^s, \tag{44}$$

where

$$\begin{aligned}K_1&=\frac{\mathrm{G}_1l^2}{20}-\frac{1}{6(l+\mu(1-l))}\left[\frac{3\mathrm{G}_1l^3}{10}\left(1+\frac{3\mu}{2l}(1-l)\right)+\frac{\mathrm{G}_2\nu}{4}(l-1)^3\right],\\ K_2&=\frac{\mathrm{G}_1l}{6}-\frac{1}{6(l+\mu(1-l))}\left[\mathrm{G}_1l^2\left(1+\frac{5\mu}{4l}(1-l)\right)+\frac{\mathrm{G}_2\nu}{4l}(l-1)^3\right],\\ K_3&=\frac{\rho l^2}{(1-l)(l+\mu(1-l))}\left[\frac{3\mathrm{G}_1l^2}{20}+\frac{\mathrm{G}_2(l-1)^2}{\rho l}\left(\frac{3l}{4}+\mu(1-l)\right)\right],\\ K_4&=\frac{\rho l^2}{(1-l)(l+\mu(1-l))}\left[\frac{\mathrm{G}_1l}{4}+\frac{\mathrm{G}_2(l-1)^2}{\rho l}\left(\frac{3l}{4}+\mu(1-l)\right)\right],\quad \rho=\frac{\rho_1}{\rho_2}.\end{aligned} \tag{45}$$

The constant α_1 is the solution of the quadratic equation

$$\mathrm{E}K_1\alpha_1^2 + \left(\frac{k\mathrm{P}_1 l}{2} + \mathrm{E}A_1^s(K_2 + K_1 l^{-1})\right)\alpha_1 + \mathrm{E}K_2 A_1^s l^{-1} = 0. \tag{46}$$

If the quantity

$$\delta = \left(\frac{k\mathrm{P}_1 l}{2} + \mathrm{E}A_1^s(K_2 + K_1 l^{-1})\right)^2 - 4\mathrm{E}^2 K_1 K_2 A_1^s l^{-1} \tag{47}$$

is positive, then there are two solutions of the equation (46), which means that there are two stationary solutions to the two-layer system. For $\delta = 0$ there is one stationary solution, and for $\delta < 0$ there are no solutions.

Remark 4. Fpr $l = \mu(1+\mu)^{-1}$ we get $N_{22} = 1$, $N_{12} = (1-l)(\rho l)^{-1}$, and the formulas (44), (45) retain their form with the replacement of μ by $\mu = l(1-l)^{-1}$.

As for the functions $Q_j(\xi)$, they are determined by the formulas

$$\begin{aligned} Q_1(\xi) &= Q_1^s + a\xi - \frac{2}{l^2}\left(\frac{\alpha_1\xi^3}{3} + A_1^s\xi^2\right), \quad 0 \leqslant \xi \leqslant l, \\ Q_2(\xi) &= b + \frac{2}{l^2}(\alpha_1 l + A_1^s)(2\xi - \xi^2), \quad l \leqslant \xi \leqslant 1, \end{aligned} \tag{48}$$

where

$$\begin{aligned} a &= \frac{4}{l^2}\left[k + \frac{2\mathrm{E}F_1(l)}{l^2}\right]^{-1}\left[(k + l - l^2)(\alpha_1 l + A_1^s) + \frac{2\mathrm{E}F_1(l)}{l}\left(\frac{\alpha_1 l}{2} + A_1^s\right)\right], \\ b &= Q_1^s + a_1 l + \frac{2}{3}(l-6)\alpha_1 - 2A_1^s(l+2)l^{-1}. \end{aligned} \tag{49}$$

In (48), (49) Q_1^s is the dimensionless temperature on the substrate at the origin of coordinates, and $F_1(l)$ is given by the equality (42) at $\alpha_1 = \gamma_1$, $\alpha_2 = \gamma_2 = \alpha_1 l + A_1^s$, $A_1^s = C_1^s$, and α_1 is a solution to the equation (46).

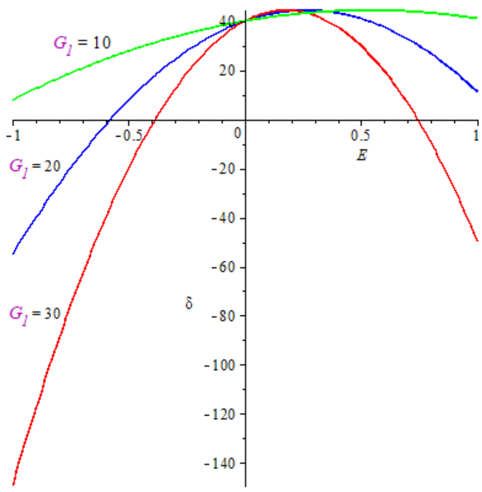

Fig. 1. Dependence $\delta(E)$ for various Grashof numbers G_1; $A_1^s = 0.1$

Figs. 1–3 shows the dependences $\delta(E)$ for various values of dimensionless parameters. All calculations are given for the transformer oil – formic acid system. The dimensionless parameters of the physical system are as follows: $\rho = 0.74$, $\nu = 15.41$, $\chi = 0.71$, $k = 0.41$, $\beta = \beta_1/\beta_1^{-1} = 1.46$, $\mathrm{P}_1 = 308.2$, $\mathrm{P}_2 = 14.2$. Fig. 1 represent the dependence $\delta(E)$ or various Grashof numbers G_1, $\mathrm{G}_2 = \beta\mathrm{G}_1$. It can be seen that as G_1 grows, the region of existence of two solutions decreases.

Fig. 2 illustrates the dependence $\delta(E)$ for various values of the dimensionless parameter A_1^s. Here, for certain values of the parameter E, as A_1^s grows, the region where there are no solutions increases. In the case when $A_1^s \leqslant 0$ there are always two solutions. Fig. 3 shows the dependence $\delta(E)$ from the geometric parameter $l = l_1 l_2^{-1} < 1$. In this case, with an increase in the thickness of the lower layer, the region of existence of two solutions increases.

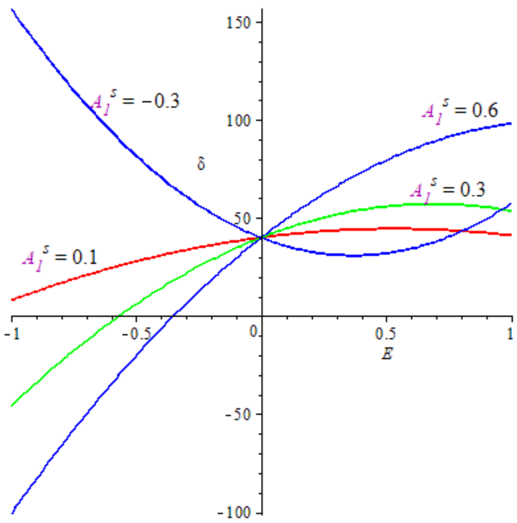

Fig. 2. Dependence $\delta(E)$ for various parameter values A_1^s

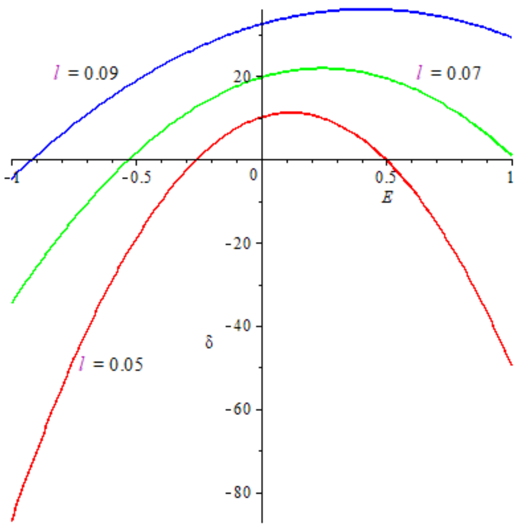

Fig. 3. Dependence $\delta(E)$ for various values of the geometric parameter l

Conclusion

In the article, the problem of three-dimensional two-layer motion with a special velocity field is reduced to the inverse conjugate problem for a system of one-dimensional integro-differential equations. In the case of a stationary flow at low Marangoni numbers, the solution is obtained in the analytical form. It is shown that, depending on the physical and geometric parameters, two stationary modes can exist. For the transformer oil - formic acid system, the effect of changes in interfacial internal energy on the number of stationary solutions has been studied.

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Об одном ползущем трехмерном конвективном движении жидкостей с изотермической границей раздела

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Аннотация. В работе рассматривается двухслойное трехмерное движение жидкостей, поле скоростей которых имеет специальный вид. Возникающая сопряжённая начально-краевая задача для модели Обербека–Буссинеска сведена к системе десяти интегродифференциальных уравнений с полными условиями на плоской поверхности раздела. Показано, что для малых чисел Марангони её стационарный аналог может иметь до двух решений, которые находятся в явном виде. Отдельно проанализирован случай, когда стационарное течение возникает за счет изменения внутренней межфазной энергии.

Ключевые слова: модель Обербека-Буссинеска, межфазная энергия, ползущее течение, обратная задача.