Relationship Between the Bergman and Cauchy-Szegö Kernels in the Domains $\tau^+(n-1)$ and $\Re_{IV}^n$

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Abstract. In this paper, a connection has been established between the Bergman and Cauchy-Szegö kernels using the biholomorphic equivalence of the domains $\tau^+(n-1)$ and the Lie ball $\Re_{IV}^n$. Moreover, integral representations of holomorphic functions in these domains are obtained.

Keywords: classical domains, Lie ball, future tube, Shilov's boundary, Jacobian, Bergman's kernel, Cauchy-Szegö's kernel, Poisson's kernel.

1. Introduction and preliminaries

The selection of classes biholomorphically equivalent domains has great importance in multidimensional analysis and its applications. It is well known that all simply connected proper open subsets of the plane $\mathbb{C}$ are conformally equivalent (Rieman mapping theorem). The situation is completely different in the multidimensional case. For instance, an open unit ball and an open unit polydisc are not biholomorphically equivalent. In fact, there does not exist any holomorphic mapping from one to the other. Therefore, it is very important to have stocks of domains that are biholomorphically equivalent to each other.

Finding the kernels of representations of holomorphic functions in domains $\mathbb{C}^n$ and in the matrix domains from $\mathbb{C}^n[m \times m]$ is a rather difficult task (see [1–4]). Usually, in classical theory, kernels of such kind are constructed in bounded symmetric domains (see [5]). One of such domain is the matrix ball. One considers the following problems for it (see [4,6]): finding the transitive group of automorphisms of a matrix ball; computing the Bergman and Cauchy-Szegö kernels for this domain; finding Carleman's formula, recovering values of a holomorphic function in a matrix ball by its values on some boundary (uniqueness) sets (see [7–9]).

By writing down explicitly the transitive group of automorphisms of the matrix ball, by direct calculation, we can find the Bergman and Cauchy-Szegö kernels for this domain. And then (using the properties of the Poisson kernel) we can find Carleman's formula, which recovers values of a holomorphic function in whole domain by its values on some boundary set of uniqueness

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(see [9–11]). Here we use the scheme from ([5, 12, 13]) for finding the Bergman and Cauchy-Szegö's kernels. In [14], the volumes of the third type matrix ball and the generalized Lie ball are calculated. The full volumes of these domains are necessary for finding kernels of integral formulas for these domains (the Bergman, Cauchy-Szegö's, Poisson kernels etc.). It is also used for the integral representation of functions holomorphic in these domains as well as in the mean value theorem and other important concepts.

Bergman spaces of bounded symmetric domains are fundamental objects in the analysis. They are equipped with natural projection, i.e. with the Bergman projection, which is defined by the property of the reproducing kernel. On the other hand, the Bergman weighted spaces are very important in harmonic analysis also. For any transitive circular domain, the Bergman kernel is equal to the ratio of the volume density to the Euclidean volume of the domain. In the book of Hua Lo-ken (see [5]) the Bergman kernels are constructed for four types of classical domains, guided only by this consideration and without referring to complete orthonormal systems. In [15], holomorphic and pluriharmonic functions for classical domains of the first Cartan type were defined, and the Laplace and Hua Lo-Ken operators were studied. Moreover, the relationship was stated between these operators.

In homogeneous domains,the groups of automorphisms can be used for finding integral formulas ([2, 3]). Domains with rich automorphism groups are often realized as matrix domains ([5, 16]). They are very useful in solving various problems in theory of functions.

In this paper, we continue to develop the analysis in the future tube and move on to the study of the Lie ball. In [17, 18] it was noted that the Lie ball can be realized as a future tube. These realizations are subject of our research. We will be interested in integral formulas with holomorphic kernels in the future tube. There are two main types of formulas for restoring holomorphic functions: the Bergman formulas where integration is carried out over the entire domain and the Cauchy-Szegö formulas where integration is carried out over some set on the boundary of the domain (usually along its skeleton). This implementation turns out to be convenient for calculating the Bergman and Cauchy-Szegö kernels.

1.   Realization of the Lie ball

We consider an n dimensional complex space $\mathbb{C}^n$, the set of all ordered n tuples of complex numbers $z = (z_1, z_2, \ldots, z_n)$. The domain $\Re_{IV}^n$ (the Lie ball (see [5])) consists of all n dimensional complex vectors z satisfying the conditions

$$\Re_{IV}^n = \left\{ z \in \mathbb{C}^n \colon |zz'|^2 - 2\bar{z}z' + 1 > 0, \, |zz'| < 1 \right\},$$

where z' is the transpose of a vector $z = (z_1, z_2, \ldots, z_n)$.

This domain is called the classical fourth type domain (according to E. Cartan's classification (see [19–21])) or the Lie ball. The Shilov boundary (the skeleton) $\Gamma_{\Re_{IV}^n}$ for the domain $\Re_{IV}^n$ is defined as follows:

$$\Gamma_{\Re_{IV}^n} = \left\{ z \in \mathbb{C}^n \colon \bar{z}z' = 1, \, |zz'| = 1 \right\}.$$

An unbounded domain of the form

$$\tau^+(n) = \left\{ w \in \mathbb{C}^{n+1} \colon (Im w_{n+1})^2 > (Im w_1)^2 + \cdots + (Im w_n)^2, \, Im w_{n+1} > 0 \right\}$$

is called the future tube in $\mathbb{C}^{n+1}$. The boundary $\partial\tau^+(n)$ of the domain $\tau^+(n)$ is defined as

$$\partial\tau^+(n) = \left\{ w \in \mathbb{C}^{n+1} \colon (Im w_{n+1})^2 = (Im w_1)^2 + \cdots + (Im w_n)^2, \, Im w_{n+1} > 0 \right\}$$

and the skeleton

$$\Gamma_{\tau^+(n)} = \left\{ w \in \mathbb{C}^{n+1} \colon Im w_1 = \cdots = Im w_n = Im w_{n+1} = 0 \right\} = \mathbb{R}^{n+1},$$

on which the boundary degenerates.

The following statement is true.

Lemma 1. *The map* $\Phi \colon \mathbb{C}_z^n \to \mathbb{C}_w^n$ *defined by the equalities*

$$w_k = \frac{-2iz_k}{\sum\limits_{j=1}^{n-1} z_j^2 + (z_n - i)^2}, \quad k = 1, \ldots, (n-1), \quad w_n = \frac{2(z_n - i)}{\sum\limits_{j=1}^{n-1} z_j^2 + (z_n - i)^2} - i, \tag{1}$$

maps biholomorphically the domain $\Re_{IV}^n$ *onto* $\tau^+(n-1)$, *while* $\Gamma_{\Re_{IV}^n}$ *goes over to* $\Gamma_{\tau^+(n-1)}$.

We call the transformation (1) "*the generalized Cayley transform*". Then, from (1) we can find the inverse map $\Psi = \Phi^{-1} \colon \mathbb{C}_w^n \to \mathbb{C}_z^n$, which is defined as

$$z_k = \frac{-2iw_k}{\sum\limits_{j=1}^{n-1} w_j^2 - (w_n + i)^2}, \quad k = 1, \ldots, (n-1), \quad z_n = i - \frac{2(w_n + i)}{\sum\limits_{j=1}^{n-1} w_j^2 - (w_n + i)^2}. \tag{2}$$

Now we calculate the Jacobians of the transformation (1) and (2). For this purpose we denote

$$W = \sum_{k=1}^{n-1} w_k^2 - (w_n + i)^2 \quad \text{and} \quad Z = \sum_{k=1}^{n-1} z_k^2 + (z_n - i)^2.$$

Lemma 2. *The Jacobians of the transformation* Φ *of the form (1) and* Φ^{-1} *of the form (2) are given by the next formulas respectively*

$$J_{\mathbb{C}}\Phi(z) = 2^n (-i)^{n+1} Z^{-n}$$

and

$$J_{\mathbb{C}}\Phi^{-1}(z) = -2^n (-i)^{n+1} W^{-n}.$$

2. Integral representation in the domain $\tau^+(n-1)$

We denote by dV the normalized Lebesgue measure in the domain $D \subset \mathbb{C}^n$ and define the Bergman space

$$A^2(D) = \left\{ f \in \mathcal{O}(D) \colon \int_D |f(z)|^2 \, dV(z) < \infty \right\}.$$

The inner product in the Bergman space is defined as:

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} dV(z).$$

Let the Bergman kernel $K_{\tau^+(n)}(w, \xi)$ of the domain $\tau^+(n)$ has the form [17]:

$$K_{\tau^+(n)}(w, \xi) = \frac{2^n (n+1)!}{\pi^{n+1} \left[\left(\frac{w - \bar{\xi}}{i} \right)^2 \right]^{n+1}}, \quad w, \xi \in \tau^+(n), \tag{3}$$

where $\left(\dfrac{w-\bar{\xi}}{i}\right)^2 = \left[\left(w_1 - \bar{\xi}_1\right)^2 + \cdots + \left(w_{n-1} - \bar{\xi}_{n-1}\right)^2 - \left(w_n - \bar{\xi}_n\right)^2\right]$. If we denote

$$\Delta\left(y\right) := y_n^2 - \left(y'\right)^2 = y_n^2 - \left(y_2^2 + y_3^2 + \ldots + y_{n-1}^2\right),$$

then the relation (3) can be written as

$$K_{\tau^+(n)}(w,\xi) = \frac{2^n\,(n+1)!}{\pi^{n+1}\Delta^{n+1}\left(\frac{w-\bar{\xi}}{i}\right)}, \quad w,\xi \in \tau^+\left(n\right).$$

It is known that the Bergman kernel for the Lie ball $\Re_{IV}^n$ has the form

$$K_{\Re_{IV}^n}\left(z,\zeta\right) = \frac{1}{V\left(\Re_{IV}^n\right)} \cdot \frac{1}{\left(1 - 2z\bar{\zeta}' + zz'\overline{\zeta\zeta'}\right)^n}, \tag{4}$$

where $V\left(\Re_{IV}^n\right) = \dfrac{\pi^n}{2^{n-1}n!}$ is the volume of the Lie ball $\Re_{IV}^n$ (see [5]).

We denote by $d\mu$, $d\nu$ and $d\eta$, $d\sigma$ the normalized Lebesgue measures in the domains $\tau^+(n-1)$, the Lie ball $\Re_{IV}^n$ and on the skeletons $\Gamma_{\tau^+(n)}$, $\Gamma_{\Re_{IV}^n}$, respectively.

Lemma 3. *Let $w = \Phi(z)$, $\xi = \Phi(\zeta)$. Then by the mapping (1) the Bergman kernel $K_{\tau^+(n-1)}(w,\xi)$ transforms as follows*

$$K_{\tau^+(n-1)}(\Phi(z),\Phi(\zeta)) = \frac{1}{4^n}\left[Z\overline{\Upsilon}\right]^n \cdot K_{\Re_{IV}^n}\left(z,\,\zeta\right), \tag{5}$$

where

$$Z = \sum_{k=1}^{n-1} z_k^2 + \left(z_n - i\right)^2, \quad \Upsilon = \sum_{k=1}^{n-1} \zeta_k^2 + \left(\zeta_n - i\right)^2.$$

Proof. Let $\Phi : \Re_{IV}^n \mapsto \tau^+\left(n-1\right)$ be biholomorphic and $\varphi \in A^2\left(\Re_{IV}^n\right)$. Then by replacing the variable $\zeta = \Phi^{-1}(\xi)$, we have

$$\int\limits_{\Re_{IV}^n} J_{\mathbb{C}}\Phi\left(z\right)K_{\tau^+(n-1)}\left(\Phi\left(z\right),\Phi\left(\zeta\right)\right)\overline{J_{\mathbb{C}}\Phi\left(\zeta\right)}\varphi\left(\zeta\right)d\nu\left(\zeta\right) =$$

$$= \int\limits_{\tau^+(n-1)} J_{\mathbb{C}}\Phi\left(z\right)K_{\tau^+(n-1)}\left(\Phi\left(z\right),\xi\right)\overline{J_{\mathbb{C}}\Phi\left(\Phi^{-1}\left(\xi\right)\right)}\varphi\left(\Phi^{-1}\left(\xi\right)\right)J_{\mathbb{R}}\Phi^{-1}\left(\xi\right)d\mu\left(\xi\right) = \tag{6}$$

$$= J_{\mathbb{C}}\Phi\left(z\right)\int\limits_{\tau^+(n-1)} K_{\tau^+(n-1)}\left(\Phi\left(z\right),\xi\right)\overline{J_{\mathbb{C}}\Phi\left(\Phi^{-1}\left(\xi\right)\right)}\varphi\left(\Phi^{-1}\left(\xi\right)\right)\frac{1}{J_{\mathbb{R}}\Phi\left(\Phi^{-1}\left(\xi\right)\right)}d\mu\left(\xi\right).$$

By the Jacobian property $\left(J_{\mathbb{R}}\Phi = |J_{\mathbb{C}}\Phi|^2\right)$ the last integral in (6) has the form:

$$J_{\mathbb{C}}\Phi\left(z\right)\int\limits_{\tau^+(n-1)} K_{\tau^+(n-1)}\left(\Phi\left(z\right),\xi\right)\overline{J_{\mathbb{C}}\Phi\left(\Phi^{-1}\left(\xi\right)\right)}\varphi\left(\Phi^{-1}\left(\xi\right)\right)\frac{1}{\left|J_{\mathbb{C}}\Phi\left(\Phi^{-1}\left(\xi\right)\right)\right|^2}d\mu\left(\xi\right) =$$

$$= J_{\mathbb{C}}\Phi\left(z\right)\int\limits_{\tau^+(n-1)} K_{\tau^+(n-1)}\left(\Phi\left(z\right),\xi\right)\left[\left(J_{\mathbb{C}}\Phi\left(\Phi^{-1}\left(\xi\right)\right)\right)^{-1}\varphi\left(\Phi^{-1}\left(\xi\right)\right)\right]d\mu\left(\xi\right).$$

After changing variables, we can see that the expression $\left(J_{\mathbb{C}}\Phi\left(\Phi^{-1}\left(\xi\right)\right)\right)^{-1}\varphi\left(\Phi^{-1}\left(\xi\right)\right)$ in square brackets in the last integrand is an element of the space $A^2\left(\tau^+\left(n-1\right)\right)$. Applying the reproducing property of $K_{\tau^+(n-1)}$, we have

$$J_{\mathbb{C}}\Phi\left(z\right)\left(J_{\mathbb{C}}\Phi\left(z\right)\right)^{-1}\varphi\left(\Phi^{-1}\left(\Phi\left(z\right)\right)\right) = \varphi\left(z\right).$$

From here it follows "*variable replacement formula*" for the Bergman kernels:

$$J_{\mathbb{C}}\Phi\left(z\right)K_{\tau^{+}(n-1)}\left(\Phi\left(z\right),\Phi\left(\zeta\right)\right)\overline{J_{\mathbb{C}}\Phi\left(\zeta\right)}=K_{\Re_{IV}^{n}}\left(z,\zeta\right).$$

Then

$$K_{\tau^{+}(n-1)}(\Phi(z),\Phi(\zeta))=[J_{\mathbb{C}}\Phi t(z)]^{-1}\cdot K_{\Re_{IV}^{n}}(z,\zeta)\cdot\left[\overline{J_{\mathbb{C}}\Phi(\zeta)}\right]^{-1}=$$

$$=2^{-n}(-i)^{-n-1}Z^{n}\cdot2^{-n}(i)^{-n-1}\overline{\Upsilon}^{n}K_{\Re_{IV}^{n}}(z,\zeta)=\frac{1}{4^{n}}\left[Z\overline{\Upsilon}\right]^{n}\cdot K_{\Re_{IV}^{n}}(z,\zeta).$$

The lemma is proved   □

In particular, when $n=1$, from formulas (4) and (5), we have

$$K_{\tau^{+}(0)}(w,\xi)=\frac{1}{\pi\Delta\left(\frac{w-\overline{\xi}}{i}\right)}=-\frac{1}{\pi\left(w-\overline{\xi}\right)^{2}}-\frac{1}{\pi\left(-i\frac{z+i}{z-i}-i\frac{\overline{\zeta}-i}{\overline{\zeta}+i}\right)^{2}}=\frac{(z-i)^{2}\left(\overline{\zeta}+i\right)^{2}}{4\pi\left(1-z\overline{\zeta}\right)^{2}}.$$

On the other hand

$$K_{\tau^{+}(0)}(w,\xi)=K_{\Re_{IV}^{1}}(z,\zeta)\frac{Z}{2}\frac{\overline{\Upsilon}}{2}=\frac{1}{\pi\left(1-z\overline{\zeta}\right)^{2}}\frac{(z-i)^{2}\left(\overline{\zeta}-i\right)^{2}}{4}=\frac{(z-i)^{2}\left(\overline{\zeta}+i\right)^{2}}{4\pi\left(1-z\overline{\zeta}\right)^{2}}.$$

Let $w=\Phi(z)$, $\xi=\Phi(\zeta)$. We have the following theorem

Theorem 1. *For any function $f\in A^{2}\left(\tau^{+}\left(n-1\right)\right)$ the formula holds:*

$$f(w)=\int\limits_{\tau^{+}(n-1)}f(\xi)K_{\tau^{+}(n-1)}(w,\xi)d\mu(\xi),\ \ w\in\tau^{+}\left(n-1\right).$$

The integral in this formula defines an orthogonal projector of the space $L^{2}(\tau^{+}\left(n-1\right))$ into the space $A^{2}\left(\tau^{+}\left(n-1\right)\right)$.

Proof. Using the change of variables, according to Lemma 3 and the Jacobian properties we have:

$$\int\limits_{\tau^{+}(n-1)}f(\xi)K_{\tau^{+}(n-1)}(w,\xi)d\mu(\xi)=\frac{1}{4^{n}}\int\limits_{\Re_{IV}^{n}}f(\Phi(\zeta))K_{\Re_{IV}^{n}}(z,\zeta)Z^{n}\overline{\Upsilon}^{n}J_{\mathbb{R}}\Phi\left(\zeta\right)d\nu(\zeta)=$$

$$=\frac{Z^{n}}{2^{n}}\int\limits_{\Re_{IV}^{n}}f(\Phi(\zeta))K_{\Re_{IV}^{n}}(z,\zeta)\frac{\overline{\Upsilon}^{n}}{2^{n}}\left|J_{\mathbb{C}}\Phi\left(\zeta\right)\right|^{2}d\nu(\zeta)=$$

$$=Z^{n}\int\limits_{\Re_{IV}^{n}}\frac{f(\Phi(\zeta))}{\Upsilon^{n}}K_{\Re_{IV}^{n}}(z,\zeta)d\nu(\zeta).$$

The last integral is the Bergman integral in the Lie ball $\Re_{IV}^{n}$ of the function $\dfrac{f(\Phi(\zeta))}{\Upsilon^{n}}$ and it is equal to $\dfrac{f(\Phi(z))}{Z^{n}}$. So, we obtain the first statement of the theorem.

Any function $g\in L^{2}(\tau^{+}\left(n-1\right))$ can be represented as $g=f+h$, where $f\in A^{2}(\tau^{+}\left(n-1\right))$ and $h\in A^{2^{\perp}}\left(\tau^{+}\left(n-1\right)\right)$ are orthogonal function:

$$\int\limits_{\tau^{+}(n-1)}f(\xi)\overline{h(\xi)}d\mu(\xi)=0.$$

We must show that

$$\int\limits_{\tau^+(n-1)} h(\xi) K_{\tau^+(n-1)}(w,\xi) d\mu(\xi) = 0.$$

So,

$$\int\limits_{\tau^+(n-1)} f(\xi)\overline{h(\xi)} d\mu(\xi) = \int\limits_{\Re^n_{IV}} f(\Phi(\zeta))\overline{h(\Phi(\zeta))} \left|2^n(-i)^{n+1}\Upsilon^{-n}\right|^2 d\nu(\zeta) =$$

$$= 2^n \int\limits_{\Re^n_{IV}} f(\Phi(\zeta))\Upsilon^{-n}\overline{h(\Phi(\zeta))\Upsilon^{-n}} d\nu(\zeta) = 0.$$

Hence, $h(\Phi(\zeta))\Upsilon^{-n} \in A^{2^\perp}(\Re^n_{IV})$, i.e.

$$\int\limits_{\Re^n_{IV}} h(\Phi(\zeta)) K_{\Re^n_{IV}}(z,\zeta)\Upsilon^{-n} d\nu(\zeta) = 0.$$

Then

$$\int\limits_{\tau^+(n-1)} h(\xi) K_{\tau^+(n-1)}(w,\xi) d\mu(\xi) = \frac{Z^n}{4^n} \int\limits_{\Re^n_{IV}} h(\Phi(\zeta))\overline{\Upsilon}^n K_{\Re^n_{IV}}(z,\zeta)\left|2^n(-i)^{n+1}\Upsilon^{-n}\right|^2 d\nu(\zeta) =$$

$$= Z^n \int\limits_{\Re^n_{IV}} h(\Phi(\zeta))\Upsilon^n K_{\Re^n_{IV}}(z,\zeta) d\nu(\zeta) = 0.$$

The theorem is completely proved.   □

We define the Cauchy-Szegö kernel $C_{\tau^+(n-1)}(w,\xi)$ as follows (see [22])

$$C_{\tau^+(n-1)}(w,\xi) = \frac{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\Delta^{\frac{n+1}{2}}(w-\xi)},$$

for $w \in \tau^+(n)$, $\xi \in \Gamma_{\tau^+(n)}$.

The kernel $C_{\tau^+(n-1)}(w,\xi)$ is a holomorphic function in w and antiholomorphic in ξ.

The proof of the following lemma is similar to the proof of Lemma 1.

Lemma 4. *Let* $w = \Phi(z)$, $\xi = \Phi(\zeta)$. *By mapping (1), the Cauchy-Szegö kernel* $C_{\tau^+(n-1)}(w,\xi)$ *transforms in the following way*

$$C_{\tau^+(n-1)}(\Phi(z),\ \Phi(\zeta)) = \frac{1}{2^n} Z^{\frac{n}{2}} \bar{\Upsilon}^{\frac{n}{2}} C_{\Re^n_{IV}}(z,\zeta),$$

where $C_{\Re^n_{IV}}(z,\zeta)$ *is the Cauchy-Szegö kernel for the Lie ball* $\Re^n_{IV}$ *(see [19]).*

$$C_{\Re^n_{IV}}(z,\zeta) = \frac{1}{V\left(\Gamma_{\Re^n_{IV}}\right)\left[(x-e^{-i\varphi}z)(x-e^{-i\varphi}z)'\right]^{\frac{n}{2}}},$$

$$\zeta = e^{i\varphi}x,\ \ x \in \mathbb{R}^n,\ \ xx' = 1,\ \ \varphi \in [0; 2\pi].$$

Proof. According to Jacobian property (see [5]) for the Cauchy-Szegö kernel we have

$$\left[C_{\Re^n_{IV}}(z,\zeta)\right]^2 = \left[C_{\tau^+(n-1)}(\Phi(z),\Phi(\zeta))\right]^2 J_{\mathbb{C}}\Phi(z)\overline{J_{\mathbb{C}}\Phi(\zeta)},$$

it follows that

$$C_{\tau^+(n-1)}\left(\Phi\left(z\right),\Phi\left(\zeta\right)\right) = \frac{1}{2^n}Z^{\frac{n}{2}}\bar{\Upsilon}^{\frac{n}{2}}C_{\Re_{IV}^n}\left(z,\zeta\right) = \frac{Z^{\frac{n}{2}}\bar{\Upsilon}^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{2^{n-1}\pi^{\frac{n+2}{2}}\left[\left(x-e^{-i\varphi}z\right)\left(x-e^{-i\varphi}z\right)'\right]^{\frac{n}{2}}}.$$

Theorem 2. *For any function $f \in H^1(\tau^+(n-1))$ the formula holds[‡]*

$$f(w) = \int\limits_{\Gamma_{\tau^+(n-1)}} f(\xi)C_{\tau^+(n-1)}(w,\xi)d\eta(\xi),\ \ w \in \tau^+(n-1).$$

Proof. It is known that the Poisson kernel for the domain $\tau^+(n)$ (see [17]) can be written in the form

$$P_{\tau^+(n)}(w,\xi) = \frac{2^n\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+3}{2}}}\frac{\Delta^{\frac{n}{2}}(Imw)}{|\Delta^n(w-\xi)|},\ \ w \in \tau^+(n),\ \ \xi \in \Gamma_{\tau^+(n)}.$$

By Lemma 3.4 from [23] we can get

$$P_{\tau^+(n-1)}\left(\Phi(z),\Phi(\zeta)\right)d\eta(\Phi(\zeta)) = P_{\Re_{IV}^n}(z,\zeta)d\sigma(\zeta), \tag{7}$$

where

$$P_{\Re_{IV}^n}(z,\zeta) = \frac{\left(1+|(z,z)|^2-2|z|^2\right)^{\frac{n}{2}}}{|(z-\zeta,z-\zeta)|^n},\ \ z \in \Re_{IV}^n,\ \zeta \in \Gamma_{\Re_{IV}^n},$$

is the Poisson kernel for the Lie ball.

On the other hand, due to the relation between the Cauchy-Szegö and Poisson kernels (see [23]) we have

$$P(w,\xi) = \frac{C(w,\xi)C(\xi,w)}{C(w,w)} = \frac{|C(w,\xi)|^2}{C(w,w)},$$

and by Lemma 2 we get that

$$P_{\tau^+(n-1)}\left(\Phi\left(z\right),\Phi\left(\zeta\right)\right) = \frac{\left|C_{\tau^+(n-1)}(w,\xi)\right|^2}{C_{\tau^+(n-1)}(w,w)} = \frac{\frac{1}{4^n}|Z|^n|\Upsilon^n|}{\frac{1}{2^n}|Z|^n}\frac{\left|C_{\Re_{IV}^n}(w,\xi)\right|^2}{C_{\Re_{IV}^n}(w,w)} =$$

$$= \frac{1}{2^n}|\Upsilon^n|P_{\Re_{IV}^n}(z,\zeta).$$

From that, we get

$$P_{\tau^+(n-1)}\left(\Phi\left(z\right),\Phi\left(\zeta\right)\right) = \frac{1}{2^n}|\Upsilon^n|P_{\Re_{IV}^n}(z,\zeta). \tag{8}$$

Now dividing the relation (7) by (8), we obtain

$$d\eta\left(\Phi\left(\zeta\right)\right) = 2^n\left|\Upsilon^{-n}\right|d\sigma\left(\zeta\right).$$

Further on, after changing variable $\xi = \Phi(\zeta)$ and taking into account Lemma 2, we have

$$\int\limits_{\Gamma_{\tau^+(n-1)}} f(\xi)C_{\tau^+(n-1)}(w,\xi)d\eta(\xi) = Z^{\frac{n}{2}}\int\limits_{\Gamma_{\Re_{IV}^n}} \frac{f(\Phi(\zeta))}{|\Upsilon^n|}\bar{\Upsilon}^{\frac{n}{2}}C_{\Re_{IV}^n}(z,\zeta)d\sigma(\zeta) =$$

$$= Z^{\frac{n}{2}}\int\limits_{\Gamma_{\Re_{IV}^n}} \frac{f(\Phi(\zeta))}{\Upsilon^{\frac{n}{2}}}C_{\Re_{IV}^n}(z,\zeta)d\sigma(\zeta).$$

[‡]The Hardy class $H^1(D)$ is defined as follows: a function f holomorphic in D belongs to $H^1(D)$, if $\sup\limits_{0\leqslant r<1}\int\limits_{S(D)}|f(r\xi)|\,d\eta < \infty$, where η is the Lebesgue measure on the skeleton $S(D)$.

The last integral is the Cauchy-Szegö integral in the Lie ball of functions $\dfrac{f(\Phi(\zeta))}{\Upsilon^{\frac{n}{2}}}$, and it is equal to $\dfrac{f(\Phi(z))}{Z^{\frac{n}{2}}}$. It gives us the statement of the theorem.

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Связь между ядрами Бергмана и Коши-Сеге в областях $\tau^+\left(n-1\right)$ и $\Re_{IV}^n$

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Аннотация. В работе с использованием биголоморфной эквивалентности областей $\tau^+\left(n-1\right)$ и шара Ли $\Re_{IV}^n$ найдена связь между ядрами Бергмана и Коши-Сеге. Получены интегральные представления голоморфных функций в этих областях.

Ключевые слова: классические области, шар Ли, труба будущего, граница Шилова, Якобиан, ядро Бергмана, ядро Коши-Сеге, ядро Пуассона.