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L^p Regularity of the Solution of the Heat Equation with Discontinuous Coefficients

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Abstract. In this paper, we consider the transmission problem for the heat equation on a bounded plane sector in L^p -Sobolev spaces. By Applying the theory of the sums of operators of Da Prato-Grisvard and Dore-Venni, we prove that the solution can be splitted into a regular part in L^p -Sobolev space and an explicit singular part.

Keywords: transmission heat equation, sums of linear operators, singular behavior, non-smooth domains.

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1. Introduction and preliminaries

Let $\mathcal{B} = G \times]0, T[$, where G is a bounded plane sector constituted of two plane sectors G_1, G_2 with respective opening ω_1 and ω_2 , separated by an interface Σ .

$$G_1 = \{(r \cos \theta, r \sin \theta); -\omega_1 < \theta < 0, 0 < r < 1\},$$

$$G_2 = \{(r \cos \theta, r \sin \theta); 0 < \theta < \omega_2, 0 < r < 1\},$$

$$\Sigma = \{(r, 0); 0 < r < 1\}.$$

In this paper we study the regularity of the solution of the following transmission problem for the heat equation

$$\partial_t u_i - \Delta u_i = g_i \quad \text{in } \mathcal{B}_i = G_i \times]0, T[; \quad i = 1, 2, \quad (1)$$

$$u_1 = u_2 \quad \text{on } \Sigma \times [0, T], \quad (2)$$

$$\alpha_1 \frac{\partial u_1}{\partial n_1} + \alpha_2 \frac{\partial u_2}{\partial n_2} = 0 \quad \text{on } \Sigma \times [0, T], \quad (3)$$

$$u_i = 0 \quad \text{on } (\partial G_i \setminus \Sigma) \times [0, T]; \quad i = 1, 2, \quad (4)$$

$$u_i(., 0) = 0 \quad \text{in } G_i; \quad i = 1, 2, \quad (5)$$

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where n_i denotes the unit normal vector to $\Sigma \times [0, T]$ directed outside $\mathcal{B}_i$, u_i means the restriction of u to $\mathcal{B}_i$, $g \in L^p(\mathcal{B})$ and α_1, α_2 are two positive real numbers such that $\alpha_1 \neq \alpha_2$.

Problem (1)–(5) is a particular case of the abstract Cauchy problem

$$\partial_t u(t) + Au(t) = f(t) \quad 0 \leq t \leq T, \quad (6)$$

$$u(0) = 0, \quad (7)$$

where $f \in L^p([0, T[; X])$, also considered as a special case of the more general operator equation

$$(A + B)u = f, \quad (8)$$

in which B is the derivative operator defined on the interval $[0, T]$ with values in a Banach function space.

The study of the abstract equation (8), where A and B are two closed linear operators with dense domains acting in a complex Banach space X , is based on the theory of sums of operators in Banach spaces. In [2], G. Da Prato and P. Grisvard proved under appropriate assumptions on the resolvents of A and B that the sum operator $A + B$ is closable. As an application, problem (6)–(7) has a strong solution that is a solution in $L^p([0, T[; X])$. In their famous paper [6], G. Dore and A. Venni showed under appropriate assumptions on the imaginary powers of A and B and if the space X is U.M.D. that $A + B$ is closed. As an application, problem (6)–(7) has the L^p maximal regularity property. We refer to [1, 4, 5, 7–12] for some applications of the theory of sums of operators.

By analogy with Grisvard [8, 9] who studied the heat equation in plane polygonal domains in L^p -Sobolev spaces and De Coster-Nicaise [5] who extended his results to the weighted L^p -Sobolev spaces setting, we show that the solution u of problem (1)–(5) is decomposable into a regular part having the optimal regularity $L^p([0, T[; PW^{2,p}(G) \cap W_0^{1,p}(G)) \cap W^{1,p}([0, T[; L^p(G))$ and a finite sum of explicit singular functions. For the sake of simplicity, we restrict ourselves to the case of two sectors G_i , $i = 1, 2$ with a common interface Σ . The result of this paper can be easily extended to the case of more than two sectors using the results from [13].

The paper is organised as follows:

In Section 2, we present the main results of the theory of the sums of operators of Da Prato-Grisvard [2] and Dore-Venni [6]. Applying this theory requires some results concerning a transmission problem with complex parameter z . This will be recalled from [1] in Section 3 and extended for z in a larger part of the complex plane. In Section 4 we apply the strategy of Da Prato-Grisvard to show existence and uniqueness of a strong solution u of problem (1)–(5) which admits a decomposition in regular and singular parts. Optimal regularity of the regular part is obtained in Section 5 by applying a Dore-Venni result.

Let us finish this introduction with some notation used in the whole paper: if D is an open subset of $\mathbb{R}^N$ ($N = 1$ or 2), we denote by $L^p(D)$, ($p > 1$) the Lebesgue spaces, and by $W^{s,p}(D)$, $s \geq 0$, the standard Sobolev spaces built on. The space $W_0^{1,p}(D)$ is defined as usual by $W_0^{1,p}(D) := \{v \in W^{1,p}(D); v = 0 \text{ on } \partial D\}$. When $p = 2$, we use the common notation $H_0^1(D)$ instead of $W_0^{1,p}(D)$.

For any separable Banach space X provided with the norm $\|\cdot\|_X$, we denote by $L^p([0, T[; X])$ the space of measurable functions v from $]0, T[$ in X such that

$$\|v\|_{L^p([0, T[; X])} = \left(\int_0^T \|v(\cdot, t)\|_X^p dt \right)^{\frac{1}{p}} < +\infty,$$

and by $W^{1,p}([0, T[; X])$ the Sobolev space of functions v in $L^p([0, T[; X])$ such that $\partial_t v$ belongs to $L^p([0, T[; X])$.

2. Sums of linear operators

For an operator P we denote by $\sigma(P)$ and $\rho(P)$ respectively its spectrum and its resolvent set.

2.1. The first strategy

We recall some results on sums of operators of Da prato-Grisvard taken from [9].

Let E be a complex Banach space and A, B two closed linear operators with dense domains $D(A)$ and $D(B)$ respectively. Their sum is defined by

$$Lx = Ax + Bx,$$

for every $x \in D(L) = D(A) \cap D(B)$.

Assumptions

H₁ There exist positive numbers $M_A, M_B, R, \theta_A, \theta_B$ such that $\theta_A + \theta_B > \pi$ and $\rho(-A)$ contains the truncated sector

$$S_A = \{\lambda; |\lambda| \geq R, |\arg \lambda| \leq \theta_A\},$$

while $\rho(-B)$ contains the truncated sector

$$S_B = \{\lambda; |\lambda| \geq R, |\arg \lambda| \leq \theta_B\},$$

and

$$\begin{aligned} \|(A + \lambda)^{-1}\| &\leq \frac{M_A}{|\lambda|}, & \forall \lambda \in S_A, \\ \|(B + \lambda)^{-1}\| &\leq \frac{M_B}{|\lambda|}, & \forall \lambda \in S_B. \end{aligned}$$

H₂ $\sigma(-A) \cap \sigma(B) = \emptyset$.

H₃ The resolvents of A and B commute, i.e.

$$(A + \lambda)^{-1}(B + \mu)^{-1} = (B + \mu)^{-1}(A + \lambda)^{-1},$$

for every $\lambda \in \rho(-A)$ and every $\mu \in \rho(-B)$.

Theorem 2.1 (Da Prato-Grisvard [2]). *Under the assumptions **H₁**, **H₂**, **H₃**, the closure $\bar{L}$ of L is invertible.*

An explicit construction of the inverse of $\bar{L}$ is given by the Dunford integral

$$(\bar{L})^{-1} = \frac{1}{2\pi i} \int_{\gamma} (A + \lambda I)^{-1} (\lambda I - B)^{-1} d\lambda,$$

where the path γ separates $\sigma(-A)$ and $\sigma(B)$ and joins $\infty e^{-i\theta_\gamma}$ to $\infty e^{i\theta_\gamma}$ where θ_γ is chosen so that $\pi - \theta_B < \theta_\gamma < \theta_A$.

The unique solution $v \in D(\bar{L})$ of the equation

$$\bar{L}v = (\overline{A + B})v = f,$$

is called the strong solution of $Lv = f$. This means the existence of a sequence $(v_n) \subset D(L)$ such that $v_n \rightarrow v$ and $Lv_n \rightarrow f$ in E .

2.2. The second strategy

Here we just recall from [6] an application of the theory of sums of operators of Dore-Venni. Let X be an U.M.D. complex Banach space and $A : D(A) \rightarrow X$ a closed linear operator with dense domain in X satisfying the following assumptions

H₄ $\rho(A) \supset]-\infty, 0]$ and there exists $M_A > 0$ such that

$$\|(A + t)^{-1}\| \leq \frac{M_A}{t+1} \quad \forall t \geq 0.$$

H₅ $A^{is} \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$ and there exist $K > 0$, τ_A such that $0 \leq \tau_A < \frac{\pi}{2}$ and

$$\|A^{is}\| \leq K e^{|s|\tau_A} \quad \forall s \in \mathbb{R},$$

where A^{is} are the complex powers of A .

If we also call A the operator induced on $E = L^p([0, T[; X])$ by the equality $(Au)(t) = A(u(t))$, it is obvious that A has the same properties in E as in X . The application of Theorems 2.1 and 3.1 of [6] gives

Theorem 2.2 (Dore-Venni [6]). *Under the assumptions **H₄** and **H₅**, the Cauchy problem (6)–(7) has the L^p maximal regularity property, that is for each $f \in L^p([0, T[; X])$, $1 < p < \infty$, it has a unique solution $u \in W^{1,p}([0, T[; X) \cap L^p([0, T[; D(A))$.*

3. Results on the transmission problem

We consider the following Helmholtz transmission problem with complex parameter z

$$-\Delta u_i + z u_i = f_i \text{ in } G_i; \quad i = 1, 2, \quad (9)$$

$$u_i = 0 \text{ on } \partial G_i \setminus \Sigma; \quad i = 1, 2, \quad (10)$$

$$u_1 = u_2 \text{ on } \Sigma, \quad (11)$$

$$\alpha_1 \frac{\partial u_1}{\partial n_1} + \alpha_2 \frac{\partial u_2}{\partial n_2} = 0 \text{ on } \Sigma, \quad (12)$$

where $f \in L^p(G)$, $1 < p < +\infty$, n_i here denotes the normal vector to Σ directed outside G_i and u_i is still the restriction of u to G_i .

Problem (9)–(12) admits the equivalent variational formulation:
Find $u \in H_0^1(G)$ such that

$$a_z(u, v) = \int_G \alpha f \bar{v} \, dx \quad \forall v \in H_0^1(G), \quad (13)$$

where

$$a_z(u, v) = \int_G \alpha \left\{ \sum_{i=1}^2 \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i} + z u \bar{v} \right\} dx,$$

$$\alpha(x) = \alpha_i > 0 \text{ for } x \in G_i; \quad i = 1, 2, \text{ with } \alpha_1 \neq \alpha_2.$$

In what follows, we use the positive constant C to denote a generic constant and may take different values in different places.

Lemma 3.1. *Let $\theta_A \in]0, \pi[$. Then problem (13) admits a unique solution $u \in H_0^1(G)$ for any $f \in L^p(G)$ and any $z \in \mathbb{C}$ with $|\arg z| \leq \theta_A$.*

Proof. Since $H_0^1(G) \hookrightarrow L^{p'}(G)$ (p' being the conjugate exponent of p), we can easily see that the antilinear form

$$\begin{aligned} K : H_0^1(G) &\longrightarrow \mathbb{C} \\ v &\longmapsto \int_G \alpha f \bar{v} dx, \end{aligned}$$

is continuous on $H_0^1(G)$, with the estimate

$$|K(v)| \leq C \|f\|_{L^p(G)} \|v\|_{H_0^1(G)}, \quad \forall v \in H_0^1(G). \quad (14)$$

It is clear that the sesquilinear form a_z is continuous on $H_0^1(G) \times H_0^1(G)$. It is also coercive observing that, for all $z \in \mathbb{C}$ with $|\arg z| \leq \theta_A$, there exists $\theta \in [0, 2\pi]$ such that $\cos \theta > 0$ and $\Re(ze^{i\theta}) \geq 0$, which implies thanks to Poincaré's inequality

$$\Re(e^{i\theta} a_z(v, v)) \geq \cos \theta \int_G \alpha |\nabla v|^2 dx \geq C \|v\|_{H_0^1(G)}^2,$$

for all $v \in H_0^1(G)$. We conclude using the Lax-Milgram lemma. $\square$

For $R > 0$ and $\theta_A \in \left] \frac{\pi}{2}, \pi \right[$ fixed, we define the sets S^+ and S_A as follows

$$S^+ = \{z \in \mathbb{C} / \Re(z) \geq 0\},$$

$$S_A = \{z \in \mathbb{C} / |z| \geq R \text{ and } |\arg z| \leq \theta_A\}.$$

Lemma 3.2. *Let $R > 0$ and $\theta_A \in \left] \frac{\pi}{2}, \pi \right[$ be fixed. Let $z \in S^+ \cup S_A$ and $u \in H_0^1(G)$ be the solution of (13), then u satisfies the estimate*

$$\|u\|_{L^p(G)} \leq C \|f\|_{L^p(G)}. \quad (15)$$

Proof. We proceed as in [3, Lemma 2.4]. By Applying (13) with $v = u$, we obtain

$$\int_G \alpha \{|\nabla u|^2 + z|u|^2\} dx = \int_G \alpha f \bar{u} dx. \quad (16)$$

Taking the real and the imaginary parts of (16) respectively, we obtain using (14)

$$\int_G \alpha |\nabla u|^2 dx + \Re(z) \int_G \alpha |u|^2 dx \leq C \|f\|_{L^p(G)} \|u\|_{H_0^1(G)}, \quad (17)$$

and

$$|\Im(z)| \int_G \alpha |u|^2 dx \leq C \|f\|_{L^p(G)} \|u\|_{H_0^1(G)}. \quad (18)$$

Case 1 : $\Re(z) \geq 0$. Due to (17) and Poincaré's inequality, we deduce that

$$\|u\|_{H_0^1(G)} \leq C \|f\|_{L^p(G)},$$

which gives (15) since $H_0^1(G) \hookrightarrow L^p(G)$ for all $1 < p < \infty$.

Case 2 : $\Re(z) < 0$. In this case $z \in S_A$, then $\Re(z) = \rho \cos \theta$, $\Im(z) = \rho \sin \theta$ with $\rho \geq R$ and $|\Im(z)| > \rho \sin \theta_A$.

Consequently, from (18) we obtain

$$\|u\|_{L^2(G)}^2 \leq C \frac{1}{\rho} \|f\|_{L^p(G)} \|u\|_{H_0^1(G)}. \quad (19)$$

As $\Re(z) < 0$, the estimate (17) gives

$$\|\nabla u\|_{L^2(G)}^2 \leq C(\|f\|_{L^p(G)}\|u\|_{H_0^1(G)} - \Re(z)\|u\|_{L^2(G)}^2).$$

Due to (19) and Poincaré's inequality this gives

$$\|u\|_{H_0^1(G)} \leq C\left(1 - \Re(z)\frac{1}{\rho}\right)\|f\|_{L^p(G)}.$$

We conclude using the inequality $-\Re(z) \leq \rho$. $\square$

For all $1 < p < \infty$, we consider the operator A_p defined by

$$D_{A_p} = \left\{ u \in H_0^1(G); \Delta u_i \in L^p(G_i), u_1 = u_2 \text{ and } \alpha_1 \frac{\partial u_1}{\partial n_1} + \alpha_2 \frac{\partial u_2}{\partial n_2} = 0 \text{ on } \Sigma \right\},$$

$$\begin{aligned} A_p : D_{A_p} &\subset L^p(G) \longrightarrow L^p(G) \\ u = (u_1, u_2) &\longmapsto (-\Delta u_1, -\Delta u_2). \end{aligned}$$

Note that $L^p(G) \equiv L^p(G_1) \times L^p(G_2)$ and $u_i = u|_{G_i}; i = 1, 2$.

Theorem 3.1 ([1]). *Let $f \in L^p(G), z \in \mathbb{C}$ with $\Re(z) \geq 0$ and $u \in H_0^1(G)$ be the solution of (13), then u satisfies the estimates*

$$\Re(z)\|u\|_{L^p(G)} \leq \|f\|_{L^p(G)},$$

$$|\Im(z)|\|u\|_{L^p(G)} \leq \frac{p}{2}\|f\|_{L^p(G)},$$

and

$$|z|\|u\|_{L^p(G)} \leq C\|f\|_{L^p(G)}. \quad (20)$$

Corollary 3.1. $-A_p$ is the infinitesimal generator of a C_0 semigroup of contraction $T(t)$ for $t \geq 0$.

As in [3], we can prove the estimate (20) for z in a larger part of the complex plane.

Corollary 3.2. *There exists $\theta_A \in \left] \frac{\pi}{2}, \pi \right[$ such that, for all $f \in L^p(G)$, all $z \in \mathbb{C}$ with $|\arg z| \leq \theta_A$ and $u \in H_0^1(G)$ solution of (13), we have*

$$|z|\|u\|_{L^p(G)} \leq C\|f\|_{L^p(G)}. \quad (21)$$

Proof. By Theorem 3.1, there exists a positive constant c such that, for all $\sigma > 0$ and $\tau \in \mathbb{R}^*$

$$|\sigma + i\tau| \|(A_p + (\sigma + i\tau)I)^{-1}\| \leq c,$$

hence, thanks to Corollary 3.1, we can apply Theorem II-5.2 in [14] to deduce the existence of $\delta \in \left] 0, \frac{\pi}{2} \right[$ and $M > 0$ such that

$$\rho(-A_p) \supset \Gamma := \left\{ z \in \mathbb{C} / |\arg z| < \frac{\pi}{2} + \delta \right\} \cup \{0\}, \quad (22)$$

and, for all $z \in \Gamma$,

$$|z| \|(A_p + zI)^{-1}\| \leq M. \quad (23)$$

This proves (21) with $\theta_A = \delta' + \frac{\pi}{2}$, where $0 < \delta' < \delta$. $\square$

In order to present the singular behavior of the variational solution of the transmission problem (9)–(12), we need the following notations.

For $s > 0$,

$$PW^{s,p}(G) := \{u \in H^1(G); u_i \in W^{s,p}(G_i), i = 1, 2\},$$

is the space of piecewise $W^{s,p}$ functions on G .

Let $S^{(m)}$ be the function defined by

$$S^{(m)} = \eta r^{\lambda_m} t_m(\theta), \quad (24)$$

where η is a radial cut-off function such that $\eta \equiv 1$ in a small ball centered at the origin and $\eta \equiv 0$ outside a larger ball of radius strictly less than 1, λ_m is a nonnegative real number and λ_m^2 , t_m are respectively the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$\begin{cases} -t_m''(\theta) = \lambda_m^2 t_m(\theta) & \text{for } \theta \in [-\omega_1, \omega_2], \theta \neq 0, \\ t_m(0^-) = t_m(0^+), \\ \alpha_2 t_m'(0^-) = \alpha_1 t_m'(0^+), \\ t_m(-\omega_1) = t_m(\omega_2) = 0. \end{cases}$$

Theorem 3.2. *If $\lambda_m \neq \frac{2}{p'}$ for all $m \in \mathbb{N}^*$, then there exists $\theta_A \in \left] \frac{\pi}{2}, \pi \right[$ such that, for all $f \in L^p(G)$, all $z \in S^+ \cup S_A$, the unique solution $u \in H_0^1(G)$ of problem (9)–(12) admits the decomposition*

$$u = u_R + \sum_{0 < \lambda_m < \frac{2}{p'}} c_m \psi_m(z), \quad (25)$$

where

$$\psi_m(z) = \begin{cases} e^{-r\sqrt{z}} S^{(m)} & \text{if } \frac{2}{p'} - 1 < \lambda_m < \frac{2}{p'}, \\ e^{-r\sqrt{z}} (1 + r\sqrt{z}) S^{(m)} & \text{if } \lambda_m \leq \frac{2}{p'} - 1, \end{cases} \quad (26)$$

$u_R \in PW^{2,p}(G)$ satisfies

$$\|u_R\|_{PW^{2,p}(G)} + |z|^{\frac{1}{2}} \|u_R\|_{W^{1,p}(G)} + |z| \|u_R\|_{L^p(G)} \leq C \|f\|_{L^p(G)}; \quad z \neq 0, \quad (27)$$

and c_m satisfies

$$|c_m| \leq C |z|^{\frac{\lambda_m}{2} - \frac{1}{p'}} \|f\|_{L^p(G)}; \quad z \neq 0, \quad (28)$$

$$\sum_{0 < \lambda_m < \frac{2}{p'}} |c_m| \left(1 + |z|^{\frac{1}{p'} - \frac{\lambda_m}{2}}\right) \leq C \|f\|_{L^p(G)}. \quad (29)$$

Proof. The proof of this Theorem stays as in [1] until the inequality (28), thanks to Theorem 3.1 and Corollary 3.2. It remains to prove the estimate (29).

From the decomposition (25), we have

$$\sum_{0 < \lambda_m < \frac{2}{p'}} c_m \psi_m(z) = u - u_R.$$

By using the estimates (15) and (27), we deduce that

$$\left\| \sum_{0 < \lambda_m < \frac{2}{p'}} c_m \psi_m(z) \right\|_{L^p(G)} \leq C \|f\|_{L^p(G)}.$$

As the space $V_S = \text{span}\left\{\psi_m(z); 0 < \lambda_m < \frac{2}{p'}\right\}$ being of finite dimension, we have

$$\sum_{0 < \lambda_m < \frac{2}{p'}} |c_m| \leq C \left\| \sum_{0 < \lambda_m < \frac{2}{p'}} c_m \psi_m(z) \right\|_{L^p(G)} \leq C \|f\|_{L^p(G)}.$$

From the previous estimate and inequality (28), we deduce that we have (29). $\square$

With the notation introduced above, we can write

$$u = (A_p + z)^{-1} f,$$

consequently the decomposition (25) implies a similar decomposition of the resolvent of A_p . Namely we may write

$$(A_p + z)^{-1} = R(z) + \sum_{0 < \lambda_m < \frac{2}{p'}} T_m(z) \otimes \psi_m(z), \quad (30)$$

where $R(z)$ is the continuous linear operator from $L^p(G)$ into $PW^{2,p}(G)$ defined by

$$R(z)f := u_R,$$

and $T_m(z)$ is the continuous linear functional on $L^p(G)$ defined by

$$\langle T_m(z), f \rangle := c_m.$$

Recall that

$$(T_m(z) \otimes \psi_m(z))(f) = \langle T_m(z), f \rangle \psi_m(z).$$

The estimates (27) and (29) imply that

$$\|R(z)\|_{L^p(G) \rightarrow PW^{2,p}(G)} + |z|^{\frac{1}{2}} \|R(z)\|_{L^p(G) \rightarrow W^{1,p}(G)} + |z| \|R(z)\|_{L^p(G) \rightarrow L^p(G)} \leq C,$$

and

$$\|T_m(z)\|_{L^{p'}(G)} \leq C \frac{1}{1 + |z|^{\frac{1}{p'} - \frac{\lambda_m}{2}}}, \quad (31)$$

for all $z \in S^+ \cup S_A$.

4. Application of the first strategy

In order to apply Theorem 2.1 to problem (1)–(5), we write it as the sum of linear operators on the Banach space $E = L^p(I; L^p(G))$ (where $I =]0, T[$) by setting

$$Au = -\{\Delta u_i\}_{i=1,2},$$

for $u \in D(A) = L^p(I; D(A_p))$,

$$Bu = \partial_t u = \{\partial_t u_i\}_{i=1,2},$$

for $u \in D(B) = \{v \in W^{1,p}(I; L^p(G)); v(., 0) = 0\}$.

Proposition 4.1. *The closure $\bar{L}$ of $L = A + B$ is invertible, i.e. for all $g \in L^p(I; L^p(G))$, there exists a unique strong solution $u \in L^p(I; L^p(G))$ of $(A + B)u = g$. In addition u is explicitly given by*

$$u = \frac{1}{2\pi i} \int_{\gamma} (A + z)^{-1} (z - B)^{-1} g \, dz. \quad (32)$$

Proof. The properties of A are those of its realization A_p . Thanks to (22) and (23), hypothesis $\mathbf{H}_1$ is fulfilled for the operator A with some θ_A in $\left] \frac{\pi}{2}, \pi \right[$, while from [2, p. 330–331], it is fulfilled for the operator B for all $\theta_B < \frac{\pi}{2}$. So there exists $M > 0$ such that

$$\|(B + \lambda I)^{-1}\| \leq \frac{M}{|\lambda|}, \quad (33)$$

for all $\lambda \in S_B = \{\lambda \in \mathbb{C} / |\arg(\lambda)| \leq \theta_B\}$.

Hence, we conclude that $\mathbf{H}_1$ is satisfied with θ_A in $\left] \frac{\pi}{2}, \pi \right[$ and $\theta_B = \frac{\pi}{2} - \delta_B$ with $0 < \delta_B < \delta < \theta_A - \frac{\pi}{2}$.

On the other hand, it's clear that $]-\infty, 0] \subset \rho(B)$. Thus the assumption $\mathbf{H}_2$ is fulfilled since A has a discrete spectrum that contains only strictly positive eigenvalues (see [1, p. 20–21]).

The commutativity assumption $\mathbf{H}_3$ follows from the fact that the variables are separate in these two operators.

Hence we can apply Theorem 2.1 to conclude. $\square$

For each t , we can write

$$[(A + zI)^{-1}h](t) = (A_p + z)^{-1}(h(t)).$$

Using the decomposition (30), the representation formula (32) can be split as follows

$$u = u_R + \sum_{0 < \lambda_m < \frac{2}{p'}} u_m, \quad (34)$$

where

$$u_R = \frac{1}{2\pi i} \int_{\gamma} R(z)[(z - B)^{-1}g]dz, \quad (35)$$

$$u_m = \frac{1}{2\pi i} \int_{\gamma} \langle T_m(z), (z - B)^{-1}g \rangle \psi_m(z) dz. \quad (36)$$

Summing up, we have prove the following theorem.

Theorem 4.1. *Suppose that $\lambda_m \neq \frac{2}{p'}$, then for all $g \in L^p(I; L^p(G))$, the problem (1)–(5) has a unique strong solution u which is in the form*

$$u = u_R + \sum_{0 < \lambda_m < \frac{2}{p'}} u_m,$$

where u_R (resp. u_m) is given by (35) (resp. (36)).

Theorem 4.2. *Let $p \geq 2$, suppose that $\lambda_m \neq \frac{2}{p'}$. Denote $\sigma_m = \frac{1}{p'} - \frac{\lambda_m}{2}$, then for all $g \in L^p(I; L^p(G))$, there exist $q_m \in W^{\sigma_m, p}(I)$ and E_m such that u_m defined by (36) can be written as*

$$u_m = (E_m *_{\mathbf{t}} q_m) S^{(m)}, \quad (37)$$

the symbol $*_{\mathbf{t}}$ means the convolution product in $\mathbf{t}$. Moreover we have

$$q_m = \frac{1}{2\pi i} \int_{\gamma} \langle T_m(z), (z - B)^{-1}g \rangle dz, \quad (38)$$

$$E_m(r, t) = \begin{cases} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} e^{-r\sqrt{i\xi}} d\xi & \text{for } \lambda_m > 1 - \frac{2}{p}, \\ \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} (1 + r\sqrt{i\xi}) e^{-r\sqrt{i\xi}} d\xi & \text{for } \lambda_m \leq 1 - \frac{2}{p}, \end{cases} \quad (39)$$

and the mappings

$$\begin{aligned} U_1 : L^p(I; L^p(G)) &\longrightarrow W^{\sigma_m, p}(I) \\ g &\longmapsto q_m, \end{aligned}$$

$$\begin{aligned} U_2 : W^{\sigma_m, p}(I) &\longrightarrow L^p(I; L^p(G_i)) \\ q_m &\longmapsto \left(\frac{\partial}{\partial t} - \Delta \right) u_{m, i}, \end{aligned}$$

are continuous.

Proof. We proceed as in [9, Proposition 6.2] and [5, Proposition 2.2]. First we consider the extension of g to $G \times \mathbb{R}$, defined by

$$\tilde{g}(x, t) = \begin{cases} g(x, t) & \text{if } t \in [0, T], \\ 0 & \text{if } t \notin [0, T], \end{cases}$$

and denote by $\tilde{u}_z = (zI - B_\infty)^{-1} \tilde{g}$, the solution of

$$\begin{cases} z\tilde{u} - \partial_t \tilde{u} = \tilde{g} & \text{in } G \times \mathbb{R}, \\ \tilde{u}(\cdot, 0) = 0 & \text{in } G, \end{cases}$$

where B_∞ is the operator, defined by

$$B_\infty u = \partial_t u \quad \text{for } u \in D(B_\infty) = \{v \in W^{1,p}([0, \infty[; L^p(G)); v(\cdot, 0) = 0\}.$$

Observe that, by uniqueness of the solution of the Cauchy problem, we have $\tilde{u}_z|_{G \times [0, T]} = (zI - B)^{-1}g$.

Consider the functions

$$\tilde{u}_m(x, t) = \frac{1}{2\pi i} \int_{\gamma} \langle T_m(z), (zI - B_\infty)^{-1} \tilde{g} \rangle \psi_m(z) dz, \quad (40)$$

$$\tilde{q}_m(t) = \frac{1}{2\pi i} \int_{\gamma} \langle T_m(z), (zI - B_\infty)^{-1} \tilde{g}(\cdot, t) \rangle dz. \quad (41)$$

We assume that $\tilde{g} \in \mathcal{D}(G \times \mathbb{R})$, a dense subspace of $L^p(G \times \mathbb{R})$. Then we can apply partial Fourier transform in t , to (40) and (41). By Fubini's theorem, we obtain

$$\begin{aligned} \mathfrak{F}_t \tilde{u}_m(x, \xi) &= \frac{1}{2\pi i} \int_{\gamma} \left\langle T_m(z), \mathfrak{F}_t \left((zI - B_\infty)^{-1} \tilde{g} \right) (\cdot, \xi) \right\rangle \psi_m(z) dz = \\ &= \frac{1}{2\pi i} \int_{\gamma} \left\langle T_m(z), \frac{\mathfrak{F}_t \tilde{g}(\cdot, \xi)}{z - i\xi} \right\rangle \psi_m(z) dz, \end{aligned}$$

and

$$\mathfrak{F}_t \tilde{q}_m(\xi) = \frac{1}{2\pi i} \int_{\gamma} \left\langle T_m(z), \frac{\mathfrak{F}_t \tilde{g}(\cdot, \xi)}{z - i\xi} \right\rangle dz.$$

The decay at infinity of $T_m(z)$ and $\psi_m(z)$ due to (31) and (26) allows us to apply Cauchy's formula. We get

$$\mathfrak{F}_t \tilde{u}_m(x, \xi) = -\langle T_m(i\xi), \mathfrak{F}_t \tilde{g}(\cdot, \xi) \rangle \psi_m(i\xi), \quad (42)$$

and

$$\mathfrak{F}_t \tilde{q}_m(\xi) = -\langle T_m(i\xi), \mathfrak{F}_t \tilde{g}(\cdot, \xi) \rangle.$$

According to (26), the identity (42) can be written as follows

$$\mathfrak{F}_t \tilde{u}_m(x, \xi) = \begin{cases} -\langle T_m(i\xi), \mathfrak{F}_t \tilde{g}(\cdot, \xi) \rangle e^{-r\sqrt{i\xi}} S^{(m)} & \text{if } \lambda_m > 1 - \frac{2}{p}, \\ -\langle T_m(i\xi), \mathfrak{F}_t \tilde{g}(\cdot, \xi) \rangle e^{-r\sqrt{i\xi}} (1 + r\sqrt{i\xi}) S^{(m)} & \text{if } \lambda_m \leq 1 - \frac{2}{p}. \end{cases} \quad (43)$$

Now we consider the function E_m defined by

$$\mathfrak{F}_t E_m(x, \xi) = \begin{cases} e^{-r\sqrt{i\xi}} & \text{if } \lambda_m > 1 - \frac{2}{p}, \\ e^{-r\sqrt{i\xi}} (1 + r\sqrt{i\xi}) & \text{if } \lambda_m \leq 1 - \frac{2}{p}. \end{cases}$$

It is clear that E_m is given by (39).

Then (43) can be seen as the Fourier transform of a convolution in t . We have

$$\tilde{u}_m = (E_m *_t \tilde{q}_m) S^{(m)}.$$

This identity is easily extended from $\tilde{g} \in \mathcal{D}(G \times \mathbb{R})$ to any $\tilde{g} \in L^p(G \times \mathbb{R})$. The identity (37) follows by observing that $\tilde{u}_m|_{G \times [0, T]} = u_m$ and $\tilde{q}_m|_{[0, T]} = \frac{1}{2\pi i} \int_{\gamma} \langle T_m(z), (z - B)^{-1} g \rangle dz = q_m$.

Let us underline that we differ from [5] in the definition of A , the operator B being the same but in Sobolev spaces instead of the weighted L^p -Sobolev spaces. This comes from the fact that α depends only on the plane variables thus the interface has no effect on the variable t . Thus, the continuity of the operators U_1 and U_2 can be shown as done in [5, Theorem 2.3 and Theorem 3.2]. $\square$

5. Application of the second strategy

Now we are able to prove the regularity of u_R .

From [1, Section 5.2], the following estimate for $R(z)$ (defined in Section 3) is derived thanks to an interpolation argument.

$$\|R(z)\|_{L^p(G) \rightarrow PW^{s,p}(G)} = O\left(\frac{1}{|z|^{1-\frac{s}{2}}}\right) \quad \forall s < 2.$$

With the help of (33), this yields

$$u_R \in L^p(I; PW^{s,p}(G)) \quad \text{for every } s < 2, \quad (44)$$

with the estimate

$$\|u_R\|_{L^p(I; PW^{s,p}(G))} \leq C(s) \|g\|_{L^p(I; L^p(G))}. \quad (45)$$

Going back to (34), we have

$$\partial_t u_{R,i} - \Delta u_{R,i} = g_i - \sum_{0 < \lambda_m < \frac{2}{p'}} (\partial_t u_{m,i} - \Delta u_{m,i}) := g_{R,i}; \quad i = 1, 2, \quad (46)$$

the function $g_R \in L^p(I; L^p(G))$ by Theorem 4.2.

We shall apply Theorem 2.2 to study the equation (46) with $X = L^p(G)$, the space E and the operator A are defined exactly as in Section 4.

By Theorem 3.1 and Lemma 3.2, the assumption $\mathbf{H}_4$ is fulfilled. It remains to check $\mathbf{H}_5$. Thanks to [1] there exists $\tau_A < \frac{\pi}{2}$ such that

$$\|A_p^{is}\| = 0(e^{|s|\tau_A}).$$

Accordingly Theorem 2.2 may be applied, then we have the existence and the uniqueness of $w_R \in W^{1,p}([0, T[; L^p(G)) \cap L^p([0, T[; D(A_p))$ solution of

$$\begin{cases} \partial_t w_R + A w_R = g_R, \\ w_R(., 0) = 0. \end{cases}$$

w_R do not coincide necessarily with u_R , so we will prove that $w_R = u_R$.

First we show that u_R is a strong solution of

$$\partial_t u_{R,i} - \Delta u_{R,i} = g_i - \sum_{0 < \lambda_m < \frac{2}{p'}} (\partial_t u_{m,i} - \Delta u_{m,i}) \text{ in } \mathcal{B}_i; \ i = 1, 2, \quad (47)$$

$$u_{R,1} = u_{R,2} \quad \text{on } \Sigma \times [0, T], \quad (48)$$

$$a_1 \frac{\partial u_{R,1}}{\partial n_1} + a_2 \frac{\partial u_{R,2}}{\partial n_2} = 0 \quad \text{on } \Sigma \times [0, T], \quad (49)$$

$$u_{R,i} = 0 \quad \text{on } (\partial G_i \setminus \Sigma) \times [0, T]; \ i = 1, 2, \quad (50)$$

$$u_{R,i}(., 0) = 0 \quad \text{in } G_i; \ i = 1, 2. \quad (51)$$

Due to Proposition 4.1, u is a strong solution of

$$\partial_t u_i - \Delta u_i = g_i \text{ in } \mathcal{B}_i; \ i = 1, 2,$$

$$u_1 = u_2 \text{ on } \Sigma \times [0, T],$$

$$a_1 \frac{\partial u_1}{\partial n_1} + a_2 \frac{\partial u_2}{\partial n_2} = 0 \text{ on } \Sigma \times [0, T],$$

$$u_i = 0 \text{ on } (\partial G_i \setminus \Sigma) \times [0, T]; \ i = 1, 2,$$

$$u_i(., 0) = 0 \text{ in } G_i; \ i = 1, 2,$$

i.e. there exists $(u_n) \subset D(A) \cap D(B)$ and $(g_n) \subset E$ such that $(A + B)u_n = g_n$, $u_n \rightarrow u$ and $g_n \rightarrow g$ in E .

Moreover, as in Section 4, for every n , we have

$$u_n = u_{n,R} + \sum_{0 < \lambda_m < \frac{2}{p'}} u_{n,m}.$$

Thanks to Theorem 4.2, we have

$$\partial_t u_{n,m,i} - \Delta u_{n,m,i} \rightarrow \partial_t u_{m,i} - \Delta u_{m,i},$$

then

$$\partial_t u_{n,R,i} - \Delta u_{n,R,i} \rightarrow g_i - \sum_{0 < \lambda_m < \frac{2}{p'}} (\partial_t u_{m,i} - \Delta u_{m,i}).$$

From the estimate (45) it follows that $u_{n,R} \rightarrow u_R$ in E .

It is obvious that w_R is a strong solution of (47)–(51). Consequently, by applying the first strategy to (47)–(51), we have by uniqueness of the strong solution that $w_R = u_R$.

This implies that $u_R \in L^p(I; D(A_p)) \cap W^{1,p}(I; L^p(G))$. With the help of (44), this yields

$$u_R \in L^p(I; D(A_p) \cap PW^{s,p}(G)).$$

Then, from [1, Lemma 5.4], we deduce that

$$u_R \in L^p(I; PW^{2,p}(G) \cap W_0^{1,p}(G)) \cap W^{1,p}(I; L^p(G)).$$

Summing up, we have proved the following Theorem.

Theorem 5.1. *Let $p \geq 2$, suppose that $\lambda_m \neq \frac{2}{p'}$ $\forall m \in \mathbb{N}^*$. Then for every $g \in L^p(I; L^p(G))$, there exists a unique solution $u \in L^p(I; L^p(G))$ to the transmission problem (1)–(5). Moreover u admits the decomposition*

$$u = u_R + \sum_{0 < \lambda_m < \frac{2}{p'}} (E_m *_t q_m) S^{(m)},$$

with $u_R \in L^p(I; PW^{2,p}(G) \cap W_0^{1,p}(G)) \cap W^{1,p}(I; L^p(G))$, where E_m (resp. $q_m \in W^{-\frac{\lambda_m}{2}+1-\frac{1}{p}}(I)$) is defined by (39) (resp. (38)) and the singular functions $S^{(m)}$ are given by (24).

References

- [1] A.Aibeche, W.Chikouche, S.Nicaise, L^p Regularity of transmission problems in dihedral domains, *Bollettino U.M.I.*, **10-B**(2007), no. 3, 633–660.
- [2] G.Da Prato, P.Grisvard, Sommes d'opérateurs linéaires et équations différentielles opérationnelles *J. Math. Pures Appl. IX Ser.*, **54**(1975), 305–387.
- [3] C.De Coster, S.Nicaise, Singular behavior of the solution of the Helmholtz equation in weighted L^p -Sobolev spaces, *Adv. Differential Equations*, **16**(2011), 165–198.
- [4] C.De Coster, S.Nicaise, Singular behavior of the solution of the periodic-Dirichlet heat equation in weighted L^p -Sobolev spaces, *Adv. Differential Equations*, **16**(2011), 221–256.
- [5] C.De Coster, S.Nicaise, Singular behavior of the solution of the Cauchy-Dirichlet heat equation in weighted L^p -Sobolev spaces, *Bull. Belg. Math. Soc. Simon Stevin*, **18**(2011), 769–780.
- [6] G.Dore, A.Venni, On the closedness of the sum of two closed operators, *Mathematische Zeitschrift*, **196**(1987), 189–201.
- [7] A.Favini, R.Labbas, K.Lemrabet, S.Maingot, Study of the limit of transmission problems in a thin layer by the sum theory of linear operators, *Rev. Mat. Complut.*, **18**(2005), 143–176.
- [8] P.Grisvard, Edge behaviour of the solution of an elliptic problem, *Math. Nachr.*, **132**(1987), 281–299.
- [9] P.Grisvard, Singular behaviour of elliptic problems in non Hilbertian Sobolev spaces, *J. Math. Pures Appl.*, **74**(1995), 3–33.
- [10] J.R.Kweon, Edge singular behavior for the heat equation on polyhedral cylinders in $\mathbb{R}^3$, *Potential Anal.*, **38**(2013), 589–610.

- [11] R.Labbas, M.Moussaoui, M.Najmi, Singular behaviour of the Dirichlet problem in Hölder spaces of the solutions to the Dirichlet problem in a cone, *Rend. Ist. Mat. Univ. Trieste*, **30**(1998), 155–179.
- [12] R.Labbas, M.Moussaoui, On the resolution of the heat equation with discontinuous coefficients, *Semigroup Forum*, **60**(2000), 187–201.
- [13] S.Nicaise, Polygonal interface problems, Peter Lang, Berlin, 1993.
- [14] A.Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Math. Sciences, Vol. 44, Springer-Verlag, New York, 1983.

L^p -регулярность решения уравнения теплопроводности с разрывными коэффициентами

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Аннотация. В этой статье мы рассмотрим задачу прохождения для уравнения теплопроводности на ограниченном плоском секторе в пространствах L^p -Соболева. Применяя теорию сумм операторов Да Прато-Грисварда и Доре-Венни, мы доказываем, что решение можно разбить на регулярную часть в пространстве L^p -Соболева и явную особую часть.

Ключевые слова: уравнение теплопередачи, суммы линейных операторов, сингулярное поведение, негладкие области.