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Exact Solution of 3D Navier–Stokes Equations

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Abstract. Procedure for constructing exact solutions of 3D Navier–Stokes equations for an incompressible fluid flow is proposed. It is based on the relations representing the previously obtained first integral of the Navier–Stokes equations. A primary generator of particular solutions is proposed. It is used to obtain new classes of exact solutions.

Keywords: incompressible fluid, motion, equation, integral, primary generator of solutions, exact solution.

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Introduction

The Navier–Stokes equations describe the motion of fluid and gaseous media in the presence of viscosity. These equations are widely used for solving practical problems in various fields. These fields traditionally include hydraulic engineering, oceanology, shipbuilding, aircraft engineering, tribology and cardiology.

The simplest version of the equations corresponds to the case of incompressible fluid motion. In this case the density and all other physical characteristics of the fluid are constant and unknowns are the components of velocity vector u , v , w and pressure p [1, 2]. In this case the Navier–Stokes equations in dimensionless variables and in the presence of the potential of external forces can be represented as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial(p + \Phi)}{\partial x} + \frac{1}{Re} \Delta u, \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial(p + \Phi)}{\partial y} + \frac{1}{Re} \Delta v, \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial(p + \Phi)}{\partial z} + \frac{1}{Re} \Delta w, \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4)$$

where Δ is the 3D Laplace operator with respect to spatial coordinates: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, Φ is the potential of external forces, Re is the Reynolds number.

The study of equations (1–4) is one of the directions of modern mathematical physics [3, 4]. However, at present many issues are not fully clarified and they require additional research. One

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of the main problems is the lack of a general constructive method of solution. How to construct solutions of 3D Navier–Stokes equations with all non-linear terms? There is no answer to this question yet but practice needs resolution of this issue.

An important step along this path is the construction of exact solutions. Some solutions are known [5–8]. Now broad classes of solutions are of particular interest. Each class of exact solutions introduces new understanding of general laws and to some extent creates a basis for developing methods to construct exact solutions.

1. Integral of the Navier–Stokes equations

The procedure for constructing an integral of equations (1–4) was proposed by author [9, 10]. So, the integral is represented by nine relations. In the most simple notation they are

$$p + \Phi + \frac{U^2}{2} + d + d_t = p_0, \quad (5)$$

$$u^2 - v^2 + \frac{2}{Re} \left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{10}}{\partial y^2} - \frac{\partial^2 \Psi_{11}}{\partial z^2} - \frac{\partial^2 \Psi_{12}}{\partial z^2} + \frac{\partial^2 \Psi_{15}}{\partial y \partial z} + \frac{\partial^2 \Psi_{14}}{\partial x \partial z} + \frac{\partial}{\partial t} \left(-\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_3}{\partial y} + \frac{\partial(\Psi_5 + \Psi_6)}{\partial z} \right), \quad (6)$$

$$v^2 - w^2 + \frac{2}{Re} \left(-\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = \frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{11}}{\partial x^2} - \frac{\partial^2 \Psi_{12}}{\partial y^2} + \frac{\partial^2 \Psi_{12}}{\partial z^2} - \frac{\partial^2 \Psi_{13}}{\partial x \partial y} - \frac{\partial^2 \Psi_{14}}{\partial x \partial z} + \frac{\partial}{\partial t} \left(\frac{\partial(\Psi_1 + \Psi_2)}{\partial x} + \frac{\partial \Psi_4}{\partial y} - \frac{\partial \Psi_6}{\partial z} \right), \quad (7)$$

$$uw - \frac{1}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = -\frac{\partial^2 \Psi_{10}}{\partial x \partial y} + \frac{1}{2} \frac{\partial}{\partial z} \left(-\frac{\partial \Psi_{15}}{\partial x} + \frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{13}}{\partial z} \right) + \frac{1}{2} \frac{\partial}{\partial t} \left(-\frac{\partial \Psi_3}{\partial x} - \frac{\partial \Psi_1}{\partial y} - \frac{\partial(\Psi_8 + \Psi_9)}{\partial z} \right), \quad (8)$$

$$vw - \frac{1}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = \frac{\partial^2 \Psi_{11}}{\partial x \partial z} + \frac{1}{2} \frac{\partial}{\partial y} \left(-\frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{14}}{\partial y} - \frac{\partial \Psi_{13}}{\partial z} \right) + \frac{1}{2} \frac{\partial}{\partial t} \left(-\frac{\partial \Psi_5}{\partial x} + \frac{\partial(\Psi_9 - \Psi_7)}{\partial y} + \frac{\partial \Psi_2}{\partial z} \right), \quad (9)$$

$$vw - \frac{1}{Re} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = -\frac{\partial^2 \Psi_{12}}{\partial y \partial z} + \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{13}}{\partial z} \right) + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial(\Psi_7 + \Psi_8)}{\partial x} + \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z} \right), \quad (10)$$

$$u = \frac{1}{2} \left(\frac{\partial}{\partial y} \left(-\frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_7}{\partial z} \right) + \frac{\partial}{\partial z} \left(-\frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_8}{\partial y} - \frac{\partial \Psi_2}{\partial z} \right) \right), \quad (11)$$

$$v = \frac{1}{2} \left(\frac{\partial}{\partial x} \left(\frac{\partial \Psi_3}{\partial x} - \frac{\partial \Psi_1}{\partial y} - \frac{\partial \Psi_7}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \Psi_9}{\partial x} + \frac{\partial \Psi_6}{\partial y} - \frac{\partial \Psi_4}{\partial z} \right) \right), \quad (12)$$

$$w = \frac{1}{2} \left(\frac{\partial}{\partial x} \left(\frac{\partial \Psi_5}{\partial x} - \frac{\partial \Psi_8}{\partial y} + \frac{\partial \Psi_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \Psi_9}{\partial x} - \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z} \right) \right). \quad (13)$$

Functions Ψ_j denote new unknowns that arise in the process of integration. In the case being considered there are fifteen functions and they complete the system of unknowns. The term

"stream pseudo-function" was introduced for them [9, 10]. Thus, a total of nineteen unknowns are introduced, namely, four major unknowns and fifteen associated unknowns.

Relation (5) contains additional terms p_0 , $\frac{U^2}{2}$, d and d_t . The first one is an additive pressure constant. Another three terms represent combinations of unknowns defined in a special way. Value $\frac{U^2}{2}$ is dimensionless velocity

$$\frac{U^2}{2} = \frac{u^2 + v^2 + w^2}{2}.$$

Values d and d_t are dissipative terms defined as

$$d = -\frac{U^2}{6} - \frac{1}{3} \left(\Delta_{xy} \Psi_{10} - \Delta_{xz} \Psi_{11} + \Delta_{yz} \Psi_{12} + \frac{\partial^2 \Psi_{13}}{\partial x \partial y} - \frac{\partial^2 \Psi_{14}}{\partial x \partial z} + \frac{\partial^2 \Psi_{15}}{\partial y \partial z} \right), \quad (14)$$

$$d_t = \frac{1}{3} \frac{\partial}{\partial t} \left(\frac{\partial(\Psi_2 - \Psi_1)}{\partial x} + \frac{\partial(\Psi_4 - \Psi_3)}{\partial y} + \frac{\partial(\Psi_6 - \Psi_5)}{\partial z} \right). \quad (15)$$

Symbols Δ_{yz} , Δ_{xz} , Δ_{xy} in (14) denote the incomplete Laplace operators with respect to spatial coordinates

$$\Delta_{yz} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_{xz} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Relations (5–13) include the major unknowns u , v , w , p , the associated unknowns Ψ_j , given potential function of external forces Φ and the Reynolds number Re . The order of derivatives for major unknowns is one. It is less than their order in original equations (1–4). Relations (5–13) represent the first integral of the Navier – Stokes equations (1–4).

The integral of equations (1–4) in the form (5–13) allows us to construct exact solutions in a new way.

2. Primary generator of solutions

The primary generator of solutions allows us to construct the set of solutions of original equations (1–4). One such primary generator is presented below.

Let us briefly analyze relations (5–13) that represent the first integral of the Navier–Stokes equations. Relations (5) and (11–13) give expressions for the major unknowns u , v , w , p in terms of associated unknowns Ψ_j , where $j = 1, 2, \dots, 15$. It is fair to conclude that these four relations determine general structure of solutions for equations (1–4). Let us note that unknowns u , v , w defined by (11–13) satisfy continuity equation (4). Relation (5) is special because it contains the unknown p . In the way of practical solution of equations this relation should be used at the last stage when all other unknowns have been already found.

When considering relations (6–13) in general, the following features attract attention [11]. In the right-hand sides of (11–13) there are derivatives of only the first nine associated unknowns Ψ_k , $k = 1, 2, \dots, 9$ but there are fifteen associated unknowns in total. Unknowns Ψ_k with $k = 10, 11, \dots, 15$ do not appear in relations (11–13). These unknowns are present in relations (6–10) in the form of linear combinations of second derivatives. It is possible to exclude these unknowns from (6–7) and to obtain general relations. The procedure for constructing such relations is briefly described below.

Let us denote the sums of all terms in (6–10) that are independent of unknowns Ψ_k , $k = 1, 2, \dots, 9$, by f_j ($j = 2, 3, \dots, 6$). So we have

$$\begin{aligned} f_2 &= u^2 - v^2 + \frac{2}{Re} \left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial t} \left(\frac{\partial \Psi_1}{\partial x} - \frac{\partial \Psi_3}{\partial y} - \frac{\partial \Psi_5}{\partial z} - \frac{\partial \Psi_6}{\partial z} \right), \\ f_3 &= v^2 - w^2 + \frac{2}{Re} \left(-\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_2}{\partial x} + \frac{\partial \Psi_4}{\partial y} - \frac{\partial \Psi_6}{\partial z} \right), \\ f_4 &= uv - \frac{1}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_8}{\partial z} + \frac{\partial \Psi_9}{\partial z} \right), \\ f_5 &= uw - \frac{1}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_7}{\partial y} - \frac{\partial \Psi_9}{\partial y} - \frac{\partial \Psi_2}{\partial z} \right), \\ f_6 &= vw - \frac{1}{Re} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) - \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \Psi_7}{\partial x} + \frac{\partial \Psi_8}{\partial x} + \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z} \right). \end{aligned} \quad (16)$$

As a result, five non-linear equations (6–10) are represented in the form

$$-\frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{10}}{\partial y^2} - \frac{\partial^2 \Psi_{11}}{\partial z^2} - \frac{\partial^2 \Psi_{12}}{\partial z^2} + \frac{\partial^2 \Psi_{15}}{\partial y \partial z} + \frac{\partial^2 \Psi_{14}}{\partial x \partial z} = f_2, \quad (17)$$

$$\frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{11}}{\partial x^2} - \frac{\partial^2 \Psi_{12}}{\partial y^2} + \frac{\partial^2 \Psi_{12}}{\partial z^2} - \frac{\partial^2 \Psi_{13}}{\partial x \partial y} - \frac{\partial^2 \Psi_{14}}{\partial x \partial z} = f_3, \quad (18)$$

$$-\frac{\partial^2 \Psi_{10}}{\partial x \partial y} + \frac{1}{2} \frac{\partial}{\partial z} \left(-\frac{\partial \Psi_{15}}{\partial x} + \frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{13}}{\partial z} \right) = f_4, \quad (19)$$

$$\frac{\partial^2 \Psi_{11}}{\partial x \partial z} + \frac{1}{2} \frac{\partial}{\partial y} \left(-\frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{14}}{\partial y} - \frac{\partial \Psi_{13}}{\partial z} \right) = f_5, \quad (20)$$

$$-\frac{\partial^2 \Psi_{12}}{\partial y \partial z} + \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{13}}{\partial z} \right) = f_6. \quad (21)$$

Let us eliminate terms with unknowns Ψ_k for $k = 10, 11, \dots, 15$. To do this we take term by term derivatives of (17–21) with respect to coordinates and then select the necessary linear combinations to exclude terms with the specified unknown. As a result, terms with unknowns Ψ_k at $k = 10, 11, \dots, 15$ are excluded from (17–18). Then we obtain two equations [11]

$$\frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_4}{\partial x^2} + \frac{\partial^2 f_4}{\partial y^2} + \frac{\partial^2 f_5}{\partial y \partial z} - \frac{\partial^2 f_6}{\partial x \partial z} = 0, \quad (22)$$

$$\frac{\partial^2 f_3}{\partial y \partial z} + \frac{\partial^2 f_4}{\partial x \partial z} - \frac{\partial^2 f_5}{\partial x \partial y} - \frac{\partial^2 f_6}{\partial y^2} + \frac{\partial^2 f_6}{\partial z^2} = 0. \quad (23)$$

Taking into account (16), it is clear that only nine unknowns are present in equations (22–23). These unknowns are Ψ_k with $k = 1, 2, \dots, 9$. This fact is obvious since u, v, w are expressed in terms of these unknowns, according to (11–13). So, system of two equations (22–23) can be considered as primary generator of solutions of 3D Navier–Stokes equations (1–4). Any set of functions $\Psi_1, \Psi_2, \dots, \Psi_9$ that satisfy this system allows one to determine all other unknowns including the major ones. Firstly u, v, w are found according to (11–13). Then using (16), f_j are defined for $j = 2, 3, \dots, 6$. Next, six unknowns $\Psi_{10}, \Psi_{11}, \dots, \Psi_{15}$ are determined with the help of (17–21). Finally, using relation (5) and taking into account (14–15), we determine unknown p . As a result, all major unknowns are determined and the main problem is solved.

3. Method implementation

As an example of the implementation of the described approach we construct a set of solutions that correspond to a cascade of plane waves into deep water provided $\Phi = 0$. Let us assume that unknowns u, v, w are represented in complex variables as linear combinations of plane waves. General structure for u, v, w is defined by relations (11–13). Let us assume that

$$\begin{aligned} \frac{\partial \Psi_1}{\partial y} - \frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_7}{\partial z} &= A(t)e^{i(n_1x+m_1y+l_1z)}, \\ -\frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_8}{\partial y} - \frac{\partial \Psi_2}{\partial z} &= B(t)e^{i(n_2x+m_2y+l_2z)}, \\ \frac{\partial \Psi_9}{\partial x} + \frac{\partial \Psi_6}{\partial y} - \frac{\partial \Psi_4}{\partial z} &= C(t)e^{i(n_3x+m_3y+l_3z)}, \end{aligned} \quad (24)$$

where i is the imaginary unit, n_k, m_k, l_k , ($k = 1, 2, 3$) are some constants and $A(t), B(t), C(t)$ are some functions of time.

Taking into account (11–13), we obtain the following expressions

$$\begin{aligned} u &= \frac{i}{2}(Am_1e^{i(n_1x+m_1y+l_1z)} + Bl_2e^{i(n_2x+m_2y+l_2z)}), \\ v &= \frac{i}{2}(-An_1e^{i(n_1x+m_1y+l_1z)} + Cl_3e^{i(n_3x+m_3y+l_3z)}), \\ w &= \frac{i}{2}(-Bn_2e^{i(n_2x+m_2y+l_2z)} - Cm_3e^{i(n_3x+m_3y+l_3z)}). \end{aligned} \quad (25)$$

So, u, v, w are defined by (25), where n_k, m_k, l_k for $k = 1, 2, 3$ are still unknown wave numbers and $A(t), B(t), C(t)$ are indeterminate functions of time.

Let us consider primary generator of solutions (22–23) and find the restrictions imposed on these equations.

Substituting (16) into (22–23) and taking into account (24) and (25), we obtain the following results. Components of two kinds are present in (22–23). Components of the first kind are linear combinations of quantities $e^{i(n_1x+m_1y+l_1z)}, e^{i(n_2x+m_2y+l_2z)}, e^{i(n_3x+m_3y+l_3z)}$. Components of the second kind are quadratic combinations of quantities $e^{i(n_1x+m_1y+l_1z)}, e^{i(n_2x+m_2y+l_2z)}, e^{i(n_3x+m_3y+l_3z)}$. Components of the first kind are mutually reduced if functions $A(t), B(t), C(t)$ satisfy the ordinary differential equations of the first order

$$\begin{aligned} \frac{dA}{dt} &= -\frac{A}{Re}(n_1^2 + m_1^2 + l_1^2), \\ \frac{dB}{dt} &= -\frac{B}{Re}(n_2^2 + m_2^2 + l_2^2), \\ \frac{dC}{dt} &= -\frac{C}{Re}(n_3^2 + m_3^2 + l_3^2). \end{aligned} \quad (26)$$

Components of the second kind are also mutually reduced and equations (22–23) are identi-

cally satisfied if wave numbers n_k, m_k, l_k satisfy the following system of six algebraic equations

$$\begin{aligned}
& 2n_1l_3(n_1+n_3)(m_1+m_3) - m_1l_3(n_1+n_3)^2 + m_1l_3(m_1+m_3)^2 - \\
& \quad - m_1m_3(m_1+m_3)(l_1+l_3) - n_1m_3(n_1+n_3)(l_1+l_3) = 0, \\
& -m_3l_2(m_2+m_3)(l_2+l_3) + n_2l_3(n_2+n_3)(l_2+l_3) - l_2l_3(n_2+n_3)^2 + l_2l_3(m_2+m_3)^2 = 0, \\
& 2m_1l_2(n_1+n_2)(m_1+m_2) + n_1l_2(n_1+n_2)^2 - n_1l_2(m_1+m_2)^2 - \\
& \quad - m_1n_2(m_1+m_2)(l_1+l_2) - n_1n_2(n_1+n_2)(l_1+l_2) = 0, \\
& -n_1l_2(n_1+n_2)(l_1+l_2) + m_1n_2(n_1+n_2)(m_1+m_2) - \\
& \quad - n_1n_2(m_1+m_2)^2 + n_1n_2(l_1+l_2)^2 = 0, \\
& -2n_2m_3(m_2+m_3)(l_2+l_3) + n_2l_3(m_2+m_3)^2 - n_2l_3(l_2+l_3)^2 + \\
& \quad + l_2l_3(n_2+n_3)(l_2+l_3) + m_3l_2(n_2+n_3)(m_2+m_3) = 0, \\
& -2n_1l_3(m_1+m_3)(l_1+l_3) + m_1l_3(n_1+n_3)(l_1+l_3) + m_1m_3(n_1+n_3)(m_1+m_3) - \\
& \quad - n_1m_3(m_1+m_3)^2 + n_1m_3(l_1+l_3)^2 = 0.
\end{aligned} \tag{27}$$

Solutions of equations (26) are easy to find. They are defined by expressions

$$\begin{aligned}
A(t) &= A(0)e^{-\frac{(n_1^2+m_1^2+l_1^2)t}{Re}}, \quad B(t) = B(0)e^{-\frac{(n_2^2+m_2^2+l_2^2)t}{Re}}, \\
C(t) &= C(0)e^{-\frac{(n_3^2+m_3^2+l_3^2)t}{Re}},
\end{aligned} \tag{28}$$

where $A(0), B(0), C(0)$ are arbitrary constants.

Preliminary analysis of system (27) shows that it has many real and complex solutions. Each set of numbers that satisfy (27) generates a solution of Navier–Stokes equations (1–4). Some special cases are presented below. Each of them can be considered as an implementation of the above approach.

4. Special cases

4.1. Solution 1. The simplest solution corresponds to the case when the wave vectors are collinear. In this case n_3, m_3, l_3 are arbitrary and not all equal to zero. In addition, the following proportionality relations are fulfilled $n_1 = \mu n_3, m_1 = \mu m_3, l_1 = \mu l_3, n_2 = \xi n_3, m_2 = \xi m_3, l_2 = \xi l_3$, where μ and ξ have arbitrary but not equal to zero values. In this case all six equations (27) are identically satisfied.

According to (25) and (28), expressions for velocities are

$$\begin{aligned}
u &= \frac{i}{2}(A(0)\mu m_3 e^{-\frac{\mu^2(n_3^2+m_3^2+l_3^2)t}{Re} + i\mu(n_3x+m_3y+l_3z)} + B(0)\xi l_3 e^{-\frac{\xi^2(n_3^2+m_3^2+l_3^2)t}{Re} + i\xi(n_3x+m_3y+l_3z)}), \\
v &= \frac{i}{2}(-A(0)\mu n_3 e^{-\frac{\mu^2(n_3^2+m_3^2+l_3^2)t}{Re} + i\mu(n_3x+m_3y+l_3z)} + C(0)l_3 e^{-\frac{(n_3^2+m_3^2+l_3^2)t}{Re} + i(n_3x+m_3y+l_3z)}), \\
w &= \frac{i}{2}(-B(0)\xi n_3 e^{-\frac{\xi^2(n_3^2+m_3^2+l_3^2)t}{Re} + i\xi(n_3x+m_3y+l_3z)} - C(0)m_3 e^{-\frac{(n_3^2+m_3^2+l_3^2)t}{Re} + i(n_3x+m_3y+l_3z)}).
\end{aligned} \tag{29}$$

According to (5), the unknown p is

$$p = p_0. \tag{30}$$

4.2. Solution 2. Analysis of algebraic equations (27) leads to the conclusion that system admits the following solution $n_1 = n_2 = 0$, $n_3 = \sqrt{3}$, $m_1 = m_2 = m_3 = \frac{1}{\sqrt{2}}$, $l_1 = l_2 = -2$, $l_3 = 1$. In this case unknowns u, v, w are defined as

$$\begin{aligned} u &= \frac{i}{2} \left(\frac{1}{\sqrt{2}} A(0) - 2B(0) \right) e^{-\frac{9}{2Re}t + i\left(\frac{1}{\sqrt{2}}y - 2z\right)}, & v &= \frac{i}{2} C(0) e^{-\frac{9}{2Re}t + i\left(\sqrt{3}x + \frac{1}{\sqrt{2}}y + z\right)}, \\ w &= -\frac{i}{2\sqrt{2}} C(0) e^{-\frac{9}{2Re}t + i\left(\sqrt{3}x + \frac{1}{\sqrt{2}}y + z\right)}. \end{aligned} \quad (31)$$

According to (5), the unknown p is defined as

$$p = p(0) + \frac{1}{4} C(0) \left(\frac{\sqrt{3}}{2} A(0) - \sqrt{6} B(0) \right) e^{-\frac{9}{2Re}t + i\left(\sqrt{3}x + \sqrt{2}y - z\right)}. \quad (32)$$

4.3. Solution 3. Equations (27) are also satisfied if $n_1 = n_2 = 0$, $n_3 = i\sqrt{3}$, $m_1 = m_2 = m_3 = i\sqrt{3}$, $l_1 = l_2 = 1$, $l_3 = 2$.

The velocities in this case are defined as

$$\begin{aligned} u &= \frac{i}{2} (i\sqrt{3}A(0) + B(0)) e^{\frac{2}{Re}t - \sqrt{3}y + iz}, & v &= iC(0) e^{\frac{2}{Re}t - \sqrt{3}x - \sqrt{3}y + 2iz}, \\ w &= \frac{1}{2} \sqrt{3} C(0) e^{\frac{2}{Re}t - \sqrt{3}x - \sqrt{3}y + 2iz}. \end{aligned} \quad (33)$$

For pressure we have the following expression

$$p = p(0) + \frac{1}{4} C(0) (i\sqrt{3}A(0) + B(0)) e^{\frac{4}{Re}t - \sqrt{3}x - 2\sqrt{3}y + 3iz}. \quad (34)$$

Conclusion

As a result of the implementation of the proposed approach new complex solutions of 3D Navier–Stokes equations (1–4) are obtained. They are defined by expressions (29–34).

Let us pay attention to the qualitative differences of the obtained solutions. Let us consider coefficients at the time t in Solution 1 and Solution 2. The following inequalities are true for these coefficients: $-\frac{\mu^2(n_3^2 + m_3^2 + l_3^2)}{Re} < 0$, $-\frac{\xi^2(n_3^2 + m_3^2 + l_3^2)}{Re} < 0$ for Solution 1 and $-\frac{9}{Re2} < 0$ for Solution 2. For Solution 3 we have $\frac{2}{Re} > 0$. Then Solution 1 and Solution 2 decay exponentially with time. On the contrary, Solution 3 increases exponentially with time. This pattern holds for both pressure and the magnitude of velocity.

The following fact is also worth attention. The pressure increases in half the time by comparison to the magnitude of velocity.

If we compare expressions for pressure (30), (32) for Solution 1 and Solution 2 then there is also a qualitative difference. The pressure does not depend on coordinates for Solution 1 whereas the pressure depends on x, y and z for Solution 2.

Let us pay attention to another interesting feature of the proposed method for constructing solutions. The above relations allow us to construct purely real solutions of the Navier–Stokes equations. To do this, let us consider relations (24–26) and assume that $n_k = -iN_k$, $m_k = -iM_k$, $l_k = -iL_k$, where $k = 1, 2, 3$, i is the imaginary unit and N_k, M_k, L_k are real numbers. In this case algebraic equations (27) retain their form but we should take N_k, M_k, L_k instead of n_k, m_k, l_k . Any set of real numbers (N_1, M_1, L_1) , (N_2, M_2, L_2) , (N_3, M_3, L_3) that satisfies equations (27) allows us to construct purely real solutions of Navier–Stokes equations (1–4).

References

- [1] L.I.Sedov, *Mekhanika sploshnoy sredy*, Part 2, Moscow, Nauka, 1970 (in Russian).
- [2] L.G.Loitsaynskiy, *Mechanics of Fluid and Gas*, Moscow, Nauka, 1987 (in Russian).
- [3] O.A.Lodizhenskaya, *The Mathematical Theory of Viscous Incompressible Fluid*, Gordon and Breach, New York, 1969.
- [4] C.L.Fefferman, *Existence and Smoothness of the Navier–Stokes Equation*, Preprint, Princeton Univ., Math. Dept., Princeton, NJ, USA, (2000).
- [5] A.D. Polynin, V.F.Zaitsev, *Handbook of Nonlinear Partial Differential Equations* (second edition), CRS Press, London, UK, 2012.
- [6] S.N.Aristov, A.D.Polynin, *Dokl. Phys.*, **54**(2009), 316–321.
DOI: 10.1134/S1028335809070039
- [7] A.V.Koptev, E.M.Pastushok, *New Solutios of 2D Navier–Stokes Equations*, *American Scientific Journal*, New York, **2**(2017), no. 17, 4–7.
- [8] A.V.Koptev, *Generator of Solution of 2D Navier–Stokes Equations*, *Journal of Siberian Federal University. Math. and Phys.*, **7**(2014), no. 3, 324–330.
- [9] A.V.Koptev, *Integrals of Navier–Stokes equations*, *Saransk: Trudy Srednevolzhskogo matematicheskogo obshchestva*, **6**(2004), no. 1, 215–225 (in Russian).
- [10] A.V.Koptev, *Integrals of Motion of an Incompressible Medium Flow. From Classic to Modern*, In: *Handbook on Navier–Stokes Equations. Theory and Applied Analysis*, Nova Science Publishers, New York, 2017, 443–459.
- [11] A.V.Koptev, *How solve 3D Navier–Stokes equations*, *Izvestia Rossiyskogo gosudarsvennogo pedagogicheskogo universiteta im. Gertsena, Saint-Petersburg*, **173**(2015), 7–15 (in Russian).

Точные решения 3D-уравнений Навье–Стокса

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Аннотация. В работе предложена процедура построения точных решений 3D-уравнений Навье–Стокса для несжимаемой жидкости. За основу принимаются соотношения, представляющие первый интеграл уравнений Навье–Стокса, ранее полученные автором. Построен первичный генератор частных решений, и с его помощью найдены новые классы точных решений.

Ключевые слова: несжимаемая жидкость, движение, уравнение, интеграл, первичный генератор решений, точное решение.