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Anisotropic Antiplane Elastoplastic Problem

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Abstract. In this work we solve an anisotropic antiplane elastoplastic problem about stress state in a body weakened by a hole bounded by a piecewise-smooth contour. We give the conservation laws which allowed us to reduce calculations of stress components to a contour integral over the contour of the hole. The conservation laws allowed us to find the boundary between the elastic and plastic areas.

Keywords: anisotropic elastoplastic problem, antiplane stress state, conservation laws.

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Introduction

Fields of shifts and stresses in the case under consideration are the following [1]

$$u = v = 0, \quad w = w\left(x, y\right) \quad \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \quad \tau_{xz} = \tau^1\left(x, y\right), \quad \tau_{yz} = \tau^2\left(x, y\right). \tag{1}$$

Here u, v, w are shift vector components, $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are stress components, x, y, z the Cartesian coordinates, axis directed parallel to the element.

In the elastic zone there are the relations

$$\frac{\partial \tau^1}{\partial x} + \frac{\partial \tau^2}{\partial y} = 0 \ (equilibrium\ equation), \tag{2}$$

$$\tau^1 = \frac{\partial w}{\partial x}, \tau^2 = G_2 \frac{\partial w}{\partial y} \ (Hooke's\ law). \tag{3}$$

Here G_i are constants called elastic moduli [2].

From (2), (3) there arise relations in the elastic zone

$$G_1 \frac{\partial^2 w}{\partial x^2} + G_2 \frac{\partial^2 w}{\partial y^2} = 0, \tag{4}$$

$$G_2 \frac{\partial \tau^1}{\partial y} = G_1 \frac{\partial \tau^2}{\partial x}. \tag{5}$$

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From (2) and (5) it follows that τ^1, τ^2 satisfy the system of linear equations

$$F_1 = \frac{\partial \tau^1}{\partial x} + \frac{\partial \tau^2}{\partial y} = 0, \quad F_2 = \frac{\partial \tau^1}{\partial y} - n\frac{\partial \tau^2}{\partial x} = 0, \tag{6}$$

where $n = G_1/G_2$.

In the plastic zone there holds the relation (2), and also

$$a_{13}(\tau^1)^2 + a_{23}(\tau^2)^2 = 1 \ (yield\ condition)\,, \tag{7}$$

$$\tau^2\frac{\partial w}{\partial x} = \tau^1\frac{\partial w}{\partial y} \ (Hencky's\ equation)\,. \tag{8}$$

Here a_{13}, a_{23} are constants called anisotropy coefficients.

On the boundary of the elastic and plastic areas the stresses and shifts are supposed to be continuous.

1. Conservation laws

By a conservation law for the system of equations (6) we shall call the relation of the form of

$$\frac{\partial A\left(x, y, \tau^1, \tau^2\right)}{\partial x} + \frac{\partial B\left(x, y, \tau^1, \tau^2\right)}{\partial y} = \omega^1 F_1 + \omega^2 F_2, \tag{9}$$

where $\omega^i = \omega^i(x, y, \tau^1, \tau^2)$ are some functions not identically zero simultaneously.

Note. A more general definition of conservation laws and their use in mechanics of a solid body being deformed can be studied for example in [3–5].

For the purposes that are set in this article a simplified formulation in the form of (9) will suit fine.

In (9) the values A, B are called conserved current components.

Let us assume that the components A, B appear as follows

$$A = \alpha^1\tau^1 + \beta^1\tau^2 + \gamma^1, \quad B = \alpha^2\tau^1 + \beta^2\tau^2 + \gamma^2, \tag{10}$$

where $\alpha^i = \alpha^i\,(x, y)\,,\ \beta^i = \beta^i\,(x, y)\,,\ \gamma^i = \gamma^i\,(x, y)$ are some smooth functions to be determined.

Let us substitute (10) into (9), as a result we obtain

$$\begin{aligned}&\alpha^1_x\tau^1 + \alpha^1\tau^1_x + \beta^1_x\tau^2 + \beta^1\tau^2_x + \gamma^1_x + \alpha^2_y\tau^1 + \alpha^2\tau^1_y + \beta^2_y\tau^2 + \beta^2\tau^2_y + \gamma^2_y = \\ &= \omega^1\left(\tau^1_x + \tau^2_y\right) + \omega^2\left(\tau^1_y - n\tau^2_x\right) = 0,\end{aligned} \tag{11}$$

where the index below stands for a derivative with respect to the corresponding variable.

From (11) we obtain

$$\alpha^1 = \omega^1, \quad \beta^1 = -n\omega^2, \quad \alpha^2 = \omega^2, \quad \beta^2 = \omega^1, \quad \alpha^1_x + \alpha^2_y = 0, \quad \beta^1_x + \beta^2_y = 0, \quad \gamma^1_x + \gamma^2_y = 0. \tag{12}$$

From (12) excluding ω^i we obtain

$$\alpha^1 = \beta^2, \quad \beta^1 = -n\alpha^2, \quad \alpha^1_x - n\beta^1_y = 0, \quad \beta^1_x + \alpha^1_y = 0, \quad \gamma^1_x + \gamma^2_y = 0. \tag{13}$$

By virtue of relations (12) the conserved current components are written as

$$A = \alpha^1\tau^1 + \beta^1\tau^2 + \gamma^1, \quad B = \frac{-\beta^1}{n}\tau^1 + \alpha^1\tau^2 + \gamma^2. \tag{14}$$

Since the right-hand part (9) is equal to zero, according to Green's formula we obtain

$$\begin{aligned}&\iint_S (A_x + B_y)\,dxdy = \oint_{\partial S} Ady - Bdx = \\ &= \oint_{\partial S} (\alpha^1\tau^1 + \beta^1\tau^2 + \gamma^1)dy - \left(\frac{-\beta^1}{n}\tau^1 + \alpha^1\tau^2 + \gamma^2\right) dx = 0,\end{aligned} \tag{15}$$

where S is the area, ∂S is its piecewise-smooth boundary. All the functions in (15) are supposed to be smooth.

2. Elastoplastic problem for an arbitrary hole in case when the plastic area surrounds the entire hole

Assume C is a piecewise-smooth contour, there is a load applied to it

$$l_1\tau^1 + l_2\tau^2 = \tau_n, \quad |\tau_n| \leqslant \sqrt{\frac{l_1^2 a_{23} + l_2^2 a_{13}}{a_{13}a_{23}}}, \tag{16}$$

where (l_1, l_2) are normal's vector components to contour C. The plastic area's contour L surrounds entirely the hole C. See Fig. 1.

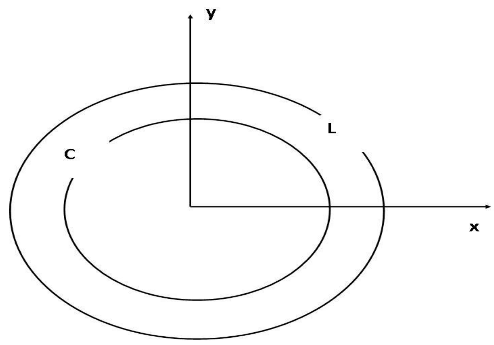

Fig. 1. Elastic-plastic border near the hole C

In this case on contour C, apart from the condition (16), also fulfilled is the yield condition (7). Thus on C there are two conditions:

$$l_1\tau^1 + l_2\tau^2 = \tau_n = \tau_n, \quad a_{13}(\tau^1)^2 + a_{23}(\tau^2)^2 = 1. \tag{17}$$

From the conditions (17) we find the stress components on contour C:

$$\tau^1 = -\frac{l_2}{l_1}\tau^2 + \frac{\tau_n}{l_1}, \quad \tau^2 = \frac{a_{13}l_2\tau_n \mp l_1\sqrt{l_1^2 a_{23} + l_2^2 a_{13} - a_{13}a_{23}\tau_n^2}}{l_1^2 a_{23} + l_2^2 a_{13}}. \tag{18}$$

From this point on, to be definite, in formulas (18) we will be selecting the upper sign.

3. The use of conservation laws to find stress components in the area

Assume the point $M\,(x_m, y_m)$ lies beyond the contour C. Let us draw a circumference with radius ε with the centre at the point M. We have $\varepsilon : (x - x_m)^2 + (y - y_m)^2 = \varepsilon^2$. Assume D is a line connecting the point M with the contour C. We obtain a closed contour consisting of the circumference ε, the segmant P and the contour C. See Fig. 2.

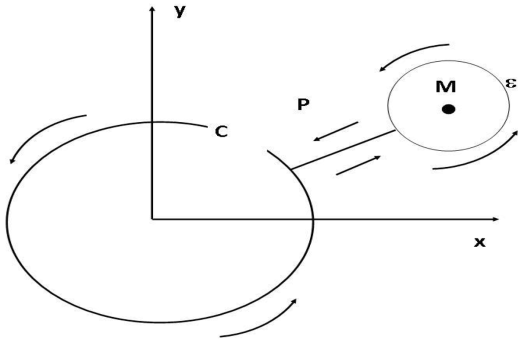

Fig. 2. Calculating the contour integral around the singular point M

From (15) we obtain

$$\oint_C Ady - Bdx + \int_{P^+} Ady - Bdx + \int_{P^-} Ady - Bdx + \oint_\varepsilon Ady - Bdx = 0. \tag{19}$$

The sum of the second and the third summands in (19) is equal to zero, because the integrals are calculated in different directions. Finally from (19) we have

$$\int_C Ady - Bdx = -\oint_\varepsilon Ady - Bdx. \tag{20}$$

Let us convert the right-hand part of equation (20) introducing parametrisation $x = \varepsilon \cos t$, $y = \varepsilon \sin t$, $0 \leqslant t \leqslant 2\pi$. As a result we have

$$\oint_\varepsilon Ady - Bdx = \varepsilon \int_0^{2\pi} (A \cos t + B \sin t)\, dt. \tag{21}$$

Assume in (15)

$$\alpha^1 = \frac{x}{x^2 + ny^2}, \beta^1 = -\frac{y}{x^2 + ny^2} \tag{22}$$

Then from (21) we obtain

$$\oint_\varepsilon A_1 dy - B_1 dx = \varepsilon \int_0^{2\pi} (A_1 \cos t + B_1 \sin t)\, dt = \int_0^{2\pi} \tau^1 dt = 2\pi\tau^1 (x_m, y_m). \tag{23}$$

The last equality in (23) is obtained with the use of the mean-value theorem with ε tending to zero.

Assume in (15)

$$\alpha^1 = \frac{\sqrt{n} y}{x^2 + ny^2}, \quad \beta^1 = \frac{1}{\sqrt{n}} \frac{x}{x^2 + ny^2}. \tag{24}$$

Then from (21) we obtain

$$\oint_\varepsilon A_2 dy - B_2 dx = \varepsilon \int_0^{2\pi} (A_2 \cos t + B_2 \sin t)\, dt = \int_0^{2\pi} \tau^2 dt = 2\pi\tau^2 (x_m, y_m). \tag{25}$$

The last equality in (25) is obtained with the use of the mean-value theorem with ε tending to zero.

From formula (20), and also from (23) and (25) we obtain

$$\int_C A_1 dy - B_1 dx = -2\pi\tau^1 (x_m, y_m), \quad \int_C A_2 dy - B_2 dx = -2\pi\tau^2 (x_m, y_m). \tag{26}$$

Conclusion

Formulas (26) offer the opportunity to find stress components in any point x_m, y_m beyond the contour C. This allows us to determine the boundary between the elastic and plastic areas. If the plasticity condition is met $a_{13}(\tau^1)^2 + a_{23}(\tau^2)^2 = 1$ at the point x_m, y_m then this point belongs to the plastic area, if in the point the condition $a_{13}(\tau^1)^2 + a_{23}(\tau^2)^2 < 1$ is met, then to the elastic area.

Note. The formulas found above allow us to solve elastoplastic problems even if the plastic contour does not entirely surrounds the contour C, provided that on the contour C the plasticity condition (7) is fulfilled.

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Анизотропная антиплоская упругопластическая задача

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Аннотация. В работе решена анизотропная антиплоская упругопластическая задача о напряженном состоянии в теле, ослабленном отверстием, ограниченном кусочно-гладким контуром. В статье приведены законы сохранения, которые позволили свести вычисления компонент тензора напряжений к криволинейному интегралу по контуру отверстия. Законы сохранения дали возможность найти границу между упругой и пластической областями.

Ключевые слова: анизотропная упругопластическая задача, антиплоское напряженное состояние, законы сохранения.