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# On the Application of the Plan Formula to the Study of the Zeta-Function of Zeros of Entire Function

**Vyacheslav I. Kuzovatov***
**Alexander M. Kytmanov**†
Institute of Mathematics and Computer Science
Siberian Federal University
Russian Federation
**Azimbai Sadullaev**‡
National university of Uzbekistan
Uzbekistan

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**Abstract.** We consider an application of the Plan formula to the study of the properties of the zeta-function of zeros of entire function. Based on this formula, we obtained an explicit expression for the kernel of the integral representation of the zeta-function in this case.

**Keywords:** zeta-function of zeros, Plan formula, integral representation.

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## Introduction

Recall (see, for example, [1, Chapter I, S. 1.9]) that the classical Plan formula has the form

$$\sum_{n=0}^{\infty} g(n) = \frac{1}{2} g(0) + \int_0^{\infty} g(\tau)\, d\tau + i \int_0^{\infty} \frac{g(it) - g(-it)}{e^{2\pi t} - 1}\, dt, \tag{1}$$

and is valid if

1. $g(\zeta)$ is regular for $\operatorname{Re} \zeta \geqslant 0, \quad \zeta = \tau + it$,
2. $\lim\limits_{t\to\infty} e^{-2\pi|t|} g(\tau + it) = 0$ uniformly for $0 \leqslant \tau < \infty$,
3. $\lim\limits_{\tau\to\infty} \int\limits_{-\infty}^{\infty} e^{-2\pi|t|} \left|g(\tau + it)\right| dt = 0$.

The Plan formula has been known for quite some time in the theory of functions of a complex variable. It is used in investigating analytical properties of functions assigned in the form of

*kuzovatov@yandex.ru
†AKytmanov@sfu-kras.ru https://orcid.org/0000-0002-7394-1480
‡sadullaev@mail.ru

progressions and for finding the sums of progressions in final form. Various generalizations of the Plan formula are obtained in works [2–4].

Among the physical applications of the classical Plan formula (some of its generalizations) one may note in the theory of quantum fields for renormalizing the tensor of the energy pulse of a scalar field in different Friedmann models of the Universe. It is also used for calculating a vacuum mean tensor of the energy-pulse of quantum fields in different complete and incomplete manifolds (the Casimir effect). A detailed presentation of these issues may be found in work [2].

In this article we consider an application of the Plan formula to the study of the properties of the zeta-function of zeros of entire function.

Regarding generalizations of the zeta-function, we note that in 1950s I. M. Gel'fand, B. M. Levitan, and L. A. Dikii (see, for example, [5–7]) studied the zeta-function associated to eigenvalues of the Sturm-Liouville operator. As it turned out, its value is connected with the trace of the operator. Their approach was further developed by V. B. Lidskii and V. A. Sadovnichii [8] who considered a class of entire functions of one variable, defined the zeta-function of their zeroes and investigated its domain of analytic continuation. S. A. Smagin and M. A. Shubin [9] constructed the zeta-functions for elliptic operators, as long as for operators of more general type, proved a possibility of meromorphic continuation of the zeta-function and gave some information on its poles.

Multidimensional results were obtained by A. M. Kytmanov and S. G. Myslivets [10]. They introduced the concept of the zeta-function associated with a system of meromorphic functions $f = (f_1, \ldots, f_n)$ in $\mathbb{C}^n$. Using the residue theory, these authors gave an integral representation for the zeta-function, but the system of functions $f_1, \ldots, f_n$ was subject to rigid constraints.

## 1. Auxiliary results

Let $f(z)$ be an entire function of order $\rho$ in $\mathbb{C}$. Consider the equation

$$f(z) = 0. \tag{2}$$

Denote by $N_f = f^{-1}(0)$ the set of all solutions to (2) (we take every zero as many times as its multiplicity). The numbers of roots is at most countable.

*The zeta-function* $\zeta_f(s)$ of Eq. (2) is defined in the following way:

$$\zeta_f(s) = \sum_{z_n \in N_f} (-z_n)^{-s},$$

where $s \in \mathbb{C}$.

In [11], using the residue theory, V. I. Kuzovatov and A. A. Kytmanov obtained two integral representation for the zeta-function constructed by zeros of an entire function of finite order on the complex plane. With the help of these representations, they described a domain which the zeta-function can be extended to.

**Theorem 1.1** ([11]). *Let $f(z)$ be an entire function of the zero order in $\mathbb{C}$ and satisfy the condition*

$$\frac{f'(z)}{f(z)} - \omega_0 = O\left(\frac{1}{|z|}\right).$$

*Suppose that $0 < Re\ s < 1$. Then*

$$\zeta_f(s) = \frac{\sin \pi s}{\pi} \int_0^\infty \left(\frac{f'(x)}{f(x)} - \omega_0\right) x^{-s}\, dx, \tag{3}$$

*where* $\omega_0$ *is the limit value of* $\dfrac{f'(x)}{f(x)}$ *at infinity.*

The method of proof of Theorem 1.1 shows that the statement remains valid in the case when $f(z)$ is an entire function of order less than 1.

Now we will give an integral representation for the zeta-function $\zeta_f(s)$ of zeros $z_n$ of $f$ which are $z_n = -q_n + is_n$, $q_n > 0$. Let us denote

$$F(f,x) = \sum_{n=1}^{\infty} e^{z_n x}. \tag{4}$$

We will assume that Re $s = \sigma > 1$ and the following conditions hold:

$$\varliminf_{n\to\infty} \frac{q_n}{n} > 0, \tag{5}$$

$$\text{the series } \sum_{n=1}^{\infty} \left(\frac{1}{q_n}\right)^{\sigma-1} \text{ converges.} \tag{6}$$

For the convergence of the series (4), using condition (5), it is necessary and sufficient (for real $x$) that $x > 0$ [11].

**Theorem 1.2** ([11])**.** *Suppose that the conditions* (5) *and* (6) *are satisfied and Re* $s > 1$*. Then*

$$\zeta_f(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} F(f,x)\, dx,$$

*where* $F(f,x)$ *is defined by formula* (4)*, and* $\Gamma(s)$ *is the Euler gamma-function.*

Our goal is to obtain an explicit expression for the kernel of the integral representation (3) in case $z_n = -\pi n^2$. This choice of zeros $z_n$ is due to the fact that for series

$$F(f,x) = \sum_{n=1}^{\infty} e^{z_n x} = \sum_{n=1}^{\infty} e^{-\pi n^2 x} := \psi(x)$$

for $x > 0$ it is known (see, for example, [12, Chapter II, S. 6]) that

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}} \left\{ 2\psi\left(\frac{1}{x}\right) + 1 \right\}.$$

# 2. The main result

**Theorem 2.1.** *Let* $f(z)$ *be an entire function of order* $\rho$ *with zeros* $z_n = -\pi n^2$*. Then for real* $x \in (0; +\infty)$ *the following holds*

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}} - \frac{1}{2x}.$$

*Proof.* We consider entire functions $f(z)$ of order $\rho$, which have the form

$$f(z) = C \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right). \tag{7}$$

The representation (7) is true, for example, for entire functions of order less than 1 or for entire functions of the first order with the additional condition (the series $\sum\limits_{n=1}^{\infty} \frac{1}{|z_n|}$ is convergent). In particular, the representation (7) is true for functions of the zero genus.

It is easy to show that in this case we obtain

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{z - z_n} \tag{8}$$

if $z \neq z_n$.

Since the order of the canonical product (7) is equal to the index of convergence $\rho_1$ of its zeros and for given values of $z_n$

$$\rho_1 = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln |z_n|} = \frac{1}{2},$$

then representations (7) and (8) are true for considered function $f(z)$.

Now we make use of a summation formula due to Plan (1). Taking

$$g(\zeta) = \frac{1}{z + \pi\zeta^2}, \quad \text{Re } z > 0,$$

in (1) we find that

$$\sum_{n=1}^{\infty} \frac{1}{z + \pi n^2} = -\frac{1}{2z} + \int_0^{\infty} \frac{d\tau}{z + \pi\tau^2}.$$

Passing in the last equality from complex $z$ to real $x \in (0; +\infty)$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{x + \pi n^2} = \frac{f'(x)}{f(x)} = -\frac{1}{2x} + \int_0^{\infty} \frac{d\tau}{x + \pi\tau^2} = -\frac{1}{2x} + \frac{1}{\pi}\int_0^{\infty} \frac{d\tau}{\tau^2 + x/\pi} =$$
$$= -\frac{1}{2x} + \frac{1}{\pi}\sqrt{\frac{\pi}{x}} \operatorname{arctg} \frac{\tau\sqrt{\pi}}{\sqrt{x}}\Big|_0^{\infty} = -\frac{1}{2x} + \frac{1}{\sqrt{\pi x}} \cdot \frac{\pi}{2} = \frac{\sqrt{\pi}}{2\sqrt{x}} - \frac{1}{2x}. \quad \square$$

**Corollary 1.** *Suppose that the conditions of Theorem 2.1 are satisfied. If $\omega_0$ is the limit value of $\frac{f'(x)}{f(x)}$ at infinity, i.e.*

$$\omega_0 = \lim_{x\to+\infty} \frac{f'(x)}{f(x)},$$

*then $\omega_0 = 0$.*

**Remark 1.** If $f$ is an arbitrary entire function of order $1 \leqslant \rho < \infty$, with zeros $z_n = -\pi n^2$, then the ratio can be represented as

$$\frac{f(z)}{\prod\limits_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)} = e^{g(z)},$$

where $g(z)$ is an entire function. Since $1 \leqslant \rho < \infty$, $g(z)$ is a polynomial, $\deg g = \rho$, and $\rho \in \mathbb{N}$ [13]. Therefore,

$$f(z) = \Pi(z)e^{g(z)}, \quad \Pi(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right),$$

and

$$\frac{f'(z)}{f(z)} = \frac{\Pi'(z)e^{g(z)} + \Pi(z)e^{g(z)}g'(z)}{\Pi(z)e^{g(z)}} = \frac{\Pi'(z)}{\Pi(z)} + g'(z).$$

Consequently in this case we take

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}} - \frac{1}{2x} + g'(x), \quad 1 \leqslant \rho < \infty.$$

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# О применении формулы Плана к исследованию дзета-функции нулей целой функции

**Вячеслав И. Кузоватов**
**Александр М. Кытманов**
Сибирский федеральный университет
Красноярск, Российская Федерация
**Азимбай Садуллаев**
Национальный университет Узбекистана
Ташкент, Узбекистан

**Аннотация.** В данной статье рассмотрено применение формулы Плана к исследованию свойств дзета-функции нулей целой функции. На основе данной формулы получено явное выражение для ядра интегрального представления дзета-функции в этом случае.

**Ключевые слова:** дзета-функция нулей, формула Плана, интегральное представление.