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Singular Quasilinear Elliptic Systems with (super-) Homogeneous Condition

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Abstract. In this paper we establish existence, nonexistence and regularity of positive solutions for a class of singular quasilinear elliptic systems subject to (super-) homogeneous condition. The approach is based on sub-supersolution methods for systems of quasilinear singular equations combined with perturbation arguments involving singular terms.

Keywords: singular system, p-Laplacian, sub-supersolution, regularity.

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Introduction

We consider the following system of quasilinear and singular elliptic equations:

$$(\mathcal{P}) \qquad \begin{cases} -\Delta_{p_1} u_1 = \lambda u_1^{\alpha_1} u_2^{\beta_1} + \delta h_1(x) \text{ in } \Omega \\ -\Delta_{p_2} u_2 = \lambda u_1^{\alpha_2} u_2^{\beta_2} + \delta h_2(x) \text{ in } \Omega \\ u_1, u_2 > 0 \text{ in } \Omega \\ u_1, u_2 = 0 \text{ on } \partial\Omega \end{cases},$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ having a smooth boundary $\partial\Omega$, $\lambda > 0$, $\delta \geqslant 0$ are parameters, and $h_i \in L^\infty(\Omega)$ is a nonnegative function. Here Δ_{p_i} stands for the p_i-Laplacian differential operator with $1 < p_i \leqslant N$. A solution of $(\mathcal{P})$ is understood in the weak sense, that is, a pair $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, which are positive a.e. in Ω and satisfying

$$\int_\Omega |\nabla u_i|^{p_i-2} \nabla u_i \nabla \varphi_i \, dx = \int_\Omega (\lambda u_1^{\alpha_i} u_2^{\beta_i} + \delta h_i) \varphi_i \, dx, \text{ for all } \varphi_i \in W_0^{1,p_i}(\Omega), \ i = 1, 2.$$

We consider the system $(\mathcal{P})$ in a singular case assuming that

$$0 < \alpha_2 < p_1^* - 1, \ \ 0 < \beta_1 < p_2^* - 1 \ \text{ and } \ -1 < \alpha_1, \, \beta_2 < 0, \tag{1}$$

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where $p_i^* = \dfrac{Np_i}{N-p_i}$. This assumption makes system $(\mathcal{P})$ be cooperative, that is, for u_1(resp. u_2) fixed the right term in the first (resp. second) equation of $(\mathcal{P})$ is increasing in u_2 (resp. u_1).

Recently, singular cooperative system $(\mathcal{P})$ with $\delta = 0$ was mainly studied in [8, 9, 20]. In [20] existence and boundedness theorems for $(\mathcal{P})$ was established by using sub-supersolution method for systems combined with perturbation techniques. In [8] one gets existence, uniqueness, and regularity of a positive solution on the basis of an iterative scheme constructed through a sub-supersolution pair. In [9] an existence theorem involving sub-supersolution was obtained through a fixed point argument in a sub-supersolution setting. The semilinear case in $(\mathcal{P})$ (i.e. $p_i = 2$) was considered in [7, 13, 21] where the linearity of the principal part is essentially used. In this context, the singular system $(\mathcal{P})$ can be viewed as the elliptic counter-part of a class of Gierer-Meinhardt systems that models some biochemical processes (see, e.g. [21]). It can be also given an astrophysical meaning since it generalizes to the system the well-known Lane-Emden equation, where all exponents are negative (see [7]). For the one dimentional case ($N = 1$) we quote [15] and the references therein. The complementary situation for the system $(\mathcal{P})$ with respect to (1) is the so-called competitive system, which has recently attracted much interest. Relevant contributions regarding this topic can be found in [9, 18, 19]. For the regular case in $(\mathcal{P})$, that is when all the exponents are positive, we refer to [6, 22], while for quasilinear systems with singular weights we cite [2, 4] and their references.

It is worth pointing out that the aforementioned works have examined the subhomogeneous case $\Theta > 0$ of singular problem $(\mathcal{P})$ where

$$\Theta = (p_1 - 1 - \alpha_1)(p_2 - 1 - \beta_2) - \beta_1\alpha_2. \tag{2}$$

The constant Θ is related to system stability $(\mathcal{P})$ that behaves in a drastically different way, depending on the sign of Θ. For instance, for $\Theta < 0$ system $(\mathcal{P})$ is not stable in the sense that possible solutions cannot be obtained by iterative methods (see [5]).

Unlike the subhomogeneous case $\Theta > 0$ studied in the above references, the novelty of this paper is to establish the existence, regularity and nonexistence of (positive) solutions for singular problem $(\mathcal{P})$ by processing the two cases: 'homogeneous' when $\Theta = 0$ and 'superhomogeneous' if $\Theta < 0$. It should be noted that throughout this paper, $\Theta < 0$ (resp. $= 0$) means that $p_i - 1 - \alpha_i - \beta_i < 0$ (resp. $= 0$).

The existence result for problem $(\mathcal{P})$ is stated as follows.

Theorem 1. *Assume (1), $\Theta < 0$ (resp. $\Theta = 0$) and suppose that*

$$\inf_{\Omega} h_1(x), \ \inf_{\Omega} h_2(x) > 0. \tag{3}$$

Then, there is $\delta_0 > 0$ (resp. $\delta_0, \lambda_0 > 0$) such that, for all $\delta \in (0, \delta_0)$, problem $(\mathcal{P})$ possesses a (positive) solution (u_1, u_2) in $C_0^{1,\beta}(\overline{\Omega}) \times C_0^{1,\beta}(\overline{\Omega})$, for certain $\beta \in (0, 1)$, verifying

$$u_i \geqslant cd(x) \quad in\ \Omega,$$

for some constant $c > 0$ and for all $\lambda > 0$ (resp. $\lambda \in (0, \lambda_0)$). Moreover, if $\Theta = \delta = 0$ and

$$\beta_1 = \frac{p_2}{p_1}(p_1 - 1 - \alpha_1) \quad or \quad \alpha_2 = \frac{p_1}{p_2}(p_2 - 1 - \beta_2), \tag{4}$$

then, there exists $\lambda_* > 0$ such that problem $(\mathcal{P})$ has no solution for every $\lambda \in (0, \lambda_*)$.

The main technical difficulty consists in the presence of singular terms in system $(\mathcal{P})$ with (1), expressed through (super-) homogeneous condition. Our approach is chiefly based on sub-supersolution method in its version for systems [3, section 5.5]. However, this method cannot be directly implemented due to the presence of singular terms in $(\mathcal{P})$ under assumption (1). So, we first disturb system $(\mathcal{P})$ by introducing a parameter $\varepsilon > 0$. This gives rise to a regularized system for $(\mathcal{P})$ depending on ε whose study is relevant for our initial problem. By applying the sub-supersolution method, we show that the regularized system has a positive solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ in $C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. It is worth noting that the choice of suitable functions with an adjustment of adequate constants is crucial in order to construct the sub-supersolution pair as well as to process the both cases $\Theta < 0$ and $\Theta = 0$. The (positive) solution (u_1, u_2) in $(W_0^{1,p_1}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,p_2}(\Omega) \cap L^\infty(\Omega))$ of $(\mathcal{P})$ is obtained by passing to the limit as $\varepsilon \to 0$. This is based on a priori estimates, Fatou's Lemma and S_+-property of the negative p_i-Laplacian. The positivity of the solution (u_1, u_2) is achieved through assumption (3) while $C^{1,\beta}$-regularity is derived from the regularity result in [11].

The rest of the paper is organized as follows. Section 1 is devoted to the existence of solutions for the regularized system. Section 2 established the proof of the main result.

1. The regularized system

Given $1 < p < +\infty$, the space $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ are endowed with the usual norms $\|u\|_p = \left(\int_\Omega |u|^p \, dx\right)^{1/p}$ and $\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p}$, respectively. We will also use the space $C_0^{1,\beta}(\overline{\Omega}) = \{u \in C^{1,\beta}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ for a suitable $\beta \in (0,1)$.

In what follows, we denote by ϕ_{1,p_i} the positive eigenfunction associated with the principal eigenvalue λ_{1,p_i}, characterized by the minimum of Rayleigh quotient

$$\lambda_{1,p_i} = \inf_{u_i \in W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\|\nabla u_i\|_{p_i}^{p_i}}{\|u_i\|_{p_i}^{p_i}}. \tag{5}$$

For a later use recall there exist constants $l_i, \hat{l}_i > 0$ such that

$$\hat{l}_1 \phi_{1,p_1}(x) \geqslant \phi_{1,p_2}(x) \geqslant \hat{l}_2 \phi_{1,p_1}(x) \text{ and } l_1 d(x) \geqslant \phi_{1,p_i}(x) \geqslant l_2 d(x) \text{ for all } x \in \Omega, \tag{6}$$

where $d(x) := dist(x, \partial\Omega)$ (see, e.g., [10]).

Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\tilde{\Omega}$ such that $\overline{\Omega} \subset \tilde{\Omega}$. Denote $\tilde{d}(x) := d(x, \partial\tilde{\Omega})$. By the definition of $\tilde{\Omega}$ there exists a constant $\rho > 0$ sufficiently small such that

$$\tilde{d}(x) > \rho \text{ in } \overline{\Omega}. \tag{7}$$

Define $w_i \in C^1(\overline{\tilde{\Omega}})$ the unique solution of the torsion problem

$$-\Delta_{p_i} w_i = 1 \text{ in } \tilde{\Omega}, \; w_i = 0 \text{ on } \partial\tilde{\Omega}, \tag{8}$$

satisfying the estimates

$$w_i(x) \geqslant c_0 \tilde{d}(x) \text{ in } \tilde{\Omega}, \tag{9}$$

for certain constant $c_0 \in (0,1)$ (see [12, Lemma 2.1]).

For a real constant $C > 1$, set

$$(\underline{u}_{i,\varepsilon}, \overline{u}_i) = (c_\varepsilon \phi_{1,p_i}, C^{-1} w_i), \; i = 1, 2, \tag{10}$$

where $c_\varepsilon > 0$ is a constant depending on $\varepsilon > 0$ such that

$$0 < c_\varepsilon < c_0 l_1^{-1} C^{-1}. \tag{11}$$

Then, by (10), (6) and (8), it is readily seen that

$$\begin{aligned}&\overline{u}_i(x) = C^{-1}w_i(x) \geqslant C^{-1}c_0\tilde{d}(x) \geqslant C^{-1}c_0 d(x) \geqslant \\ &\geqslant l_1^{-1}C^{-1}c_0\phi_{1,p_i}(x) \geqslant c_\varepsilon\phi_{1,p_i}(x) = \underline{u}_{i,\varepsilon}(x) \text{ in } \overline{\Omega}, \text{ for } i = 1,2.\end{aligned}$$

For every $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 < 1$, let introduce the auxiliary problem

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} -\Delta_{p_1}u_1 = \lambda(u_1+\varepsilon)^{\alpha_1}(u_2+\varepsilon)^{\beta_1} + \delta h_1(x) & \text{in } \Omega \\ -\Delta_{p2}u_2 = \lambda(u_1+\varepsilon)^{\alpha_2}(u_2+\varepsilon)^{\beta_2} + \delta h_2(x) & \text{in } \Omega \\ u_1, u_2 = 0 & \text{on } \partial\Omega \end{cases},$$

which provides approximate solutions for the initial problem $(\mathcal{P})$.

Lemma 1. *Assume (1) and $h_1, h_2 \neq 0$ in Ω. Then, if $\Theta < 0$ (resp. $\Theta = 0$), there is a constant $\delta_0 > 0$ (resp. $\delta_0, \lambda_0 > 0$) such that for all $\delta \in (0, \delta_0)$, $(\overline{u}_1, \overline{u}_2)$ in (10) is a supersolution of $(\mathcal{P}_\varepsilon)$ for all $\lambda > 0$ (resp. $\lambda \in (0, \lambda_0)$) and all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Assume $\Theta < 0$ and set $\varepsilon_0 = C^{-1}$,

$$\delta_0 = \frac{1}{2}\min_{i=1,2}\left\{\frac{1}{C^{p_i-1}\|h_i\|_\infty}\right\}. \tag{12}$$

On account of (1), (7)–(10) and (12), for all $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_0)$, one derives

$$\begin{aligned}&(\overline{u}_1+\varepsilon)^{-\alpha_1}(\overline{u}_2+\varepsilon)^{-\beta_1}(-\Delta_{p_1}\overline{u}_1 - \delta h_1) \geqslant \overline{u}_1^{-\alpha_1}(\overline{u}_2+\varepsilon_0)^{-\beta_1}(-\Delta_{p_1}\overline{u}_1 - \delta\|h_1\|_\infty) \geqslant \\ &\geqslant C^{\alpha_1+\beta_1}(c_0\tilde{d}(x))^{-\alpha_1}(\|w_2\|_\infty+1)^{-\beta_1}(C^{-(p_1-1)} - \delta_0\|h_1\|_\infty) \geqslant \\ &\geqslant C^{\beta_1-(p_1-1-\alpha_1)}(c_0\rho)^{-\alpha_1}(\|w_2\|_\infty+1)^{-\beta_1}(1-\delta_0 C^{p_1-1}\|h_1\|_\infty) \geqslant \\ &\geqslant \frac{1}{2}C^{\beta_1-(p_1-1-\alpha_1)}(c_0\rho)^{-\alpha_1}(\|w_2\|_\infty+1)^{-\beta_1} \geqslant \lambda \text{ in } \overline{\Omega},\end{aligned}$$

and similarly

$$\begin{aligned}&(\overline{u}_1+\varepsilon)^{-\alpha_2}(\overline{u}_2+\varepsilon)^{-\beta_2}(-\Delta_{p_2}\overline{u}_2 - \delta h_2) \geqslant (\overline{u}_1+\varepsilon_0)^{-\alpha_2}\overline{u}_2^{-\beta_2}(-\Delta_{p_2}\overline{u}_2 - \delta\|h_2\|_\infty) \geqslant \\ &\geqslant C^{\alpha_2+\beta_2}(\|w_1\|_\infty+1)^{-\alpha_2}(c_0\tilde{d}(x))^{-\beta_2}(C^{-(p_2-1)} - \delta_0\|h_2\|_\infty) = \\ &= C^{\alpha_2-(p_2-1-\beta_2)}(\|w_1\|_\infty+1)^{-\alpha_2}(c_0\rho)^{-\beta_2}(1-\delta_0 C^{p_2-1}\|h_2\|_\infty) \geqslant \\ &\geqslant \tfrac{1}{2}C^{\alpha_2-(p_2-1-\beta_2)}(\|w_1\|_\infty+1)^{-\alpha_2}(c_0\rho)^{-\beta_2} \geqslant \lambda \text{ in } \overline{\Omega},\end{aligned}$$

for all $\lambda > 0$, provided $C > 1$ is sufficiently large. This shows that $(\overline{u}_1, \overline{u}_2)$ is a supersolution pair for problem $(\mathcal{P}_\varepsilon)$. If $\Theta = 0$, by repeating the argument above, the same conclusion can be drawn for $\lambda \in (0, \lambda_0)$ with a constant $\lambda_0 > 0$ that can be precisely estimated. This completes the proof. □

Lemma 2. *Assume (1) and $\Theta \leqslant 0$ hold. Then, $(\underline{u}_{1,\varepsilon}, \underline{u}_{2,\varepsilon})$ is a subsolution of $(\mathcal{P}_\varepsilon)$ for all $\lambda, \delta > 0$ and every $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Fix $\varepsilon \in (0, \varepsilon_0)$. From (10) and (1), we obtain

$$\begin{aligned}&(\underline{u}_{1,\varepsilon}+\varepsilon)^{-\alpha_1}(\underline{u}_{2,\varepsilon}+\varepsilon)^{-\beta_1}(-\Delta_{p_1}\underline{u}_{1,\varepsilon} - \delta h_1) \leqslant \\ &\leqslant c_\varepsilon^{p_1-1}(c_\varepsilon\phi_{1,p_1}+\varepsilon_0)^{-\alpha_1}(c_\varepsilon\phi_{1,p_2}+\varepsilon)^{-\beta_1}\lambda_{1,p_1}\phi_{1,p_1}^{p_1-1} \leqslant \\ &\leqslant c_\varepsilon^{p_1-1}(\phi_{1,p_1}+\varepsilon_0)^{-\alpha_1}(c_\varepsilon\phi_{1,p_2}+\varepsilon)^{-\beta_1}\lambda_{1,p_1}\phi_{1,p_1}^{p-1} \leqslant \\ &\leqslant c_\varepsilon^{p_1-1}\varepsilon^{-\beta_1}(\|\phi_{1,p_1}\|_\infty+1)^{-\alpha_1}\lambda_{1,p_1}\|\phi_{1,p_1}\|_\infty^{p_1-1} \leqslant \lambda \text{ in } \overline{\Omega}\end{aligned} \tag{13}$$

and similarly

$$\begin{aligned}&(\underline{u}_{1,\varepsilon}+\varepsilon)^{-\alpha_2}(\underline{u}_{2,\varepsilon}+\varepsilon)^{-\beta_2}(-\Delta_{p_2}\underline{u}_{2,\varepsilon}-\delta h_2)\leqslant\\ &\leqslant(\underline{u}_{1,\varepsilon}+\varepsilon)^{-\alpha_2}(\underline{u}_{2,\varepsilon}+\varepsilon_0)^{-\beta_2}(\Delta_{p_2}\underline{u}_{2,\varepsilon})=\\ &=c_\varepsilon^{p_2-1}(c_\varepsilon\phi_{1,p_1}+\varepsilon)^{-\alpha_2}(\phi_{1,p_2}+\varepsilon_0)^{-\beta_2}\lambda_{1,p_2}\phi_{1,p_2}^{p_2-1}\leqslant\\ &\leqslant c_\varepsilon^{p_2-1}\varepsilon^{-\alpha_2}(\|\phi_{1,p_2}\|_\infty+\varepsilon_0)^{-\beta_2}\lambda_{1,p_2}\|\phi_{1,p_2}\|_\infty^{p_2-1}\leqslant\lambda\ \text{ in }\overline{\Omega},\end{aligned}\tag{14}$$

provided $c_\varepsilon > 0$ is sufficiently small. Gathering (13) and (14) together yields

$$-\Delta_{p_i}\underline{u}_{i,\varepsilon}\leqslant\lambda(\underline{u}_{1,\varepsilon}+\varepsilon)^{\alpha_i}(\underline{u}_{2,\varepsilon}+\varepsilon)^{\beta_i}+\delta h_i\ \text{ in }\overline{\Omega},$$

proving that $(\underline{u}_{1,\varepsilon},\underline{u}_{2,\varepsilon})$ in (10) is a subsolution pair for problem $(\mathcal{P}_\varepsilon)$. □

We state the following result regarding the regularized system.

Theorem 2. *Assume (1) and $h_1, h_2 \neq 0$ in Ω. Then*

(a) *If $\Theta < 0$ (resp. $\Theta = 0$) there exist a constant $\delta_0 > 0$ (resp. $\delta_0, \lambda_0 > 0$) such that for all $\delta \in (0, \delta_0)$ system $(\mathcal{P}_\varepsilon)$ has a (positive) solution $(u_{1,\varepsilon}, u_{2,\varepsilon}) \in C_0^{1,\beta}(\overline{\Omega})\times C_0^{1,\beta}(\overline{\Omega})$, $\beta \in (0,1)$, satisfying*

$$u_{i,\varepsilon}(x)\leqslant\overline{u}_i(x)\text{ in }\Omega,\tag{15}$$

for all $\lambda > 0$ (resp. $\lambda \in (0, \lambda_0)$), and every $\varepsilon \in (0, \varepsilon_0)$.

(b) *For $\Theta \leqslant 0$ and under assumption (3), if $\delta > 0$, there exists a constant $c_0 > 0$, independent of ε, such that all solutions $(u_{1,\varepsilon}, u_{2,\varepsilon})$ of system $(\mathcal{P}_\varepsilon)$ verify*

$$u_{i,\varepsilon}(x)\geqslant c_0 d(x)\text{ for a.a. }x\in\Omega,\ \text{ for all }\varepsilon\in(0,\varepsilon_0).\tag{16}$$

Proof. On the basis of Lemmas 1 and 2 together with [3, section 5.5] there exists a solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ of problem $(\mathcal{P}_\varepsilon)$, for every $\varepsilon \in (0, \varepsilon_0)$. Moreover, applying the regularity theory (see [16]), we infer that $(u_{1,\varepsilon}, u_{2,\varepsilon}) \in C_0^{1,\beta}(\overline{\Omega})\times C_0^{1,\beta}(\overline{\Omega})$ for a suitable $\beta \in (0,1)$. This proves (a).

Now, according to (3), let $\sigma > 0$ be a constant such that $\inf_\Omega h_1(x)$, $\inf_\Omega h_2(x) > \sigma$. Define z_i the only positive solution of

$$-\Delta_{p_i}z_i=\delta\sigma\ \text{ in }\Omega,\ z_i=0\ \text{ on }\partial\Omega,$$

which is known to satisfy $z_i(x) \geqslant c_2 d(x)$ in Ω. Then it follows that $-\Delta_{p_i}u_\varepsilon \geqslant -\Delta_{p_i}z_i$ in Ω, $u_{i,\varepsilon} = z_i$ on $\partial\Omega$, for all $\varepsilon \in (0, \varepsilon_0)$, and therefore, the weak comparison principle ensures the assertion (b) holds true. □

2. Proof of the main result

Set $\varepsilon = \dfrac{1}{n}$ with any positive integer $n > 1/\varepsilon_0$. From Theorem 2 with $\varepsilon = \dfrac{1}{n}$, there exists $u_{i,n} := u_{i,\frac{1}{n}}$ such that

$$\langle-\Delta_{p_i}u_{i,n},\varphi_i\rangle=\lambda\int_\Omega\Big(u_{1,n}+\frac{1}{n}\Big)^{\alpha_i}\Big(u_{2,n}+\frac{1}{n}\Big)^{\beta_i}\varphi_i\,dx+\delta\int_\Omega h_i\varphi_i dx,\tag{17}$$

for all $\varphi_i \in W_0^{1,p_i}(\Omega), i = 1,2$. Taking $\varphi_1 = u_{1,n}$ in (17), since $\alpha_1 < 0 < \beta_1$, we get

$$\begin{aligned}
\|u_{1,n}\|_{1,p_1}^{p_1} &= \lambda \int_\Omega \left(u_{1,n} + \frac{1}{n}\right)^{\alpha_1} \left(u_{2,n} + \frac{1}{n}\right)^{\beta_1} u_{1,n}\, dx + \int_\Omega \delta h_1 u_{1,n}\, dx \leqslant \\
&\leqslant \lambda \int_\Omega u_{1,n}^{\alpha_1+1} (u_{2,n}+1)^{\beta_1}\, dx + \delta \|h_1\|_\infty \int_\Omega u_{1,n}\, dx \leqslant \\
&\leqslant \lambda \int_\Omega \overline{u}_1^{\alpha_1+1} (\overline{u}_2+1)^{\beta_1}\, dx + \delta \|h_1\|_\infty \int_\Omega \overline{u}_1\, dx \leqslant \\
&\leqslant \lambda|\Omega|(\|\overline{u}_1\|_\infty^{\alpha_1+1} (\|\overline{u}_2\|_\infty + 1)^{\beta_1} + \delta \|h_1\|_\infty \|\overline{u}_1\|_\infty).
\end{aligned} \tag{18}$$

Hence, $\{u_{1,n}\}$ is bounded in $W_0^{1,p_1}(\Omega)$. Similarly, we derive that $\{u_{2,n}\}$ is bounded in $W_0^{1,p_2}(\Omega)$. We are thus allowed to extract subsequences (still denoted by $\{u_{i,n}\}$) such that

$$u_{i,n} \rightharpoonup u_i \text{ in } W_0^{1,p_i}(\Omega),\ i = 1,2. \tag{19}$$

The convergence in (19) combined with Rellich embedding Theorem and (15)–(16) entails

$$c_0 d(x) \leqslant u_i(x) \leqslant \overline{u}_i(x) \text{ in } \Omega. \tag{20}$$

Inserting $\varphi_i = u_{i,n} - u_i$ in (17) yields

$$\langle -\Delta_{p_i} u_{i,n}, u_{i,n} - u_i \rangle = \int_\Omega \left[\lambda\Big(u_{1,n} + \frac{1}{n}\Big)^{\alpha_i} \Big(u_{2,n} + \frac{1}{n}\Big)^{\beta_i} + \delta h_i\right](u_{i,n} - u_i)\, dx.$$

We claim that

$$\lim_{n\to\infty} \langle -\Delta_{p_i} u_{i,n}, u_{i,n} - u_i \rangle \leqslant 0.$$

Indeed, from (15), (16) and (10), we have

$$\begin{aligned}
&\left|\left(\Big(u_{1,n} + \frac{1}{n}\Big)^{\alpha_1} \Big(u_{2,n} + \frac{1}{n}\Big)^{\beta_1} + \delta h_1\right)(u_{1,n} - u_1)\right| \leqslant \\
&\leqslant \big(u_{1,n}^{\alpha_1} (u_{2,n}+1)^{\beta_1} + \delta h_1\big)(|u_{1,n}| + |u_1|) \leqslant \\
&\leqslant 2\big((c_0 d(x))^{\alpha_1} (\overline{u}_2 + 1)^{\beta_1} + \delta h_1\big)\overline{u}_1 \leqslant \\
&\leqslant 2\big((c_0 d(x))^{\alpha_1} (\|\overline{u}_2\|_\infty + 1)^{\beta_1} + \delta \|h_1\|_\infty\big) \|\overline{u}_1\|_\infty \leqslant \hat{C}_0 d(x)^{\alpha_1} \text{ in } \Omega,
\end{aligned}$$

with some positive constant $\hat{C}_0$. Then, (1) together with Lemma in [14, page 726] imply that

$$\left(\Big(u_{1,n} + \frac{1}{n}\Big)^{\alpha_1} \Big(u_{2,n} + \frac{1}{n}\Big)^{\beta_1} + \delta h_1\right)(u_{1,n} - u_1) \in L^1(\Omega). \tag{21}$$

Using (19), (21) and applying Fatou's Lemma, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \sup \int_\Omega \left(\left(u_{1,n} + \frac{1}{n}\right)^{\alpha_1} \left(u_{2,n} + \frac{1}{n}\right)^{\beta_1} + \delta h_1\right)(u_{1,n} - u_1)\, dx \leqslant \\
&\leqslant \int_\Omega \lim_{n\to\infty} \sup \left(\left(\left(u_{1,n} + \frac{1}{n}\right)^{\alpha_1} \left(u_{2,n} + \frac{1}{n}\right)^{\beta_1} + \delta h_1\right)(u_{1,n} - u_1)\right) dx \to 0,
\end{aligned}$$

showing that $\lim_{n\to\infty} \sup \langle -\Delta_{p_1} u_{1,n}, u_{1,n} - u_1 \rangle \leqslant 0$. Likewise, we prove that

$$\lim_{n\to\infty} \sup \langle -\Delta_{p_2} u_{2,n}, u_{2,n} - u_2 \rangle \leqslant 0.$$

Then the S_+-property of $-\Delta_{p_i}$ on $W_0^{1,p_i}(\Omega)$ (see, e.g., [17, Proposition 3.5]) guarantees that

$$u_{i,n} \longrightarrow u_i \text{ in } W_0^{1,p_i}(\Omega),\ i = 1, 2. \tag{22}$$

On account of (17), besides (22), the next step is to verify that

$$\lim_{n\to\infty} \int_\Omega \left(u_{1,n} + \frac{1}{n}\right)^{\alpha_i} \left(u_{2,n} + \frac{1}{n}\right)^{\beta_i} \varphi_i \, dx = \int_\Omega u_1^{\alpha_i} u_2^{\beta_i} \varphi_i \, dx, \tag{23}$$

for all $\varphi_i \in W_0^{1,p_i}(\Omega)$. By (15), (16), (1) and (20), it holds

$$\left| \left(u_{1,n} + \frac{1}{n}\right)^{\alpha_1} \left(u_{2,n} + \frac{1}{n}\right)^{\beta_1} \varphi_1 \right| \leqslant (c_0 d(x))^{\alpha_1} (\|\overline{u}_2\|_\infty + 1)^{\beta_1} |\varphi_1|$$

and

$$\left| \left(u_{1,n} + \frac{1}{n}\right)^{\alpha_2} \left(u_{2,n} + \frac{1}{n}\right)^{\beta_2} \varphi_2 \right| \leqslant (\|\overline{u}_1\|_\infty + 1)^{\alpha_2} (c_0 d(x))^{\beta_2} |\varphi_2|.$$

Then, by (1) together with Hardy-Sobolev inequality (see, e.g., [1, Lemma 2.3]), assertion (23) stem from Lebesgue's dominated convergence Theorem. Hence we may pass to the limit in (17) to conclude that (u_1, u_2) is a solution of problem $(\mathcal{P})$ satsifying (20). Furthermore, using (1), (20) and (10), one has

$$\begin{aligned} & u_1^{\alpha_1} u_2^{\beta_1} + \delta h_1 \leqslant u_1^{\alpha_1} \overline{u}_2^{\beta_1} + \delta \|h_1\|_\infty \leqslant \\ & \leqslant (c_0 d(x))^{\alpha_1} \|\overline{v}\|_\infty^{\beta_1} + \delta \|h_1\|_\infty d(x)^{\alpha_1 - \alpha_1} \leqslant \\ & \leqslant C_1' d(x)^{\alpha_1} \text{ for all } x \in \Omega \end{aligned} \tag{24}$$

and

$$\begin{aligned} & u_1^{\alpha_2} u_2^{\beta_2} + \delta h_2 \leqslant \overline{u}_1^{\alpha_2} u_2^{\beta_2} + \delta \|h_2\|_\infty \leqslant \\ & \leqslant \|\overline{u}_1\|_\infty^{\alpha_2} (c_0 d(x))^{\beta_2} + \delta \|h_2\|_\infty d(x)^{\beta_2 - \beta_2} \leqslant \\ & \leqslant C_2' d(x)^{\beta_2} \text{ for all } x \in \Omega, \end{aligned} \tag{25}$$

for certain positive constants C_1' and C_2'. Hence, (1) enable us to apply Lemma 3.1 in [11] to infer that $(u, v) \in C_0^{1,\beta}(\overline{\Omega}) \times C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$.

We are left with the task of determining the nonexistence result stated in Theorem 1. Arguing by contradiction and assume that (u_1, u_2) is a positive solution of problem $(\mathcal{P})$ with $\delta = 0$. Multiplying in $(\mathcal{P})$ by u_i, integrating over Ω, applying Young inequality with $\alpha_1, \beta_2 > -1$, we get

$$\int_\Omega |\nabla u_1|^{p_1} \, dx = \lambda \int_\Omega u_1^{\alpha_1+1} u_2^{\beta_1} dx \leqslant \lambda \int_\Omega \left(\frac{\alpha_1 + 1}{p_1} u_1^{p_1} + \frac{p_1 - 1 - \alpha_1}{p_1} u_2^{\frac{\beta_1 p_1}{p_1 - 1 - \alpha_1}} \right) dx \tag{26}$$

and

$$\int_\Omega |\nabla u_2|^{p_2} \, dx = \lambda \int_\Omega u_1^{\alpha_2} u_2^{\beta_2+1} dx \leqslant \lambda \int_\Omega \left(\frac{p_2 - 1 - \beta_2}{p_2} u_1^{\frac{\alpha_2 p_2}{p_2 - 1 - \beta_2}} + \frac{\beta_2 + 1}{p_2} u_2^{p_2} \right) dx. \tag{27}$$

Adding (26) with (27), according to (4), this is equivalent to

$$\|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_2\|_{p_2}^{p_2} \leqslant \lambda \left[\left(\frac{\alpha_1 + 1}{p_1} + \frac{p_2 - 1 - \beta_2}{p_2} \right) \|u_1\|_{p_1}^{p_1} + \left(\frac{\beta_2 + 1}{p_2} + \frac{p_1 - 1 - \alpha_1}{p_1} \right) \|v\|_{p_2}^{p_2} \right]. \tag{28}$$

Since $\Theta = 0$, observe from (4) that

$$\begin{cases} \dfrac{\alpha_1 + 1}{p_1} + \dfrac{p_2 - 1 - \beta_2}{p_2} = \dfrac{\alpha_1 + \alpha_2 + 1}{p_1} \\ \dfrac{\beta_2 + 1}{p_2} + \dfrac{p_1 - 1 - \alpha_1}{p_1} = \dfrac{\beta_1 + \beta_2 + 1}{p_2}. \end{cases} \tag{29}$$

Then gathering (5), (28) and (29) together yields

$$\left(\lambda_{1,p_1} - \frac{\alpha_1+\alpha_2+1}{p_1}\lambda\right)\|u_1\|_{p_1}^{p_1} + \left(\lambda_{1,p_2} - \frac{\beta_1+\beta_2+1}{p_2}\lambda\right)\|u_2\|_{p_2}^{p_2} \leqslant 0$$

which is a contradiction for

$$\lambda < \lambda_* = \min\left\{\frac{p_1}{\alpha_1+\alpha_2+1}\lambda_{1,p_1}, \frac{p_2}{\beta_1+\beta_2+1}\lambda_{1,p_2}\right\}.$$

Thus, problem $(\mathcal{P})$ has no solution for $\lambda < \lambda_*$, which completes the proof.

References

[1] C.O.Alves, F.J.S.A.Corrêa, *Appl. Math. Comput.*, **185**(2007), 727–736.

[2] S.Boulaaras, R.Guefaifia, T.Bouali, *Indian J. Pure Appl. Math.*, **49**(2018), no. 4, 705–715.

[3] S.Carl, V.K.Le, D.Motreanu, Nonsmooth variational problems and their inequalities. Comparaison principles and applications, Springer, New York, 2007.

[4] Z.Deng, R.Zhang, Y.Huang, *Math. Meth. App. Sci.*, **40**(2017), no. 5, 1538–1552.

[5] P.Clément, J.Fleckinger, E.Mitidieri, F.de Thelin, *J. Diff. Eqts.*, **166**(2000), 455–477.

[6] C.Chen, *Nonl. Anal.*, **62**(2005), no. 4, 751–756.

[7] M.Ghergu, *J. Funct. Anal.*, **258**(2010), 3295–3318.

[8] J.Giacomoni, J.Hernandez, A.Moussaoui, *Contemporary Math.*, **540**(2011), 79–94.

[9] J.Giacomoni, J.Hernandez, P.Sauvy, *Advances Nonl. Anal.*, **2**(2013), 1–41.

[10] J.Giacomoni, I.Schindler, P.Takac, *A. Sc. N. Sup. Pisa*, **6**(2007), no. 5, 117–158.

[11] D.D.Hai, *J. Math. Anal. Appl.*, **383**(2011), 619–626.

[12] D.D.Ha i, H.Wang, *J. Math. Anal. Appl.*, **330**(2007), 186–194.

[13] J.Hernandez, F.J.Mancebo, J.M.Vega, *Adv. Diff. Eqts.*, **13**(2008), 857–880

[14] A.C.Lazer, P.J.Mckenna, *Proc. American Math. Soc.*, **111**(1991), no. 3, 721–730.

[15] Y-H.Lee, X.Xu, *Bull. Malays. Math. Sci. Soc.*, **42**(2019), no. 3, 1143–1159.

[16] G.M.Lieberman, *Nonlinear Anal.*, **12**(1988), 1203–1219.

[17] D.Motreanu, V.V.Motreanu, N.S.Papageorgiou, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **10**(2011), no. 5, 729–755.

[18] D.Motreanu, A.Moussaoui, *Act. Math. Sci.*, **34**(2014), no. 3, 905–916.

[19] D.Motreanu, A.Moussaoui, *Applied Math. Letters*, **38**(2014), 33–37.

[20] D.Motreanu, A.Moussaoui, *Complex Variables and Elliptic Equations* , **59**(2014), 285–296. DOI: 10.1080/17476933.2012.744404

[21] A.Moussaoui, B.Khodja, S.Tas, *Nonlinear Anal.*, **71**(2009), 708–716.

[22] R.S.Rodrigues, *Electronic Journal of Differential Equations*, (2013), no. 192, 1–10.

Сингулярные квазилинейные эллиптические системы с (супер-)однородным условием

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Аннотация. В данной работе мы устанавливаем существование (несуществование) и регулярность положительных решений для класса сингулярных квазилинейных эллиптических систем, подчиняющихся (супер-)однородному условию. Подход основан на методах субсуперрешений для систем квазилинейных сингулярных уравнений в сочетании с аргументами возмущения, включающими сингулярные члены.

Ключевые слова: сингулярная система, p-лапласиан, субсуперрешение, регулярность.