Directional Characteristics of Circular Scanning Aperture

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In this paper explicit analytical expressions are derived which describe basic directional characteristics of circular scanning aperture: radiation pattern for different kinds of amplitude distribution, beamwidth, part of radiated power contained in the main lobe, directivity. Consideration is based on the theory of continuous ring radiator within the scalar approximation.

Keywords: scanning, circular aperture, ring antenna

Характеристики направленности круглой сканирующей апертуры

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В данной статье приведены явные аналитические выражения, описывающие основные характеристики направленности круглой сканирующей апертуры: диаграмма направленности для различных видов амплитудного распределения, ширина главного лепестка, часть излучаемой мощности, содержащейся в главном лепестке, коэффициент направленного действия. Рассмотрение основано на теории непрерывного кольцевого излучателя в скалярном приближении.

Ключевые слова: сканирование, круговая апертура, кольцевая антенна.
1. Introduction

Directional properties of the radiating circular aperture are studied in the literature in detail [1, 2]. However, all presented results usually describe a uniform-phase aperture, while characteristics of the scanning aperture are determined numerically. In this paper we present explicit analytical expressions describing the main directional characteristics of circular scanning aperture: radiation pattern for different kinds of amplitude distribution, beamwidth, part of radiated power contained in the main lobe, directivity. Also we analyze the directivity changing while scanning. All consideration is based on the theory of continuous ring radiator within the scalar approximation.

2. Directional characteristics

Radiation pattern

Radiation pattern of circular aperture can be written in the form of Huygens-Green integral [3]:

\[
\begin{align*}
\mathbf{f}(\theta, \varphi) &= \int_0^R \int_0^{2\pi} \mathbf{I}(r, \varphi') e^{ikr\cos(\varphi'-\varphi)} r \sin \theta dr d\varphi',
\end{align*}
\]

where \(\mathbf{I}(r, \varphi')\) – field amplitude and phase distribution over the aperture, \(k = 2\pi/\lambda\), \((\theta, \varphi)\) – angles of spherical coordinate system, \(r, \varphi'\) – integration point coordinates (Fig. 1).

Various kinds of amplitude and phase distribution are considered in [4]. For circular scanning aperture examination we suppose axially symmetric amplitude distribution:

\[
|\mathbf{I}(r, \varphi')| = I(r),
\]

and phase distribution which establishes maximum radiation in direction \((\theta_0, \varphi_0)\), i.e.

\[
\Psi(r, \theta_0, \varphi_0, \varphi') = k r \sin \theta_0 \cos(\varphi_0 - \varphi').
\]

Then

\[
\begin{align*}
\mathbf{f}(\theta, \varphi) &= \int_0^R \int_0^{2\pi} I(r) r e^{ikr\cos(\varphi'-\varphi)} \sin \theta dr d\varphi'.
\end{align*}
\]

Internal integral here represents radiation pattern of a continuous ring radiator, and it can be written as [5]:

\[
\begin{align*}
\mathbf{F}(r, \theta, \theta_0, \varphi_0) &= J_0 \left(k r \sqrt{\sin^2 \theta + \sin^2 \theta_0 - 2 \sin \theta \sin \theta_0 \cos(\varphi - \varphi_0)}\right),
\end{align*}
\]

Fig. 1. Circular aperture
where $J_0(\zeta)$ is the Bessel function of the first kind of order zero. Then radiation pattern of circular aperture takes the form:

$$ f(\theta, \varphi) = \int_0^R \left[ I(r) r J_0(kr \cdot \delta(\theta, \varphi, \theta_0, \varphi_0)) \right] dr $$

where

$$ \delta(\theta, \varphi, \theta_0, \varphi_0) = \sqrt{\sin^2 \theta + \sin^2 \theta_0 - 2 \sin \theta \sin \theta_0 \cos(\varphi - \varphi_0)}. $$

Now we can obtain the expression of radiation pattern for scanning circular aperture with uniform amplitude distribution $I(r) = I_0$:

$$ F(\theta, \varphi, \theta_0, \varphi_0) = \frac{2J_1(kR\delta(\theta, \varphi, \theta_0, \varphi_0))}{kR\delta(\theta, \varphi, \theta_0, \varphi_0)}, $$

(2)

where $J_1(\zeta)$ is the Bessel function of the first kind of first order.

For axially symmetric amplitude distributions which taper towards aperture edge:

$$ I(r) = (1 - \Delta) + \Delta \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^n, $$

(3)

where $1 - \Delta$ – excitation level at the aperture edge, $n = 1, 2, \ldots$, we can obtain analogically

$$ f(\theta, \varphi) = (1 - \Delta) \Lambda_1(u) + \frac{\Delta}{n+1} \Lambda_{n+1}(u), $$

(4)

where $\Lambda_n(u) = J_n(u)n! \left( \frac{u}{2} \right)^n$, $u = kR\delta(\theta, \varphi)$, $J_n$ – the Bessel function of the first kind of $n$-th order.

Expressions (2), (4) represent radiation patterns of circular aperture for arbitrary direction of radiation maximum ($\theta_0, \varphi_0$). Fig. 2 demonstrates cut section of radiation pattern in $\varphi = 0$ plane for aperture with $R = 4\lambda$ and amplitude distribution of (3), where $\Delta = 1$ and $n = 1$.

Using of presented expressions, we can determine other directional properties of circular aperture: beamwidth for arbitrary direction of main lobe; directivity; fraction of radiated power concentrated within the main lobe.

**Beamwidth of scanning circular aperture**

Beamwidth of circular aperture radiation pattern in different cut sections can be derived as [5]:

$$ 2\Delta \theta = \frac{0.365\lambda}{R\sqrt{0.5 \cos \theta_0}}, \quad 0 \leq \theta_0 < \frac{\pi}{2}, \quad \varphi = \varphi_0 = \text{const}. $$

(5)

Here $R$ – aperture radius. Fig. 3 represents change of beam width depending on the direction of the radiation maximum $\theta_0$ ($\varphi_0 = 0$).

**Fraction of power, concentrated in main lobe**

Fraction of radiated power, concentrated within main lobe of circular aperture radiation pattern can be written as
In [3] it is stated that factor \((1 + \cos \theta) / 2\) is excluded from (1). This factor for small \(\theta\) can be approximated as \(\sqrt{\cos \theta}\). It allows us to make a suggestion that circular aperture with uniform amplitude distribution consists of “elementary” radiators with radiation patterns in the form of \(\sqrt{\cos \theta}\). In this way we can obtain:

\[
P_0 \approx \frac{8\pi}{k^2 R^2} \int_0^\Delta \int_0^\Delta F^2(\theta, \phi) \sin \theta \sin \phi d\phi.
\]

By changing of variable \(t = kR\sin \theta\):
\[ P_0 \simeq \frac{8\pi}{k^2 R^2} \int_0^\infty \frac{J_1^2(t)}{t} \, dt. \]  

For power allocated in the whole radiation pattern, supposing radiation in upper semi-sphere only \( \Delta \theta = \pi/2 \),

\[ P_\Sigma \simeq \frac{8\pi}{k^2 R^2} \int_0^\infty \frac{J_1^2(t)}{t} \, dt. \]

These expressions can be transformed in the following manner. Since

\[ \int_0^x x J_0^2(x) \, dx = \frac{x^2}{2} \left[ J_0^2(x) + J_1^2(x) \right] \]

and

\[ \int_0^x x J_n^2(x) \, dx = \frac{1}{2n} \left[ 1 + J_0^2(x) + J_n^2(x) - 2 \sum_{k=1}^{n} J_k^2(x) \right], \]

([4], p. 38), we can get

\[ \int_0^x \frac{1}{x} J_1^2(x) \, dx = \frac{1}{2} \left[ 1 + J_0^2(x) - J_1^2(x) \right]. \]

By integrating of (6) and using the results from [5], we can write:

\[ P_0 \simeq \frac{4\pi}{(kR)^2} \left[ 1 + J_0^2(kR \sin \Delta \theta) - J_1^2(kR \sin \Delta \theta) \right], \]  

\[ (7) \]

\[ P_\Sigma \simeq \frac{4\pi}{(kR)^2} \left[ 1 + J_0^2(kR) - J_1^2(kR) \right]. \]  

\[ (8) \]

Fraction of power, concentrated within main lobe, will be equal to:

\[ \frac{P_0}{P_\Sigma} = \frac{1 + J_0^2(kR \sin \Delta \theta) - J_1^2(kR \sin \Delta \theta)}{1 + J_0^2(kR) - J_1^2(kR)}. \]  

\[ (9) \]

Half beamwidth can be found from (5):

\[ \Delta \theta = \frac{0.365 \lambda \sqrt{2}}{2R}. \]

For large apertures \( R \gg \lambda \) half beamwidth has small values \( \Delta \ll 1 \). Therefore in (9) we can substitute sinus by its argument. Considering that \( \lim_{R \to \infty} J_0(t) \to 0 \):

\[ \frac{P_0}{P_\Sigma} = 1 + J_0^2\left(0.365 \pi \sqrt{2}\right) - J_1^2\left(0.365 \pi \sqrt{2}\right), \]

or

\[ \frac{P_0}{P_\Sigma} = 0.869. \]
**Directivity**

Taking into account definition of $P_\Sigma$ in (8) we can derive directivity of circular aperture with uniform phase distribution:

$$D = \frac{4\pi}{P_\Sigma} = \frac{(kR)^2}{1 + J_0^2(kR) - J_1^2(kR)}, \text{ while } \frac{R}{\lambda} >> 1.$$  

For apertures with large electrical radii $\lim_{R\to\infty} J_{0,1}(kR) \to 0$, then obtain well known formula [1]

$$D = 4\pi \frac{S_d}{\lambda^2},$$

where $S_d$ – aperture physical area.

Let’s examine changing of circular aperture directivity while scanning. Without loss of generality in (2) we always can assign $\varphi_0 = 0$, then radiation pattern takes a form:

$$F(\theta, \varphi, \theta_0) = \frac{2 J_1[kR\delta(\theta, \varphi, \theta_0)]}{kR\delta(\theta, \varphi, \theta_0)}.$$

Directivity can be found numerically for every certain $\theta_0$. Results are presented in Fig. 4.

### 3. Conclusion

In this paper we derived explicit analytical expressions, which can be used for defining of directional properties of circular scanning aperture without numerical computing.

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![Graph](image_url)  

**Fig. 4. Directivity of scanning circular aperture**
References


