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On a New Embedding Theorem in Analytic Bergman Type Spaces in Bounded Strictly Pseudoconvex Domains of n-dimensional Complex Space

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Some new sharp assertions concerning Carleson type embeddings in analytic spaces on bounded strictly pseudoconvex domains with smooth boundary will be provided. We extend previously known in the unit ball assertions.

Keywords: pseudoconvex domains, analytic functions, mixed-norm spaces.

Introduction

Let $D = \{z : \rho(z) < 0\}$ be as usual a bounded strictly pseudoconvex domain in C^n with C^∞ boundary. We assume that the strictly plurisubharmonic function ρ is of class C^∞ in a neighbourhood of $\bar{D}$, that, $-1 \leqslant \rho(z) < 0$, $z \in D$, $|\partial\rho| \geqslant c_0 > 0$ for $|\rho| \leqslant r_0$.

Denote by $\mathcal{O}(D)$ or $(H(D))$ the space of all analytic functions on D. Following [1] let also $A^{p,q}_{\delta,k} = \{f \in H(D) : \|f\|_{p,q,\delta,k} < \infty\}$, where

$$\|f\|_{p,q,\delta,k} = \left(\sum_{|\alpha| \leqslant k} \int_0^{r_0} \left(\int_{\partial D_r} |D^\alpha f|^p d\sigma_r \right)^{q/p} r^{\delta \frac{q}{p} - 1} dr \right)^{1/q}$$

be the mixed norm space in D. Here $D_r = \{z \in C^n : \rho(z) < (-r)\}$, ∂D_r it is boundary $d\sigma_r$ the normalized surface measure on ∂D_r and by dr we denote the normalized volume element on interval from 0 to r, $0 < p < \infty$, $0 < q \leqslant \infty$, $\delta > 0$, $k = 0, 1, 2...$ and

$$\|f\|_{p,\infty,\delta,k} = \sup \left\{ \left(\sum_{|\alpha| \leqslant k} (r^\delta) \int_{\partial D_r} |D^\alpha f|^p d\sigma_r \right)^{1/p} : 0 < r < r_0 \right\}$$

(for $p, q < 1$ it is quazinorm), where D^α is a derivative of f (see [1]).

For $p = q$ we have

$$\|f\|_{p,\delta,k} = \left(\sum_{|\alpha| \leqslant k} \int_D |D^\alpha f|^p (-\rho)^{\delta - 1} d\zeta \right)^{1/p} ; \ \delta > 0, \ k \geqslant 0.$$

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Some interesting properties of these classes can be seen in assertions which we listed below, and which will be partially used by us in the proof of the main theorem of this note. Note for $p = q$ and $k = 0$ we get as usual the classical analytic Bergman spaces $A^p_\delta(D)$ in bounded strictly pseudoconvex domains with smooth boundary (see for another direct definition below and, for example [1] and various references there). Note the study of properties of analytic function spaces on bounded strictly pseudoconvex domains with smooth boundary was an active research area for last several decades (see, for example, [1, 2] and [3, 4] and also various references there). The main problem we consider in this paper is classical (see, for example, [2, 4] and various references there). We wish to find sharp conditions on positive Borel measure μ in D so that

$$\int_D |f(z)|^p d\mu(z) \leqslant c\|f\|_Y, Y \subset \mathcal{O}(D) \text{ (or } H(D)),$$

where Y is a certain fixed quazinormed subspace of $\mathcal{O}(D)$ (or $H(D)$), and $0 < p < \infty$, and c is a positive constant. For classical case when $Y = A^p_\alpha(D)$ in various domains(unit disk, polydisk, unit ball and bounded pseudoconvex domains with smooth boundary), where $0 < p < \infty,\ \alpha > 0$, these type of problems were considered and solved before by various authors (see, for example, [1, 3–7], and various references there). Note nevertheless in more complicated domains solutions of such type of problems are still unknown. The plan of this note is the following. We provide in next section some basic facts from theory of analytic spaces in bounded pseudoconvex domains with smooth boundary. The last and the main section is devoted to our main new sharp embedding theorem for mentioned mixed norm analytic function spaces in bounded strictly pseudoconvex domains with smooth boundary. Note in our proof we used some machinery which was recently developed in [1–3]. Note also as motivation for this note we considered some results from our previous paper [6] where all results of this note were proved in less general situation, namely in case of unit ball in C^n (a simplest model of D domains we consider in this paper). The main intention of this note is to generalize them to arbitrary bounded strictly pseudoconvex domains with smooth boundary in C^n. We use at each point the same line of arguments in proof as in [6], but in this more general setting.

Finally, as usual, we denote by C or c with indexes various positive constants which appear in various estimates below.

1. Preliminary propositions

In this introduction-type section we provide some known lemmas and assertions, some of them will be needed by us later. All these estimates and assertions were previously known and appropriate citation will be found below. The main facts are taken from [1] and [3]. We shall use in particular the following notations (see [2, 3]):

- $\delta\colon D \to R^+$ will denote the Euclidean distance from the boundary, that is $\delta(z) = d(z, \partial D)$, $z \in D$;
- given two non-negative functions $f,\ g\colon D \to R^+$ we shall write $f \preceq g$ to say that there is $C > 0$ such that $f(z) \leqslant Cg(z)$ for all $z \in D$. The constant C is independent of $z \in D$, but it might depend on other parameters (r, θ, etc.);
- given two strictly positive functions $f,\ g\colon D \to R^+$ we shall write $f \approx g$ if $f \preceq g$ and $g \preceq f$, that is if there is $C > 0$ such that $C^{-1}g(z) \leqslant f(z) \leqslant Cg(z)$ for all $z \in D$;
- ν will be the Lebesgue measure (sometimes also v);

- given $1 \leqslant p \leqslant +\infty$, the *Bergman space* $A^p(D)$ is the Banach space $L^p(D) \cap \mathcal{O}(D)$, endowed with the L^p-norm;
- more generally, given $\beta \in R$ we introduce the *weighted Bergman space*
$$A^p(D, \beta) = L^p(\delta^{\beta-1}\nu) \cap \mathcal{O}(D);$$
endowed with the norm $\|f\|_{p,\beta} = \left[\int_D |f(\zeta)|^p \delta(\zeta)^{\beta-1}\, d\nu(\zeta)\right]^{1/p}$ if $1 \leqslant p < \infty$, and with the norm $\|f\|_{\infty,\beta} = \|f\delta^\beta\|_\infty$ if $p = \infty$;
- $K\colon D \times D \to C$ will be the Bergman kernel of D; K_α is a weighted Bergman kernel ([8]). The Bergman Kernel will be also denoted by us K_0, so that $K = K_0$;
- for each $z_0 \in D$ we shall denote by $k_{z_0}\colon D \to C$ the *normalized Bergman kernel* defined by $k_{z_0}(z) = \dfrac{K(z, z_0)}{\sqrt{K(z_0, z_0)}} = \dfrac{K(z, z_0)}{\|K(\cdot, z_0)\|_2}$;
- given $r \in (0, 1)$ and $z_0 \in D$, we shall denote by $B_D(z_0, r)$ the Kobayashi ball of center z_0 and radius $\dfrac{1}{2}\log\dfrac{1+r}{1-r}$.

Let us now recall some vital results for our D domains. The first two give information about the shape of Kobayashi balls:

Lemma 1 ([3], Lemma 2.1)**.** *Let $D \subset\subset C^n$ be a bounded strictly pseudoconvex domain, and $r \in (0, 1)$. Then*
$$\nu\big(B_D(\cdot, r)\big) \approx \delta^{n+1} ,$$
here the constant depends on r.

Lemma 2 ([3], Lemma 2.2)**.** *Let $D \subset\subset C^n$ be a bounded strictly pseudoconvex domain. Then there is $C > 0$ such that $\dfrac{C}{1-r}\delta(z_0) \geqslant \delta(z) \geqslant \dfrac{1-r}{C}\delta(z_0)$ for all $r \in (0, 1)$, $z_0 \in D$ and $z \in B_D(z_0, r)$.*

We shall also need the existence of suitable coverings by Kobayashi balls:

Definition 1. *Let $D \subset\subset C^n$ be a bounded domain, and $r > 0$. An* r-lattice *in D is a sequence $\{a_k\} \subset D$ such that $D = \bigcup_k B_D(a_k, r)$ and there exists $m > 0$ such that any point in D belongs to at most m balls of the form $B_D(a_k, R)$, where $R = \dfrac{1}{2}(1 + r)$.*

The existence of r-lattices in bounded strictly pseudoconvex domains is ensured by the following:

Lemma 3 ([3], Lemma 2.5)**.** *Let $D \subset\subset C^n$ be a bounded strictly pseudoconvex domain. Then for every $r \in (0, 1)$ there exists an r-lattice in D, that is there exist $m \in \mathbf{N}$ and a sequence $\{a_k\} \subset D$ of points such that $D = \bigcup_{k=0}^{\infty} B_D(a_k, r)$ and no point of D belongs to more than m of the balls $B_D(a_k, R)$, where $R = \dfrac{1}{2}(1 + r)$.*

We mention an submean estimate for non-negative plurisubharmonic functions on Kobayashi balls:

Lemma 4 ([3], Corollary 2.8)**.** *Let $D \subset\subset C^n$ be a bounded strictly pseudoconvex domain. Given $r \in (0, 1)$, set $R = \dfrac{1}{2}(1 + r) \in (0, 1)$. Then there exists a $K_r > 0$ depending on r such that*
$$\forall z_0 \in D\ \forall z \in B_D(z_0, r) \qquad \chi(z) \leqslant \frac{K_r}{\nu\left(B_D(z_0, r)\right)} \int_{B_D(z_0,R)} \chi\, d\nu$$
for every non-negative plurisubharmonic function $\chi\colon D \to \mathbf{R}^+$.

Lemma 5 (see [1])**.** *If* $0 < p_0 \leqslant p_1 < \infty$, $0 < q \leqslant \infty$, $0 < q_0 < q_1 \leqslant \infty$, $0 < \delta_0 < \infty$, k, $m = 0, 1, \ldots$ *then*

- $A^{p_0,q_0}_{\delta_0,k} \subset A^{p_0,q_1}_{\delta_0,k}$;
- $A^{p_0,q_1}_{\delta_0,k} \subset A^{p_1,q_1}_{\delta_0',k}$ *if* $\dfrac{n+\delta_0'}{p_1} = \dfrac{n+\delta_0}{p_0}$;
- $A^{p_0,q_1}_{\delta_0,k} = A^{p_0,q_1}_{\delta_0+mp_0,k+m}$.

We will need the following proposition (see [8]) for K_α weighted Bergman kernel. Note it is a reproducing kernel of A^2_α Hilbert space and it is depend on the defining function $\rho($ or $r)$. For $\alpha = 0$ it is the ordinary Bergman kernel. Note also the complete analogue of the proposition below in the unit disk, unit polydisk and unit ball is a well known fact [6, 7].

Proposition 1 (see [6, 7])**.** *Let* $\alpha > -1$, $s > 0$, $0 < t < \dfrac{1}{2}\sup_{\Omega} r$. *Then*

$$\int_{\{x : r(x)=t\}} |K_\alpha(x,y)|^s d\sigma(x) \asymp (r(y)+t)^{n-q};\ n-q<0,\ q=(n+\alpha+1)s.$$

Here $d\sigma$ denotes the $(2n-1)$-dimensional Hausdorff measure in C^n.

2. On embedding theorems in bounded strictly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$.

This main section is devoted to a new sharp Carleson-type embedding theorem for analytic mixed norm spaces in bounded strictly pseudoconvex domains with smooth boundary. Note in the unit ball this result can be seen in [6]. Note also for classical Bergman spaces in bounded strictly pseudoconvex domains with smooth boundary this results is also known [1]. This proof below appears as combination of our previous arguments in case of unit ball [6], new embedding theorems for analytic mixed norm spaces taken from [1] and properties of so-called r-lattices from [2, 3] which we provided above. We need first a definition. Let $D \subset\subset C^n$ be a bounded strictly pseudoconvex domain with smooth boundary, $\theta \in R$, then θ Carleson measure is a finite positive Borel measure on D such that $\mu(B_D(\cdot, r)) \leqslant c\nu(B_D(\cdot, r))^\theta$ for all $r \in (0,1)$ where the constant c might depend on r. We list two sharp known embeddings for these domains. We denote below by A^p a Bergman space without weight.

Theorem 2.1 ([4])**.** *Let $D \subset C^n$ be a bounded strictly pseudoconvex domain with smooth boundary. Let μ be a positive Borel measure on D, $f \in H(D)$. Let $1 \leqslant p < \infty$. We have* $\int_D |f|^p d\mu \leqslant c\|f\|^p_{A^p}$ *iff* $\mu(B_D(a_k, r)) \leqslant \nu(B_D(a_k, r))$, $r \in (0,1)$ *or iff* $\mu(B_D(\cdot, r)) \leqslant c\nu(B_D(\cdot, r))$ *or iff* $\mu(B_D(a_k, r)) \leqslant c(\delta^{n+1}(a_k))$ *for certain sequence* $\{a_k\}$ *which is r-lattice for D.*

This vital theorem was extended recently.

Theorem 2.2 ([2, 3])**.** *Let $D \subset C^n$ be a bounded strictly pseudoconvex domain with smooth boundary in C^n. Let μ be a positive Borel measure on D, $f \in H(D)$. Let* $1 - \dfrac{1}{n+1} < \theta < 2$, $1 \leqslant p < \infty$. *Then the following assertions are equivalent:*

1) $\int_D |f(z)|^p d\mu(z) \leqslant \int_D |f(z)|^p \delta^{(n+1)(\theta-1)}(z) dv(z)$;

2) μ is θ-Carleson measure;

3) for every $r \in (0,1)$ and every r-lattice $\{a_k\}$ in D one has $\mu(B_D(a_k,r)) \leqslant [\nu(B_D(a_k,r))]^{\theta}$, $r \in (0,1)$;

4) there exists $r_0 \in (0,1)$ so that for every r_0-lattice $\{a_k\}$ in D, $\mu(B_D(a_k,r_0)) \leqslant c[\nu(B_D(a_k,r_0))]^{\theta}$.

The following theorem is a new sharp embedding theorem for mixed norm $A^{p,q}_{\tau,0}$ spaces. Note using a direct relation between $B_D(a_k,r)$ Kobayashi balls and $\delta(a_k)$ function (see [3]) we can easily reformulate in our theorem below conditions on finite Borel μ measure in various manner as we see in Theorems 2.1 and 2.2 above.

Theorem 2.3. *Let D be a bounded strictly pseudoconvex domains with smooth boundary. Let μ be positive finite Borel measure on D, $f \in H(D)$. Let $\{a_k\}$ be r-lattice. Assume $q < p$ or $q = p$, $r \leqslant p$. Then we have*

$$\left(\int_D |f(z)|^p d\mu(z)\right)^{1/p} \leqslant c_0 \|f\|_{A^{q,\tau}_{v,0}} \tag{1}$$

if and only if $\mu(B_D(a,r)) \leqslant c_1 \delta^{\frac{np}{q}+\frac{vp}{q}}(a)$, $a \in D$ or if and only if $\mu(B_D(a_k,r)) \leqslant c_2 \delta^{\frac{np}{q}+\frac{vp}{q}}(a_k)$ for $k = 1,2,\dots$ and for some constants c_1, c_2 for all $v < \left(\frac{2q}{p}-1\right)n + 2\frac{q}{p}$ if $q < p$ and for all $v < (n+2)\frac{p}{r}$, if $q = p$, $r \leqslant p$.

Proof. We follow the proof taken from [6], and the proof of Theorem 3.3 from [3], in [6] the complete analogue of our theorem in the unit ball case can be seen. Let first $q < p$. Let the embedding in formulation of our theorem holds. We need an appropriate test function. Put as test function the following analytic function. $\tilde{f}(z) = (\delta(a)^{n+\alpha+1})[K(z,a)]^{\frac{l}{(n+1)p}}$; $z, a \in D$, $l = 2(n+1+\alpha)$, where α is a large enough positive number (we can choose α even so that $\dfrac{l}{(n+1)p}$ is a large integer). Following the proof of parallel implication in Theorem 3.3 from [3] and using the estimate from below of K(or K_0) Bergman kernel on Kobayashi balls $B_D(a_k,r)$ obtained recently in [2,3] we have by properties of r-lattice (see also Lemmas 1, 2, 4)

$$M = \left(\int_D |\tilde{f}(z)|^p d\mu(z)\right)^{1/p} \geqslant \left(\int_{B_D(a,r)} \delta(a)^{n+1+\alpha} |K(a,z)|^{\frac{r_0}{n+1}} |(d\mu(z)\right)^{1/p} \geqslant$$

$$\geqslant (\mu(B_D(a,r)))^{1/p} (\delta(a))^{-\frac{1+\alpha+n}{p}}; \qquad r_0 = 2(n+1+\alpha); \qquad a \in D,\ r \in (0,1)$$

The fact that $\|\tilde{f}\|_{A^{q,r}_{v,0}} \leqslant c\delta^k(a)$, $a \in D$, $k = -\dfrac{n+1+\alpha}{p} + \dfrac{n}{q} + \dfrac{v}{q}$ follows from Proposition 1 above and the fact that for $t > s > 0$, $C > 0$

$$\int_0^C (r+x)^{-t-1} x^s dx \leqslant c r^{s-t},\ r > 0.$$

To get the reverse we follow arguments of proof of related assertion from [6]. We use first an embedding from Lemma 5 (second embedding there) and then use Theorem 2.2. (see also [6]), making easy calculation with indexes. Let us turn to second $p = q$ case in our theorem. Note now for second $p = q$ case we can use the same test function and repeat all arguments for $q < p$ case to get the proof of one direction in $p = q$ case. To show the reverse in the same $p = q$ case we just use another embedding from Lemma 5 (the first embedding of Lemma 5) and then again Theorem 2.2 and follow arguments of previous $q < p$ case (see also [6] for parallel arguments in the unit ball case). □

Remark 1. We remark finally in [9] these properties of recently invented in [2, 3] r-lattices were also used by authors for solutions of some other problems in this context for analytic function spaces in bounded strictly pseudoconvex domains with smooth boundary.

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Новая теорема вложения в аналитических пространствах Бергмана в ограниченных псевдовыпуклых областях n-мерного комплексного пространства

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Получены новые точные утверждения относительно вложений типа Карлесона в пространствах аналитических функций в ограниченных сторогопсевдовыпуклых областях с гладкой границей. Данные теоремы обобщают ранее полученные в единичном шаре точные результаты.

Ключевые слова: строгопсевдовыпуклые области, аналитические функции, пространства со смешанной нормой.