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Risk Aversion for Defining Elliptic Acceptance Sets in the Model of Generalized Coherent Risk Measures

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Within the framework of generalized coherent risk measures the properties of acceptance sets are examined. The class of elliptic cones is developed for representing individual preferences. The article presents the method of defining an appropriate elliptic cone using values of risk aversion (for p-norms in the space of risks).

Keywords: generalized coherent risk measures, risk aversion, acceptance set, preference relation, elliptic cone.

Introduction

Generalized coherent risk measures represent the generalization of the classical coherent risk measures, which best known examples are Expected Shortfall [1] and Distorted probabily [2].

The paper [3] presents a procedure of calculating risk measure values using a given acceptance set and a norm in the space of risks. But the question of defining an acceptance set according to individual preferences is still open.

This paper considers one of the possible ways of defining such set, which is based on some assumptions of preference properties and on using a functional of risk aversion.

1. Preference relation and its properties

Consider a probability space $(\Omega, \mathcal{A}, P)$, where Ω is a reference set, $\mathcal{A}$ is a σ-algebra specified on Ω, P is a probability measure, specified on the sets of $\mathcal{A}$.

A **Risk** X on $(\Omega, \mathcal{A})$ is any measurable mapping from Ω to $\mathbf{R}$ (a random variable).

The values of risks can be interpreted as profits or losses earned by a certain person.

The set of all risks on $(\Omega, \mathcal{A})$ we denote by $\mathcal{X}$.

Partial order relation $\leqslant$ on a certain set M is a reflexive transitive antisymmetric binary relation on this set. If an order relation is moreover a complete relation the order is called linear.

There are several ways of defining orders on the set $\mathcal{X}$.

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1.1. Stochastic dominance

Denote by $\mathcal{F}$ the set of all distribution functions, by F_X the distribution function of a random variable X:

$$F_X(x) = P(X \leqslant x).$$

Let $\mathcal{F}_k$ be a set of all distribution functions with finite values of k-th moments:

$$\mathcal{F}_k = \{F \in \mathcal{F} : \ |\mu_k^F| < \infty\}, \quad \mu_k^F = \int_{-\infty}^{\infty} t^k dF(t).$$

For a given $F \in \mathcal{F}$ specify a sequence of functions $F^{(k)}, \ k = 1, 2, \ldots$:

$$F^{(1)}(x) = F(x), \quad F^{(k+1)}(x) = \int_{-\infty}^{x} F^{(k)}(t)dt, \quad -\infty < x < \infty.$$

Suppose $F, Q \in \mathcal{F}_k$. We say that Q has k-**order stochastic dominance** over F ($F \leqslant_k Q$), if

$$F^{(k)}(x) \geqslant Q^{(k)}(x), \quad -\infty < x < \infty.$$

We can also introduce strict stochastic dominance. Suppose $F, Q \in \mathcal{F}_k$. We say that Q **strictly dominates** F *with the order* k ($F <_k Q$), if

$$F \leqslant_k Q \quad \text{and} \quad \exists \, x \in \mathbf{R} : \ F^{(k)}(x) > Q^{(k)}(x).$$

By means of first-order stochastic dominance we can determine an order relation $\leqslant_1$ ($<_1$) on $\mathcal{X}$. Risk Y **(strictly) stochastically dominates** over risk X: $X \leqslant_1 Y$ ($X <_1 Y$) if

$$F_X \leqslant_1 F_Y \quad (F_X <_1 F_Y).$$

1.2. Coordinatewise order on the set of risks

Suppose $|\Omega| = n$. Then we can submit a σ-algebra $\mathcal{A}$ in the form of $\mathcal{A} = 2^{\Omega}$. Probability measures P on a measurable space can be represented as elements of the standard simplex in $\mathbf{R}^n$:

$$S^n = \{P = (p^1, \ldots, p^n) \in \mathbf{R}^n :$$
$$p^1 \geqslant 0, \ldots, p^n \geqslant 0, \ p^1 + \cdots + p^n = 1\}.$$

The set of all risks $\mathcal{X}$ is isomorphic to $\mathbf{R}^n$. Renumbering elements of Ω in some arbitrary way: $\Omega = \{\omega^1, \ldots, \omega^n\}$, we denote $P(\omega^i) = p^i$, $X(\omega^i) = X^i$, $i = 1, \ldots, n$. We identify random variables $X \in \mathcal{X}$ with vectors $X = (X^1, \ldots, X^n) \in \mathbf{R}^n$.

We assume that $X \leqslant Y$ if $X^i \leqslant Y^i$ for all $i = 1, \ldots, n$. This order is also partial.

If a probability space $(\Omega, \mathcal{A}, P)$ with finite Ω is fixed, then the orders $\leqslant$ and $\leqslant_1$ on $\mathcal{X}$ are consistent — from $X \leqslant Y$ follows that $X \leqslant_1 Y$.

1.3. Risk measures consistent with preferences

A **_Preference relation_** $\preceq$ on a certain set M is a complete transitive binary relation on M. An equivalence relation is defined as follows:

$$X \sim Y, \text{ if } X \preceq Y \text{ и } Y \preceq X. \tag{1}$$

Suppose that a preference relation $\preceq$ on $\mathcal{X}$ reflects an individual attitude to risk of a certain investor.

Usually market insiders strike for higher returns, thus we claim that preference relation should be consistent with the order $\leqslant$ on $\mathcal{X}$: for any $X, Y \in \mathcal{X} : X \leqslant Y$ the inequality $X \preceq Y$ should be fulfilled.

An arbitrary functional $\rho : \mathcal{X} \to \mathbf{R}$ is called a **risk measure**.

We say that a preference relation is represented on $\mathcal{X}$ by a measure $\rho : \mathcal{X} \to \mathbf{R}$ if one of the following conditions holds:

$$\rho(X) \leqslant \rho(Y), \text{ if } X \preceq Y, \quad X, Y \in \mathcal{X}; \tag{2}$$

$$\rho(X) \leqslant \rho(Y), \text{ if } Y \preceq X, \quad X, Y \in \mathcal{X}. \tag{3}$$

Hereinafter we deal with risk measures that represent preference relations like in (2). From (1) and (2) it follows that for $X \sim Y$ $\rho(X) = \rho(Y)$.

2.   Risk aversion

For most preferences a property that is called risk aversion is typical. We can informally define it as a disposition of a person to accept a bargain with an uncertain payoff and the mean a rather than another bargain with a certain value a.

Preference relation $\preceq$ possesses the **property of risk aversion** if for any arbitrary nondegenerate risk $\Delta : E\Delta = 0$ and an arbitrary $a \in \mathbf{R}$ it holds:

$$a + \Delta \prec a. \tag{4}$$

In terms of a risk measure ρ that represents the preference $\preceq$ on $\mathcal{X}$ we can note: $\rho(a + \Delta) < \rho(a)$.

For any $a \in \mathbf{R}$ we denote W_a the distribution function of a degenerate distribution $P(\xi = a) = 1$. If $\preceq$ is consistent with stochastic dominance $\leqslant_1$, then $W_a \preceq W_b$ when $a < b$. We assume that the strict preference relation $W_a \prec W_b$ also holds.

Preference relation $\preceq$ on $\mathcal{F}$ is called **regular** if it is consistent with the first stochastic dominance $\leqslant_1$, for all $a, b \in \mathbf{R} : a < b$ it holds that $W_a \prec W_b$ and in every equivalence class $K \in \mathcal{F}/\sim$ there is only one degenerate distribution.

If regular preference relation on $\mathcal{X}$ is defined by the risk measure ρ then

$$\rho(a + \Delta) = \rho(a - c), \quad c > 0.$$

The value c (which usually depends on a and Δ) can be used as a quantitative estimator of risk aversion which was presented in [4].

3.   Generalized coherent risk measures

Suppose $|\Omega| = n$. Then risk $X = (X_1, \ldots, X_n)$ is a vector in $\mathbf{R}^n$.

A risk is called *acceptable* for an investor if he agrees to work with it without investing any capital. The set of all acceptable to an investor risks we denote by A ($A \subset \mathcal{X}$).

An acceptance set A satisfies the following axioms:
A1: $C_+ \subset A$, $C_+ = \{X \in \mathcal{X} : X \geqslant 0\}$

A2: $A \bigcap C_- = \emptyset$, $C_- = \{X \in \mathcal{X} : X < 0\}$

A3: A is a convex cone (if $X \in A$, $Y \in A$, then $\alpha_1 X + \alpha_2 Y \in A$, $\alpha_1, \alpha_2 \geqslant 0$).

A **generalized coherent risk measure** f_A, associated with A is determined by

$$f_A(X) = f_{A, \|\cdot\|}(X) = \delta_A(X) \inf_{Y \in \partial A} \|X - Y\|,$$

$$\delta_A(X) = \begin{cases} 1, & X \in A, \\ -1, & X \in A^c \end{cases},$$ (5)

where ∂A is a boundary of A.

The functional $f_A(X)$ exhibits the following properties:

M) monotonicity:

$$f_A(X) \leqslant f_A(Y), \quad \forall X, Y \in \mathcal{X}, X \leqslant Y;$$

PH) positive homogeneity:

$$f_A(\lambda X) = \lambda f_A(X), \; \forall \lambda \geqslant 0, X \in \mathcal{X};$$

S) superadditivity:

$$f_A(X + Y) \geqslant f_A(X) + f_A(Y), \quad \forall X, Y \in \mathcal{X};$$

Sh) the shortest path property:

$\forall X \in \mathcal{X} \; \exists \; X'(X) \in \partial A$ that $\|X - X'(X)\| = \inf\limits_{Y \in \partial A} \|X - Y\|$ and

$$f_A(X + \lambda u(X)) = f_A(X) + \lambda, \quad -\infty < \lambda \leqslant \lambda_A(X), \quad \text{where } \lambda_A(X) > 0,$$

A vector of the shortest path is determined as the follows:

– for the case of a strictly convex norm: $\|\cdot\|$ $u(X) = \delta_A(X) \dfrac{X - X'_A(X)}{\|X - X'_A(X)\|}$;

– for the norm $\|\cdot\|_\infty$ $u(X) = I = (1, 1, \ldots, 1)$;

– for the norm $\|\cdot\|_1$ $u(X) = e_i$,

where e_i — any of the vectors of the standard basis of the space $\mathbf{R}^n$, which complies with

$$ke_i(X) = \inf_{k_j e_j} \{\|X - (X + k_j e_j)\| : \quad (X + k_j e_j) \in \partial A\}.$$

For the given generalized coherent risk measure $f(X)$ we can define an associated acceptance set as follows:

$$A_f = \{X \in \mathcal{X} : \; f(X) \geqslant 0\}.$$ (6)

A border of an acceptance set associated with f can be determined as

$$\partial A_f = \{X \in \mathcal{X} \quad f(X) = 0\}.$$

Coherent risk measure is a particular case of generalized coherent risk measures corresponding to the norm $\|\cdot\| = \|\cdot\|_\infty$.

3.1. Risk aversion for generalized coherent risk measures

For generalized coherent risk measures

$$c_{a,\Delta} = \begin{cases} a - \frac{f(aI+\Delta)}{f(I)}, & aI + \Delta \in A, \\ a + \frac{f(aI-\Delta)}{f(-I)}, & aI + \Delta \notin A \end{cases}$$

In particular,

$$c_{0,\Delta} = \frac{f_A(\Delta)}{f_A(-I)} \tag{7}$$

Since $f(aX) = af(X), a \geqslant 0$, then

$$c_{0,a\Delta} = a \cdot c_{0,\Delta},$$

hence we can limit ourselves to studying the risk aversion functional at $\Delta : \|\Delta\| = 1$ to examine the functional.

In the special case when generalized coherent risk measures degenerate into classical coherent risk measures by the translation invariance property [5] we get $\rho(a + \Delta) = \rho(\Delta) + a$, and $\rho(a - c) = a - c$.

This means that for coherent risk measures the value of risk aversion doesn't depend on a:

$$c_\Delta = -\rho(\Delta).$$

4. Inverse problem of risk theory for generalized coherent risk measures

The inverse problem can be described as a risk measure development in accord with individual preferences using some known preference characteristics. This problem can be reduced to selection of the most appropriate representative of a considered class.

In this paper the regarded problem is being solved for the class of generalized coherent risk measures. From the Definition 5 it follows that it suffices to define the appropriate acceptance set and to choose the appropriate norm in $\mathcal{X}$.

The axiomatic characterization of an acceptance set describes its most general properties and defines an extensive class of all possible acceptance sets. To select an explicit sample we should narrow down the examined class as much as possible, relying on the investor's preferences properties.

4.1. Properties of acceptance sets for some preferences

The papers [6] and [7] present the following properties of acceptance sets for preferences consistent with stochastic dominance:

I. Symmetry for the uniform distribution. If $P = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ and vector $X = (X^1, X^2, \ldots, X^n) \in A$, then any vector Y with the components found by permutation of the vector X components also lies in A (the cone is symmetric with respect to the axes of coordinates).

Remark 4.1. *As it follows from the definition, risk is a mapping from $(\Omega, \mathcal{F})$ to $\mathbf{R}$ and in general terms doesn't depend on probability P defined on the sets of $\mathcal{F}$. But there are some characteristics of risks which depend on the values of probability measure — for example distribution functions. Thus if the preferences are consistent with stochastic dominance then for different probability measures risks may be differently ordered by preference, that may influence on the acceptance set configuration.*

II. Dependence of a probability measure. We denote by $\mathcal{X}_P$ and $\mathcal{X}_Q$ the same set of all risks $\mathcal{X}$, but considered with different probability measures P and Q, and the acceptance sets we obtain in these cases by A_P and A_Q. By X_P we denote vector $(X_1, \ldots, X_n) \in \mathcal{X}_P$, by X_Q vector $(X_1, \ldots, X_n) \in \mathcal{X}_Q$.

Then if $P \neq Q$ and $\exists\, X_P$ such that $X_P <_1 X_Q$ (or $X_Q <_1 X_P$), we have $A_P \neq A_Q$.

III. Reduction of a cone dimensionality. For an acceptance cone A_P, corresponding with a probability measure $P = (p_1, \ldots, p_{k-1}, 0, p_{k+1}, \ldots, p_n)$, is valid that:

$$X = (X^1, \ldots, X^{k-1}, X^k, X^{k+1}, \ldots, X^n) \in A_P \Rightarrow$$
$$\Rightarrow Y = (X^1, \ldots, X^{k-1}, y, X^{k+1}, \ldots, X^n) \in A_P \ \forall y \in \mathbf{R}.$$

IV. Confluence of a cone to a semispace. An acceptance cone A_P, corresponding with $P = (p_1, \ldots, p_n)$, where $p_k = 1$; $p_i = 0$, $i \neq k$, may be defined by the inequality

$$X^k \geqslant 0.$$

For a preference, possessing the risk aversion property, the following property holds:

V. Positivity of mean values for acceptable risks. For all $X \in A, X \neq 0$ it holds, that $EX > 0$.

5. Elliptic acceptance cone

Consider preferences consistent with stochastic dominance and possessing the property of risk aversion.

It is clear from the properties II, III, IV that an acceptance set is not constant — it changes when the dimension of the risk space and the probability measure change.

There is the following idea: to develop such a model of an acceptance set that will allow only once estimating required parameters for a concrete individuum, automatically reconstruct his acceptance cone when the probability measure and the dimension of the risk space change.

It is quite obvious that the greater is the probability of some result (p_i) the less preferable are the negative values of profit corresponding to such result (X_i).

It can be assumed that the most preferable for an investor risks lie on the half line λP, $\lambda \geqslant 0$.[†]

While in the axiomatics of generalized coherent risk measures we determine acceptance of a risk by its farness from the acceptance cone border, in case of validity of the mentioned assumption we may estimate a risk by its distance to the half line λP. We consider the risk is the "better" the nearer it lies to the risk of the same hyperplane $(P, X) = \lambda$ on the half line λP. For taking into account the influence of p_i on acceptance of X_i we should assign a weight number inversely related to p_i to an permissible deviation of acceptable risks in the line of i-th compoment.

We define an acceptance set as a convex cone with hyperplane sections $(P, X) = a$, $a \geqslant 0$ which appear to be ellipsoids with semiaxis $r(p_i)$, $i = 1, \ldots, n$.

According to the geometric interpretation of the function $r(p)$, $p \in [0, 1]$ we call it an **axial function**. It is the only characteristic of the cone that reflects an individual attitude to risk and defines the dependence of such attitude on probabilities.

[†]In the strict sense, vector $P = (p_1, \ldots, p_n)$ of probabilities is a vector of the space $\mathcal{X}^*$, dual to $\mathcal{X}$, that is why hereinafter we denote by P the vector in $\mathcal{X}$, with the same components as the vector of probability in X^*.

5.1. Elliptic acceptance cone for the norm $\| \cdot \|_2$ in the space of risks

For the case $\| \cdot \| = \| \cdot \|_2$ an elliptic cone can be defined by the inequality

$$\sum_{i=1}^{n} \frac{(X^i - (P,X)np_i)^2}{r^2(p_i)} \leqslant (P,X)^2, \quad X \in \mathbf{R}^n \tag{8}$$

Theorem 5.1. *If the following hypothesis*

$$\begin{aligned} &1. \quad (P,X) \geqslant 0; \\ &2. \quad r(p) \geqslant \sqrt{\frac{n}{p^2} + n^3 p^2}. \end{aligned} \tag{9}$$

holds, an elliptic cone A, defined by (8), is an acceptance cone for some preference.

Proof. To prove that A is an acceptance cone it suffices to show that the Axioms **A1**–**A3** hold.

1. First we prove that A satisfies **A2**: $A \cap C_- = \emptyset$. Consider an arbitrary $X \in C_-$

$$X^i < 0 \quad \forall i = 1, 2, \ldots, n \quad \Rightarrow (P,X) < 0,$$

   therefore, X doesn't satisfy (9).

2. Then we prove **A3**: A is a convex set.

   Let $X \in A$. Then $(P,X) = a > 0$ and

$$\sum_{i=1}^{n} \frac{(X^i - anp_i)^2}{r^2(p_i)} \leqslant a^2 \tag{10}$$

   The set $\{Y : (P,Y) = a\}$ also satisfies (10) and it forms a n-dimensional ellipsoid E_a, which is a convex set.

   Suppose that $X' = \lambda X, \quad \lambda \geqslant 0$. Then $(P, X') = \lambda a$. X' also belongs to A (It can be verified by substituting in(8)).

   Moreover, it belongs to the ellipsoid $E_{\lambda a}$

$$\frac{(X^1 - \lambda anp_1)^2}{r^2(p_1)} + \cdots + \frac{(X^n - \lambda anp_n)^2}{r^2(p_n)} \leqslant \lambda a^2, \tag{11}$$

   like all other vectors $Y' = \lambda Y, \quad Y \in E_a$. Hence, A is a cone and all its hyperplane sections $(P, X) = a$, $a > 0$ are ellipsoids.

   Therefore, A is a convex set.

3. At last we prove **A1**: $C_+ \subset A$.

   Consider a basis $e = \{e_i, \ i = 1, \ldots, n : \ e_i^i = 1, \ e_i^j = 0\}$.

   Any vector $X \in C_+$ can be represented as a convex linear combination of the elements of the basis e:

$$X = X^1 e_1 + X^2 e_2 + \cdots + X^n e_n, \quad X^i \geqslant 0, \ i = 1, \ldots, n$$

   Since Axiom **A3** is satisfied, we can assert that $C_+ \subset A$, if $e_i \in A \quad \forall i = 1, \ldots, n$.

   Then we prove that $e_1 \in A$ (for the rest e_i the proof is similar). Substitute coordinates of e_1 in (8):

$$\frac{(1-np_1{}^2)^2}{r^2(p_1)} + \frac{(np_1p_2)^2}{r^2(p_n)} + \cdots + \frac{(np_1p_n)^2}{r^2(p_n)} = \frac{1-2np_1{}^2+n^2p_1^4}{r^2(p_1)} + n^2p_1^2\left(\frac{p_2^2}{r^2(p_2)} + \cdots + \frac{p_n^2}{r^2(p_n)}\right) \leqslant$$

$$\leqslant \frac{1-2np_1^2+n^2p_1^4}{\frac{n}{p_1^2}+n^3p_1^2} + n^2p_1^2\left(\frac{p_2^2}{\frac{n}{p_2^2}+n^3p_2^2} + \cdots + \frac{p_n^2}{\frac{n}{p_n^2}+n^3p_n^2}\right) \leqslant \frac{p_1^2}{n} + n^2p_1^2\frac{n-1}{n^3} = p_1^2 = (e_1,P)^2$$

Hence, $e_j \in A \quad j = 1,\ldots,n$, thus, $C_+ \subset A$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.   Elliptic acceptance cone for the norm $\|\cdot\|_1$ in the space of risks

Consider a set

$$\sum_{i=1}^{n} \frac{|X_i - (P,X)\cdot n\cdot p_i|}{r(p_i)} \leqslant (P,X) \tag{12}$$

Suppose it fulfils the conditions

$$\begin{aligned}&1.\quad (P,X) \geqslant 0\\&2.\quad r(p) \geqslant \frac{n}{p}(1+np^2)\end{aligned} \tag{13}$$

Theorem 6.1. *The set of risks A, defined by (12) and (13) is an acceptance set for some preference.*

Proof.

1. $C_- \cap A = \emptyset$. The proof is similar to the proof of 1. in theorem 5.1

2. We can prove, that A is a convex cone.

   Suppose $X: (P,X) = a > 0$ and $X \in A$, that means

   $$\sum_{i=1}^{n} \frac{|X_i - a\cdot n\cdot p_i|}{r(p_i)} \leqslant a.$$

   Consider a set $A_a = \{Y \in A: (P,Y) = a\}$.

   We demonstrate that A_a is a convex set.

   Consider any $X,Y \in A$, that fulfil the condition $(P,X) = (P,Y) = a$ and examine the risk $(1-\alpha)X + \alpha Y$, $\alpha \in [0,1]$.

   $$\sum_{i=1}^{n} \frac{|(1-\alpha)X_i + \alpha Y_i - (P,(1-\alpha)X + \alpha Y)\cdot n\cdot p_i|}{r(p_i)} =$$

   $$= \sum_{i=1}^{n} \frac{|(1-\alpha)(X_i - anp_i) + \alpha(Y_i - anp_i)|}{r(p_i)} \leqslant (1-\alpha)\sum_{i=1}^{n} \frac{|X_i - anp_i|}{r(p_i)} + \alpha\sum_{i=1}^{n} \frac{|Y_i - anp_i|}{r(p_i)} = a$$

   So, $(1-\alpha)X + \alpha Y \in A_a$, hence the set A_a is convex.

   Using the positive homogeneity property for norms, we can demonstrate that for $X \in A$ risks $\lambda X \in A$ if $\lambda \geqslant 0$.

   Hence, A is a cone with hyperplane sections $(P,X) = a > 0$, which are convex sets.

   Thus, A is a convex cone.

3. We prove that $C_+ \subset A$.

Since A is a convex cone, if $e_1, \ldots e_n \in A$, then also $\lambda_1 e_1 + \ldots \lambda_n e_n \in A \quad \forall \lambda_i \geqslant 0$.

Let us show that $e_1 \in A$ (For the rest e_i the proof is similar):

Substituting coordinates of e_1 in (12) and denoting the result by D we obtain:

$$D = \frac{|1 - np_1^2|}{r(p_1)} + \frac{np_1 p_2}{r(p_2)} + \cdots + \frac{np_1 p_n}{r(p_n)} = \frac{|1 - np_1^2|}{r(p_1)} + np_1 \left(\frac{p_2}{r(p_2)} + \cdots + \frac{p_n}{r(p_n)} \right)$$

Denote $D_1 = \dfrac{|1 - np_1^2|}{r(p_1)}$.

Suppose that $1 - np_1^2 \geqslant 0$.

$$D_1 \leqslant \frac{1 - np_1^2}{\frac{n}{p_1}(1 + np_1^2)} < \frac{p_1}{n}.$$

If $1 - np_1^2 < 0$, then

$$D_1 \leqslant \frac{np_1^2 - 1}{\frac{n}{p_1}(np_1^2 + 1)} < \frac{p_1}{n}.$$

Consider a function $h(p) = \dfrac{p}{\frac{n}{p}(1 + np^2)} = \dfrac{p^2}{n(1 + np^2)}$.

$$h'(p) = \frac{2pn(1 + np^2) - 2n^2 p^3}{n^2(1 + np^2)^2} = \frac{2p}{n(1 + np^2)^2},$$

$$h'(p) = 0 \Longleftrightarrow p = 0, \quad h'(p) > 0 \quad p \in (0, 1].$$

The function $h(p)$ is nonincreasing on $(0, 1]$ and reaches a minimum in $p = 1$.

$$h(1) = \frac{1}{n(1 + n)} \quad \Rightarrow \quad h(p) \leqslant \frac{1}{n(1 + n)}.$$

$$D \leqslant \frac{p_1}{n} + np_1 \left(\frac{p_2}{r(p_2)} + \cdots + \frac{p_n}{r(p_n)} \right) \leqslant \frac{p_1}{n} + np_1 \frac{n - 1}{n(1 + n)} = p_1 \frac{n^2 + 1}{n^2 + n} < p_1 = (P, e_1).$$

Hence, $e_1 \in A$. □

7. Elliptic acceptance set for the norm $\| \cdot \|_\infty$ in the space of risks

Consider a set

$$\max_{i=1,\ldots,n} \frac{|X_i - (P, X) \cdot n \cdot p_i|}{r(p_i)} \leqslant (P, X). \tag{14}$$

Suppose that it fulfils the conditions:

$$
\begin{aligned}
&1. \quad (P, X) \geqslant 0, \\
&2. \quad r(p) \geqslant \frac{1 + np^2}{p}.
\end{aligned}
\tag{15}
$$

Theorem 7.1. *A set of risks A, defined by (14) and (15) is an acceptance set for some preference.*

Proof.

1. $C_- \cap A = \emptyset$. $C_- \cap A = \emptyset$. The proof is similar to the proof of 1 in Theorem 5.1.

2. A is a convex cone.

   Suppose $X : (P, X) = a > 0$ и $X \in A$, i. e.

   $$\max_{i=1,\ldots,n} \frac{|X_i - a \cdot n \cdot p_i|}{r(p_i)} \leqslant a.$$

   Consider a set $A_a = \{Y \in A : (P, Y) = a\}$.

   We can prove that A_a is a convex set.

   Consider arbitrary $X, Y \in A$, which fulfil $(P, X) = (P, Y) = a$ and examine the risk $(1-\alpha)X + \alpha Y, \ \alpha \in [0, 1]$.

   $$\max_{i=1,\ldots,n} \frac{|(1-\alpha)X_i + \alpha Y_i - anp_i|}{r(p_i)} = \max_{i=1,\ldots,n} \left| \frac{(1-\alpha)(X_i - anp_i)}{r(p_i)} + \frac{\alpha(Y_i - anp_i)}{r(p_i)} \right| \leqslant$$

   $$\leqslant \max_{i=1,\ldots,n} (1-\alpha)\frac{|X_i - anp_i|}{r(p_i)} + \max_{i=1,\ldots,n} \alpha\frac{|Y_i - anp_i|}{r(p_i)} = a.$$

   Then $(1-\alpha)X + \alpha Y \in A_a$, hence the set A_a is convex.

   Using the positive homogeneity property for norms, we can demonstrate that for $X \in A$ it also holds that $\lambda X \in A$ if $\lambda \geqslant 0$.

   Thus, A is a cone with hyperplane sections $(P, X) = a > 0$, which are convex sets.

   So, A is a convex cone.

3. We prove that $C_+ \subset A$.

   As A is a convex cone if $e_1, \ldots e_n \in A$, then also $\lambda_1 e_1 + \ldots \lambda_n e_n \in A \quad \forall \lambda_i \geqslant 0$.

   We demonstrate, that $e_1 \in A$ (for the rest e_i the proof if similar). Substitute e_1 in (14) and denote the result by D:

   $$D = \max \left\{ \frac{|1 - np_1^2|}{r(p_1)}, \frac{np_1 p_2}{r(p_2)}, \ldots, \frac{np_1 p_n}{r(p_n)} \right\}.$$

   If $D = \dfrac{|1 - np_1^2|}{r(p_1)}$, then using the condition (15) we get:

   $$D \leqslant \frac{|1 - np_1^2|}{1 + np_1^2}p_1 \leqslant p_1.$$

   If $D = \dfrac{np_1 p_k}{r(p_k)}$, where $k = 2, \ldots, n$, then

   $$D \leqslant p_1 \frac{np_k^2}{1 + np_k^2} \leqslant p_1.$$

   We obtain, that e_1 fulfills (14). □

8.   Axial function determination by the functional of risk aversion

If we take an axial function $r(p)$ complied with (9) from some one-parameter family, then it is sufficient to know the value of the function only at one point for complete definition of the corresponding elliptic acceptance cone.

Let us take $p = \dfrac{1}{n}$ in place of such a point. We determine the value

$$r_0 = r\left(\frac{1}{n}\right)$$

examining the elliptic cone corresponding to the uniform distribution $P = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ (we can call such cone a **sphere cone**).

For the norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ a sphere cone can be defined by the inequality

$$\left\| X - \frac{(I, X)}{n} I \right\| \leqslant \frac{(I, X)}{n} r_0.$$

8.1. Axial function for the norm $\|\cdot\|_2$

In [6] it was proved that

$$r_0 = \frac{\sqrt{1 - n c_{0,\Delta}^2}}{c_{0,\Delta}}. \tag{16}$$

If we know a risk aversion value in the zero risk we can find a value of $r(p)$ at $p = \dfrac{1}{n}$, then determine the unknown parameter of $r(p)$ and define a corresponding acceptance cone.

It was shown that as an axial functions for the cone (8) we can take one of the following:

1. *Power axial function*

$$r_1(p, m_1) = \frac{\sqrt{n + n^3}}{p^{m_1}}, \quad m_1 \geqslant 1.$$

2. *Exponential axial function*

$$r_2(p, m_2) = \sqrt{n + n^3} \cdot e^{\frac{m_2(1 - p)}{p}}, \quad m_2 \geqslant M \approx 0.203.$$

3. *Logarithmic axial function*

$$r_3(p, m_3) = \frac{n^2}{p} \ln\left(\frac{m_3(1 - p)}{p} + e\right), \quad m_3 \geqslant 0.$$

All the considered functions decrease by p and increase by m_i.

A function $R(p) = \sqrt{\dfrac{3}{p^2} + 3^3 p^2}$ — defines an infimum of axial functions values.

8.2. Axial function for the norm $\|\cdot\|_1$

Consider $I = (1, 1, \ldots, 1)$. Its norm is equal to $\|I\| = n$.

Lemma 8.1. *For the risk I the nearest (in the sense of the norm $\|\cdot\|_1$) risk I' lying on the border ∂A, can be obtained by translation of I along one on the standard basis vectors:*

$$f(I) = \|I - I'\|, \quad I' = I - \frac{r_0 n}{2(n-1) + r_0} e_j \quad (j \in \{1, 2, \ldots n\}).$$

Proof. Denote

$$\phi = \frac{r_0 n}{2(n-1) + r_0}. \tag{17}$$

Fix arbitrary index j. Consider a vector $I' = I - \phi e_j$.

$$\|I - I'\| = \phi.$$

$$\left\| I' - \frac{(I', I)}{n} I \right\| = \left\| I - \phi e - j - \frac{(I - \phi e_j)}{n} I \right\| = \left\| I - \frac{(I, I)}{n} I + \frac{\phi}{n} I - \phi e_j \right\| = \frac{\phi}{n} \|I - n e_j\| =$$

$$= \frac{\phi}{n}(|1 - n| + 1 + \cdots + 1) = \frac{\phi \cdot 2(n-1)}{n} = r_0 \cdot \frac{2(n-1)}{2(n-1) + r_0}.$$

$$\frac{(I', I)}{n} r_0 = \frac{(I - \phi e_j, I)}{n} r_0 = \frac{(I, I)}{n} r_0 - \frac{\phi}{n} r_0 = \frac{r_0}{n}(n - \phi) = r_0 \cdot \frac{2(n-1)}{2(n-1) + r_0}$$

We get that $\left\| I' - \frac{(I', I)}{n} I \right\| = \frac{(I', I)}{n} r_0$, hence, $I' \in \partial A$.

Now we prove that I' is a vector of ∂A, the nearest to I.

Consider any vector $Y = I - \alpha_1 e_1 - \cdots - \alpha_n e_n$, such that $\|I - Y\| = \|\alpha_1\| + \cdots + \|\alpha_n\| < \phi$.

$$\left\| Y - \frac{(Y, I)}{n} I \right\| = \left\| I - \alpha_1 e_1 - \cdots - \alpha_n e_n - \frac{(I - \alpha_1 e_1 - \cdots - \alpha_n e_n, I)}{n} I \right\| =$$

$$= \left\| I - \frac{(I, I)}{n} + \frac{\alpha_1 + \cdots + \alpha_n}{n} I - \alpha_1 e_1 - \cdots - \alpha_n e_n \right\| = \left| \frac{(1-n)\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} \right| + \cdots +$$

$$+ \left| \frac{\alpha_1 + \cdots + \alpha_{n-1} + (1-n)\alpha_n}{n} \right| \leqslant \frac{1}{n}((n-1)|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| + \cdots + |\alpha_1| + \cdots + (n-1)|\alpha_n|) =$$

$$= \frac{2(n-1)}{n}(|\alpha_1| + \cdots + |\alpha_n|) < \frac{2(n-1)}{n} \phi.$$

$$\frac{(Y, I)}{n} r_0 = \frac{(I - \alpha_1 e_1 - \cdots - \alpha_n e_n, I)}{n} r_0 = \frac{r_0}{n}(n - (\alpha_1 + \cdots + \alpha_n)) \geqslant \frac{r_0}{n}(n - (|\alpha_1| + |\alpha_n|)) >$$

$$> \frac{r_0}{n}(n - \phi) = \frac{2(n-1)\phi}{n - \phi} \cdot \frac{n - \phi}{n} = \frac{2(n-1)}{n} \phi.$$

So,

$$\left\| Y - \frac{(Y, I)}{n} I \right\| < \frac{(Y, I)}{n} r_0,$$

thus $Y \in A \setminus \partial A$.   □

Now then, $f(I) = \dfrac{r_0 n}{2(n-1) + r_0}$. Note that $f(I) > 1$ (If follows from(13) and (17)).

Now pass on to detecting a relationship between r_0 and risk aversion.

Theorem 8.1. *For an elliptic acceptance cone in the space of risks with the norm $\|\cdot\|_1$*

$$r_0 = \frac{2(n-1)(1 - c_{0,I_\Delta})}{c_{0,I_\Delta}}. \tag{18}$$

Proof.

Let for definiteness $e_j = e_n$.

$$I' = I - f(I)e_n.$$

$$I' = (1, \ldots, 1, 1 - f(I)); \quad \|I'\| = n - 1 + |1 - f(I)| = n + f(I) - 2.$$

$$(I, I') = n - 1 + 1 - f(I) = n - f(I).$$

By the shortest path property $f(I + \beta e_n) = f(I) + \beta e_n, \quad \beta \leqslant 0$.

Consider such a vector $I_\Delta = I + \beta e_n$, that $(I, I_\Delta) = 0$ (it means, that it lies in the plane $EX = 0$):

$$(I, I + \beta e_n) = 0 \quad \Rightarrow \quad \beta = -n, \quad I_\Delta = (1, \ldots, 1, 1 - n).$$

On the other hand, $I_\Delta = I' - \|I' - I_\Delta\| e_n$, that is why $f(I_\Delta) = f(I') - \|I' - I_\Delta\|$.

By the risk aversion definition for generalized coherent risk measures

$$c_{0,I_\Delta} = \frac{f(I_\Delta)}{f(-I)} = \frac{\|I_\Delta - I'\|}{n} \quad \Rightarrow \quad \|I_\Delta - I'\| = c_{0,I_\Delta} \cdot n,$$

$$\|I_\Delta\| = n - 1 + |1 - n| = 2(n - 1).$$

$$\|I - I_\Delta\| = n,$$

$$\|I - I'\| = \|I - I_\Delta\| - \|I\Delta - I'\| = n(1 - c_{0,I_\Delta}).$$

But earlier we got that $\|I - I'\| = f(I) = \dfrac{r_0 n}{2(n-1) + r_0}$. That establishes the theorem.   □

8.3. Axial function for the norm $\|\cdot\|_\infty$

Theorem 8.2. *For an elliptic acceptance cone in the space of risks with the norm $\|\cdot\|_\infty$*

$$r_0 = \frac{1}{c_{0,\Delta}}. \tag{19}$$

Proof. Consider a risk $\Delta: \ E\Delta = 0, \ \|\Delta\|_\infty = 1$. Since the vector $I = (1, , \ldots, 1)$ is the vector of the shortest path for the norm $\|\cdot\|_\infty$, the nearest to Δ vector in ∂A is $\Delta' = \Delta + f(\Delta)I$.

$$(\Delta', I) = (\Delta, I) + f(\Delta)(I, I) = f(\Delta)n = c_{0,\Delta} n.$$

Suppose $\tilde{\Delta} = \dfrac{(I, \Delta')}{n} I$.

Since $\Delta' \in \partial A$,

$$\left\| \Delta' - \tilde{\Delta} \right\| = \frac{(\Delta', I)}{n} r_0 = c_{0,\Delta} r_0.$$

Since $(\Delta, \Delta - \Delta') = 0,\ (\tilde{\Delta}, \tilde{\Delta} - \Delta') = 0,\ (\tilde{\Delta}, \Delta) = 0,$

$$\Delta = \Delta' - \tilde{\Delta} \quad \Rightarrow \quad \|\Delta\| = \|\Delta' - \tilde{\Delta}\| = c_{0,\Delta} r_0.$$

This yields the proposition of the theorem. $\hfill\square$

Power axial function

$$r(p, m) = \frac{np + 1}{p^m},\ m > 1. \tag{20}$$

We show now that this function can be used as an axial function for defining an acceptance cone in the space $\mathcal{X}$ with the norm $\|\cdot\|_\infty$. Function $r(p, m)$ fulfills the condition (15) because:

1. For $p \in (0, 1]$ $\dfrac{np + 1}{p^m} - \dfrac{1 + np^2}{p} = \dfrac{np(1 - p^m) + (1 - p^{m-1})}{p^m} > 0.$

2. $\displaystyle\lim_{p \to 0} \left(\frac{np + 1}{p^m} - \frac{1 + np^2}{p} \right) = \lim_{p \to 0} \frac{np + 1 - p^{m-1}}{p^m} = +\infty.$

Thus, a cone with an axial function (20) is an acceptance cone.

8.4. Example of defining an acceptance set by the given risk aversion value

Suppose that somebody's preference relation possesses the risk aversion property, is consistent with stochastic dominance and can be characterized by the norm $\|\cdot\|_\infty$ in the space of risks.

We propose an investor to take a lottery ticket for k\$. By the lottery he may either gain 2k\$ or gain nothing with the same probability . The investor should measure the minimal value of premium, he would demand for buying such a lottery ticket.

Let the investors answer be $\alpha = 0.25\$$.

This game corresponds to a risk Δ with distribution:

Δ	-1	1
P	0.5	0.5

$E\Delta = 0,\ \|\Delta\| = 1$, hence $c_{0,\Delta} = \alpha/k$. It follows that the value of an axial function at the point $p = 1/n$ is

$$r_0 = \frac{1}{c_{0,\Delta}} = 4.$$

If the investors preferences could be defined by an axial function $r(p, m) = \dfrac{np + 1}{p^m},\ m \geqslant 1$, then

$$m = \frac{\ln c(np + 1)}{\ln p} = 1.$$

It is clear that the given as an example procedure of risk aversion estimating only by one investor's answer is not enough reliable because the procedure presumes the exact match of the investor's answer to his individual attitude to risk.

For implementation of the discussed method of inverse problem solving we should define a more reliable procedure for risk aversion detecting either by using statistics of earlier made decisions, or by more explicit questionnaires.

Conclusion

The method of inverse problem solving presented in this paper allows to define individual acceptance sets and therefore individual functionals of generalized coherent risk measures utilizing one of the preferences characteristics — value of risk aversion. Individual risk measures can be used in solving different applied problems when the individual attitude to risk should be taken into account, for example, in portfolio building.

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Неприятие риска для нахождения эллиптических конусов приемлемых рисков в модели обобщенных когерентных мер риска

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В рамках модели обобщенных когерентных мер риска исследованы свойства множеств приемлемых рисков. Предложен класс эллиптических конусов для описания индивидуальных предпочтений. Построена методика нахождения эллиптического конуса приемлемых рисков по значениям функции неприятия риска (для случая гельдеровых норм в пространстве рисков).

Ключевые слова: обобщенные когерентные меры риска, неприятие риска, множество приемлемых рисков, отношение предпочтения, эллиптический конус.