In this paper we give a full description for divisors of elementary differentials of all kinds. An analog of Appell's expansion formula for univalent functions on a variable torus is obtained. All basic type of vector bundles of meromorphic differentials of integer order over a Teichmüller space for torus are studied.

Keywords: univalent meromorphic differentials of integer order, divisors, vector bundles over Teichmüller space for torus.

Introduction

Univalent differentials (of order $q = 1$ and $q = 2$ in particular) even on a fixed surface have found a lot of applications in mathematical physics (algebraic-geometric integration of nonlinear equations in the works of S.P. Novikov, I.M. Krichever), in theoretical physics (R. Dick), and also in analytic number theory in the works by H.M. Farkas and I. Kra [1].

The main difference of the results of this paper from the classical ones found in the books by J. Springer [2], H.M. Farkas and I. Kra [1] and in other books on the geometric function theory on a compact Riemann surface is that we consider all objects on a variable compact Riemann surface $F_\mu$ of genus $g = 1$ (torus) [3, 4]. For the general theory of univalent differentials a big role is played by so called elementary differentials of integer order $q$ that have the minimal number of poles: either one pole of order $\geq 2$, or two simple poles, and depend holomorphically on the modules of the torus $F_\mu$. For the first time we give a full description for divisors of elementary abelian $q$-differentials of all kinds. An analog of Appel’s expansion formula for univalent functions on a variable torus is obtained. We study also all basic types of vector bundles of meromorphic differentials of integer order $q \neq 1$ over a Teichmüller space for torus.

Preliminaries

Let $F_0$ be a fixed compact Riemann surface of genus $g = 1$, $F_0 = \mathbb{C}/\Gamma$, where $\Gamma$ is a group with two generators $A_1(z) = z + \omega$, $B_1(z) = z + \omega'$, $\text{Im} \frac{\omega'}{\omega} > 0$. Let $\mu_0 = \frac{\omega'}{\omega}$. The fundamental group of the surface $F_0$ has an algebraic representation

$$\Gamma \cong \pi_1(F_0) = \langle a_1, b_1 : a_1b_1 = b_1a_1 \rangle.$$

The class $[F_0, \{a_1, b_1\}]$ of conformally equivalent marked compact Riemann surfaces of genus one is uniquely defined by a complex parameter (module) $\mu_0 = \frac{\omega'}{\omega}$, which lies in the upper half plane $H = \{z \in \mathbb{C} : \text{Im} z > 0\}$. Here $F_0 = \mathbb{C}/\Gamma_0$ where $\Gamma_0$ is the group generated by two generators

$$A_{01}(z) = z + 1, \quad B_{01}(z) = z + \mu_0.$$
Every other class $[F_\mu, \{a^n_1, b^n_1\}]$ of conformally equivalent marked compact Riemann surfaces of genus one is uniquely defined by a complex parameter (module) $\mu \in H$ and $F_\mu = \mathbb{C}/\Gamma_\mu$ where $\Gamma_\mu$ is generated by $A_\mu(z) = z + 1$, $B_\mu(z) = z + \mu$. Moreover, there is a quasiconformal mapping $\bar{f}_\mu : F_0 \to F_\mu$, and its lifting $f_\mu : \mathbb{C} \to \mathbb{C}$ on the universal covering surface gives an isomorphism between the marked group $\Gamma_0$ and the marked group $\Gamma_\mu = f_\mu \Gamma_0 f_\mu^{-1}$ with $a^n_1 = \bar{f}_\mu(a_1)$, $b^n_1 = \bar{f}_\mu(b_1)$.

The Teichmüller space $T_1 = T_1(F_0)$, of the classes $[F_\mu, \{a^n_1, b^n_1\}]$ of conformally equivalent marked compact Riemann surfaces of genus one can be parametrized by points from $\overline{\mathbb{H}}$.

Next, for every natural number $n \geq 1$ there is a fiber bundle over $T_1$ such that its fibre over $\mu \in T_1$ as the space of all integer divisors of degree $n$ on $F_\mu$. Locally holomorphic sections of this bundle define on every $F_\mu$ an integer divisor $D^\mu$ of degree $n$ that holomorphically depends on $\mu$ [5, p. 261, 268].

**Definition.** A $q$-differential $\phi$ with respect to the group $\Gamma$ on $\mathbb{C}$ is a differential $\phi(z)dz^q$ such that

$$\phi(Tz)(T^*z)^q = \phi(z), \quad z \in \mathbb{C}, \quad T \in \Gamma.$$

In particular, for $q = 0$, this is a meromorphic function with respect to $\Gamma$.

Let $D$ be a divisor on $F$. Introduce following the spaces: $L(D; F)$ of meromorphic functions $f$ on $F$ such that $(f) \geq D$, and $\Omega^q(D; F)$ of meromorphic $q$-differentials $\omega$ on $F$ such that $(\omega) \geq D$. Denote by $r(D) = \text{dim}_{\mathbb{C}} L(D; F)$ and $i_q(D) = \text{dim}_{\mathbb{C}} \Omega^q(D; F)$ the dimensions of these complex vector spaces.

**Theorem (Riemann-Roch)** [1, p. 73]. Let $F$ be a compact Riemann surface of genus $g = 1$. Then for every divisor $D$ on $F$

$$r(D^{-1}) = \text{deg } D + i(D).$$

**Theorem (Riemann-Roch for $q$-differentials)** [4, p. 43]. For every $q \in \mathbb{Z}$ on a compact Riemann surface $F$ of genus one

$$i_q(D) = -\text{deg } D + r(1/D).$$

**Theorem (Abel)** [1, p. 93; 4, p. 67]. Let $[F; \{a_1, b_1\}]$ be a marked compact Riemann surface of genus one and

$$D = \frac{P_1^{\alpha_1} \cdots P_m^{\alpha_m}}{Q_1^{\beta_1} \cdots Q_s^{\beta_s}}$$

be a divisor of degree zero on $F$. Then there exists a function $f$ on $F$ with

$$(f) = D \Leftrightarrow \varphi(D) = \sum_{j=1}^m \alpha_j \varphi(P_j) - \sum_{k=1}^s \beta_k \varphi(Q_k) = 0$$

in $J(F) = \varphi(F)$, where $\varphi$ is the Jacobi mapping from $F$ to $J(F)$.

1. **Univalent elementary $q$-differentials on a variable torus**

In this section we establish the general form of elementary univalent $q$-differentials on the torus $F_\mu$. 

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Let us find first the general form of $q$-differentials $\tau_{q;Q}^{(m)}$ with the only pole $Q = Q(\mu)$ exactly of order $m \geq 2$ on $F_\mu$, $q \in \mathbb{Z}$.

By the Riemann-Roch theorem for $q$-differentials on $F_\mu$ [4, p.43] we find the dimension

$$i_q \left( \frac{1}{Q^m} \right) = \dim \mathcal{O}(1) F_\mu = -\deg D + r(Q^m),$$

where $D = \frac{1}{Q^m}$. Hence $i_q \left( \frac{1}{Q^m} \right) = m \geq 2$. Here $r(Q^m) = 0$, so $\deg(Q^m) = m > 0$ under our conditions. This can also be proved by contradiction: if there existed a function $g$ on $F_\mu$ such that $\deg(g) = \deg(Q^m)$, then $0 = \deg(g) \geq \deg(Q^m) \geq 2$.

Since $\deg Q^{m-1} = m - 1 \geq 1 > 0$,

$$i_q \left( \frac{1}{Q^{m-1}} \right) = -\deg \left( \frac{1}{Q^{m-1}} \right) + r(Q^{m-1}) = m - 1.$$

Therefore, $i_q \left( \frac{1}{Q^m} \right) = i_q \left( \frac{1}{Q^{m-1}} \right) + 1$. Hence there exists a $q$-differential $\tau_{q;Q}^{(m)}$ with the pole exactly of order $m$ at the point $Q$ on $F_\mu$, i.e. the divisor $\left( \tau_{q;Q}^{(m)} \right) = \frac{R_1 \cdots R_m}{Q^m}$ on $F_\mu$, $R_j \neq Q, j = 1, \ldots, m$.

Construct now such a differential explicitly: $\tau_{q;Q}^{(m)} = f dz^q$, $q \in \mathbb{Z}$, where $dz$ is a holomorphic differential on $F_\mu$ that depends holomorphically on $\mu$. The univalent function $f$ has the divisor $(f) = \frac{R_1 \cdots R_m}{Q^m}$, since $(dz) = 1$. By the Abel theorem [4] we get the equation

$$\varphi_{P_0}(\mu)(R_1 \cdots R_m) - \varphi_{P_0}(\mu)(Q^m) = 0$$

in the Jacobi manifold $J(F_\mu)$, where $P_0$ is an initial point different from $Q$. We understand this equation as an equality in the variable Jacobian $J(F_\mu)$, i.e. in the fibre of the universal Jacobi bundle that lies over the marked surface $F_\mu$. Therefore

$$\varphi(R_1) = \varphi(Q^m) - \varphi(R_2 \cdots R_m). \quad (1)$$

Thus, for zeros of the function $f$ we have $m - 1 \geq 1$ free parameters that can be arbitrarily chosen on $F_\mu$ locally holomorphically depending on $\mu$. By the theorem of C.Earle [5, p. 268] we can choose the divisor $R_2 \cdots R_m$ in such a way that it does not contain the point $Q$ on $F_\mu$ and is a locally holomorphic section of the bundle of integer divisors of degree $m - 1$ over the Teichmüller space $T_1$.

Solving the Jacobi problem in the universal bundle over $T_1$, we find the divisor $R_1$ on $F_\mu$, which is a unique solution to the equation (1) [1, p. 95, 97]. Here the point $R_1 \neq Q$ and $R_1$ depends holomorphically on our parameter, since the right hand side in (1) was chosen as holomorphically depending on $\mu$. Indeed, if $R_1 = Q$ then consider the divisor $D = R_2 \cdots R_m$ with $m - 1$ free points. By the theorem on free points [1, p. 125] we have the inequality

$$m - 1 + 1 \leq \left( \frac{1}{D} \right) = m - 1 + i(D),$$

and hence $1 \leq i(D)$. Therefore we see that there exists a differential $\omega \neq 0$, $(\omega) \geq D$. Consequently, we have an impossible inequality

$$0 = \deg(\omega) \geq \deg D = m - 1 \geq 1.$$
**Theorem 1.1.** On a variable torus $F_{\mu}$ for every natural number $m > 1$, $q \in \mathbb{Z}$ there exists an elementary $q$-differential $\tau_{q,Q}^{(m)}$ with the pole at the point $Q = Q(\mu) \in F_{\mu}$ exactly of order $m$ locally holomorphically depending on $\mu$, whose divisor is of the form
\[
\left(\tau_{q,Q}^{(m)}\right) = \frac{R_1 \cdots R_m}{Q^m},
\]
where
\[
\varphi(R_1) = \varphi(Q^m) - \varphi(R_2 \cdots R_m).
\]
Here the divisors $R_2 \cdots R_m$ and $Q = Q(\mu)$ are chosen as locally holomorphic sections of the bundle of integer divisors over $T_1$ of degrees $m - 1$ and $1$ respectively for $\mu$ from a sufficiently small neighborhood $U(\mu_0) \subset T_1$.

**Corollary 1.1.** Under the assumptions of theorem 1.1 there exists a $q$-differential
\[
z_{q,Q}^{(m)} = \left(\frac{1}{z^m} + O(1)\right) dz^q
\]
in a neighborhood of the point $Q$ on $F_{\mu}$.

**Proof.** For every $q \in \mathbb{Z}$, $m > 1$, there exists a $q$-differential
\[
z_{q,Q}^{(m)} = \left(\frac{c_{-m}}{z_m} + \cdots + \frac{c_{-1}}{z^1} + O(1)\right) dz^q, \ c_{-m} \neq 0
\]
in a neighborhood of the point $Q$ on $F_{\mu}$. The Abelian 1-differential $\tau_{q,Q}^{(m)}$ has the residue $c_{-1} = 0$ at the point $Q$ by the residue theorem on $F_{\mu}$. For $m = 2$ we have a $q$-differential
\[
z_{q,Q}^{(2)} = \frac{1}{c_{-2}} \tau_{q,Q}^{(2)} = \left(\frac{1}{z^2} + O(1)\right) dz^q.
\]
By induction, for every $m > 1$ we can get a $q$-differential
\[
z_{q,Q}^{(m)} = \left(\frac{1}{z^m} + O(1)\right) dz^q
\]
in a neighborhood of the point $Q$ on $F_{\mu}$. Moreover, such $q$-differential can be obtained by differentiating with respect to the parameter $z(Q)$ from the formula
\[
z_{q,Q}^{(m)} = \frac{1}{(-m+1) \cdots (-2)} \left[z_{q,Q}^{(2)} \right]^{(m-2)}._Q.
\]
Thus, we have proved the corollary. \qed

**Remark 1.1.** For every $q \in \mathbb{Z}$ by the Riemann-Roch theorem for $q$-differentials we have the equality
\[
i_q(D) = -\deg D + r\left(\frac{1}{D}\right)
\]
and $i_q(1) = 1$. Therefore $i_q\left(\frac{1}{Q}\right) = 1 + r(Q) = 1$. Also $i_q\left(\frac{1}{Q}\right) = r\left(\frac{1}{Q}\right) = 1$, where the first equality follows from the isomorphism given by division by the differential $dz^q$ on the torus $F_{\mu}$.

Because of that we have $i_q\left(\frac{1}{Q}\right) = 1 = i_q(1)$. Therefore, there is no a $q$-differential $\tau_{q,Q}$ on the torus $F_{\mu}$ with the only pole at $Q$ exactly of order one for every $q \in \mathbb{Z}$. This fact can be also proved by using the residue theorem for abelian differentials of order one on $F_{\mu}$ \cite{7, 8}.

Now we establish the general form for univalent $q$-differentials $\tau_{q,Q;Q_2}$ of the third kind with exactly two simple poles at different points $Q_1 = Q_1(\mu)$ and $Q_2 = Q_2(\mu)$ on $F_{\mu}$ that depend holomorphically on the parameter $\mu$. 

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Proposition 1.1. On a variable torus $F_\mu$ for every integer $q$ there exists an elementary $q$-differential $\tau_{Q_1, Q_2}$ of the third kind with exactly two simple poles at different points $Q_1, Q_2$, and $Q_3 = Q_2(\mu)$ on $F_\mu$ locally holomorphically depending on $\mu$ with the divisor $(\tau_{Q_1, Q_2}) = R_1 R_2 Q_1 Q_2$ where $\varphi(R_1) = \varphi(Q_1 Q_2) - \varphi(R_2)$ in $J(F_\mu)$. Here the points $R_2, Q_1 = Q_1(\mu)$, $Q_2 = Q_2(\mu)$ can be chosen as locally holomorphic sections of the bundle of integer divisors of degree one over $T_1$ for $\mu$ from a sufficiently small neighborhood $U(\mu_0) \subset T_1$.

Proof. For $q \in \mathbb{Z}$, set $\tau_{Q_1, Q_2} = \tau_{Q_1, Q_2} dz^q$, where $\tau_{Q_1, Q_2}$ is the classical abelian differential of the third kind on $F_\mu$ that depends holomorphically on $\mu$ [1, p.51; 6].

Such a differential $\tau_{Q_1, Q_2}$ can also be taken as $\tau = f dz^q$, where $f$ is a univalent function with the divisor $(f) = R_1 R_2 Q_1 Q_2$. By Abel’s theorem we have the equality

\[ \varphi(R_1) = \varphi(Q_1 Q_2) - \varphi(R_2) \]

in $J(F_\mu)$. The divisor $R_1$ is the only solution to the equation (2). Moreover, we can take the points such that $R_1 \neq Q_1, Q_2, j = 1, 2$. Indeed, if $R_1 = Q_1$ for $R_2 \neq Q_1, Q_2$, then $\varphi(R_2) = \varphi(Q_2)$ and $R_2 = Q_2$. We arrive at a contradiction which proves the preposition. \(\square\)

2. An analog of Appell’s expansion formula for meromorphic functions on a variable torus

In this section we find an analog of Appell’s formula where the terms (summands) have poles only at one point on $F_\mu$ and depend holomorphically on $\mu$.

Let $f$ be a function on a variable torus $F_\mu$ with $s$ simple poles $Q_1, Q_2, \ldots, Q_s$ and residues $c_1, \ldots, c_s$ at them respectively. Consider the expression

\[ f_1 = f - c_1 T^{(1)}_{Q_1} - \ldots - c_s T^{(1)}_{Q_s}, \]

where $T^{(1)}_{Q_k}(z) = -\int T^{(2)}_{Q_k}$ is a branch of the elementary abelian integral of the second kind [1, p.51] with only simple pole at $Q_k$ and the residue +1 at $Q_k$ depends holomorphically on $\mu$, $k = 1, \ldots, s$. Then $f_1$ is an abelian integral of the first kind on the torus $F_\mu$. Therefore

\[ f_1 = C_1 \int dz + C = C_1 z + C \] on $F_\mu$.

Theorem 2.1. Let $f$ be a function on a variable torus $F_\mu$ with simple poles $Q_1, \ldots, Q_l$ and residues $c_1, \ldots, c_l$ at them, and poles at $Q_{l+1}, \ldots, Q_s$ with multiplicities $n_{l+1}, \ldots, n_s, n_k \geq 2, k = l+1, \ldots, s$, and given principal parts at them. Then

\[ f = C_1 z + C + \sum_{j=1}^{l} c_j T^{(1)}_{Q_j} + \sum_{k=l+1}^{s} \left[ A_{k,s} T^{(1)}_{Q_k} + A_{k,2} \frac{\partial T^{(1)}_{Q_k}}{\partial Q_k} + \frac{A_{k,3}}{2!} \frac{\partial^2 T^{(1)}_{Q_k}}{\partial Q_k^2} + \ldots + \frac{A_{k,n_k}}{(n_k - 1)!} \frac{\partial^{n_k - 1} T^{(1)}_{Q_k}}{\partial Q_k^{n_k - 1}} \right], \]

where $C_1, C$ are complex numbers and

\[ f = \frac{A_{k,n_k}}{(z - z(Q_k))^n_k} + \ldots + \frac{A_{k,2}}{(z - z(Q_k))^2} + \frac{A_{k,1}}{z - z(Q_k)} + O(1) \]

in a punctured neighborhood of $Q_k$, $k = l+1, \ldots, s$, on $F_\mu$, and all terms depend holomorphically on $\mu$. 

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Proof. If $Q_k$ is a pole of order $n_1, n_1 \geq 2$, then in the previous formula the term $c_1 T_Q^{(1)}$ is replaced by the sum

$$A_{11} T_{Q_1}^{(1)} + A_{12} \frac{\partial T_{Q_1}^{(1)}}{\partial Q_1} + A_{13} \frac{\partial^2 T_{Q_1}^{(1)}}{\partial Q_1^2} + ... + A_{1n_1} \frac{\partial^{n_1-1} T_{Q_1}^{(1)}}{(n_1-1)!} \frac{\partial Q_1^{n_1-1}}{Q_1},$$

where $A_{kj}$ are the coefficients of the principal part of the Laurent series for the function $f$ in a punctured neighborhood of the point $Q_k, j = 1, ..., n_k(Q_k), k = l + 1, ..., s$. Indeed, in a neighborhood of the point $Q_k$ we have the expansions $T_{Q_k}^{(1)} = \frac{1}{z - z(Q_k)} + O(1), z(Q_k) = a_k$;

$$(T_{Q_k}^{(1)})^{(m)}_{a_k} = \frac{1}{(z - a_k)^2} + O(1); \quad (T_{Q_k}^{(1)})^{(n)}_{a_k} = \frac{n!}{(z - a_k)^{m+1}} + O(1), 1 \leq m \leq n_k(Q_k) - 1,$$

where $n_k(Q_k)$ is the order of the pole at the point $Q_k$ for $f, k = l + 1, ..., s$. The theorem is proved. \(\square\)

3. The space of meromorphic $q$-differentials on a variable torus

Denote by $\Omega^q \left( \frac{1}{P_1 \ldots P_i \ldots P_{l+1} \ldots P_n}; F_\mu \right)$ the space of $q$-differentials on $F_\mu$ that are multiples of the divisor $\frac{1}{P_1^{\alpha_1} \ldots P_i^{\alpha_i} P_{l+1} \ldots P_n}$, where $q \in \mathbb{Z}, \alpha_1, ..., \alpha_i \geq 2, n \geq 1, 0 \leq l \leq n$, and the points $P_1, ..., P_n$ are pairwise distinct, and by $\Omega^q(1; F_\mu)$ the subspace of holomorphic $q$-differentials on $F_\mu$. The divisor $\frac{1}{P_1^{\alpha_1} \ldots P_i^{\alpha_i} P_{l+1} \ldots P_n}$ is chosen as a locally holomorphic section of the bundle of integer divisors of degree $\alpha_1 + \cdots + \alpha_i + n - l$ over $T_1$.

By the Riemann-Roch theorem for $q$-differentials we find the dimensions of these spaces. For every $q$ we have $\dim \Omega^q(1; F_\mu) = 1$, and

$$i_q \left( \frac{1}{P_1^{\alpha_1} \ldots P_i^{\alpha_i} P_{l+1} \ldots P_n} \right) = - \deg \left( \frac{1}{P_1^{\alpha_1} \ldots P_i^{\alpha_i} P_{l+1} \ldots P_n} \right) + r \left( P_1^{\alpha_1} \ldots P_i^{\alpha_i} P_{l+1} \ldots P_n \right) =$$

$$= \alpha_1 + \cdots + \alpha_i + n - l \quad (\geq 1).$$

Therefore

$$\dim \Omega^q \left( \frac{1}{P_1^{\alpha_1} \ldots P_i^{\alpha_i} P_{l+1} \ldots P_n}; F_\mu \right) / \Omega^q(1; F_\mu) = \alpha_1 + \cdots + \alpha_i + n - l - 1 \quad (\geq 1).$$

Consider the following collections of $q$-differentials:

$$\tau_q^{(2)}_{P_1, P_1}, \ldots, \tau_q^{(\alpha_1)}_{P_1, P_1}, \ldots, \tau_q^{(2)}_{P_i, P_i}, \ldots, \tau_q^{(\alpha_i)}_{P_i, P_i}, \tau_q; P_l, P_2, ..., \tau_q; P_l P_n, \text{ for } l \geq 1;$$

$$\tau_q; P_l P_2, ..., \tau_q; P_l P_n, \text{ for } l = 0. \quad (4)$$

Let us show that the coset classes of $q$-differentials from (4) are linearly independent over $\mathbb{C}$. Assume that there exists a linear combination of differentials from (4)

$$C_1^{(2)} \tau_q^{(2)}_{P_1, P_1} + \cdots + C_i^{(\alpha_i)} \tau_q^{(\alpha_i)}_{P_i, P_i} + \cdots + C_l^{(2)} \tau_q^{(2)}_{P_l, P_l} + \cdots + C_n^{(\alpha_i)} \tau_q^{(\alpha_i)}_{P_l, P_l} + C_2 \tau_q; P_l P_2 + \cdots + C_n \tau_q; P_l P_n = \omega,$$

where $\omega$ is a holomorphic $q$-differential, such that not all its coefficients are zeroes.

The coefficients $C_1^{(2)} = \cdots = C_i^{(\alpha_i)} = 0$, since in the right hand side the points $P_1, ..., P_l$ are not poles of order $\geq 2$. We are left with the equality

$$C_2 \tau_q; P_l P_2 + \cdots + C_n \tau_q; P_l P_n = \omega.$$
Since the points $P_2, \ldots, P_n$ are not singular for the right hand side, $C_2 = \cdots = C_n = 0$. Thus, the coset classes for $q$-differentials from (4) is a base for the quotient space.

Let us now show that the collection (5) is linearly independent over $C$. Suppose that there exists a linear combination $C_2 \tau_{q; P_2} + \cdots + C_n \tau_{q; P_n} = \omega$, where $\omega$ is a holomorphic $q$-differential, such that not all its coefficients are zeroes. The coefficients $C_2 = \cdots = C_n = 0$, since $P_2, \ldots, P_n$ are not singular for the right hand side. Therefore the coset classes of $q$-differentials from (5) form a base for the quotient space. Thus, we have proved the following theorem.

**Theorem 3.1.** The vector bundle $E_1 = \bigcup \Omega^q \left( \frac{1}{P_1^{\alpha_1} \cdots P_{l+1}^{\alpha_{l+1}} \cdots P_n^{\alpha_n}}; F_\mu \right) / \Omega^q \left( 1; F_\mu \right)$ of rank $d = \alpha_1 + \cdots + \alpha_l + n - l - 1$, where $\alpha_1, \ldots, \alpha_l \geq 2$, $n \geq 1$, $0 \leq l \leq n$, $q \in Z$, over $T_1$ is complex analytic equivalent to the direct product $T_1 \times \mathbb{C}^d$, and the coset classes of $q$-differentials from the collections (4), (5) give a base of locally holomorphic sections of this bundle over $T_1$.

Consider the collection of $q$-differentials

$$dz^q; \tau_{q; P_1}^{(1); \alpha_1}, \ldots, \tau_{q; P_1}^{(\alpha_1); \alpha_1}, \tau_{q; P_2}^{(2); \alpha_1}, \ldots, \tau_{q; P_1}^{(\alpha_1); \alpha_1}, \ldots, \tau_{q; P_1}^{(1); \alpha_1}, \tau_{q; P_2}^{(2); \alpha_1}, \ldots, \tau_{q; P_1}^{(\alpha_1); \alpha_1}.$$

(6)

Let us show that $q$-differentials from (6) are linearly independent over $C$. Assume again that there exists a linear combination

$$C_1 dz^q + C_1^{(1); \alpha_1} \tau_{q; P_1}^{(1); \alpha_1} + \cdots + C_1^{(\alpha_1); \alpha_1} \tau_{q; P_1}^{(\alpha_1); \alpha_1} + C_2^{(1); \alpha_1} \tau_{q; P_2}^{(1); \alpha_1} + \cdots + C_2^{(\alpha_1); \alpha_1} \tau_{q; P_2}^{(\alpha_1); \alpha_1} + \cdots + C_n^{(1); \alpha_1} \tau_{q; P_n}^{(1); \alpha_1} = 0,$$

such that not all its coefficients are zeroes. The coefficients $C_1^{(\alpha_1); \alpha_1} = \cdots = C_n^{(\alpha_1); \alpha_1} = 0$ and $C_2 = \cdots = C_n = 0$, since in the right hand side there are no singular points. So we have $C_1 dz^q = 0$, which implies $C_1 = 0$. Therefore the collection (6) of $q$-differentials is a base for the space

$$\Omega^q \left( \frac{1}{P_1^{\alpha_1} \cdots P_{l+1}^{\alpha_{l+1}} \cdots P_n^{\alpha_n}}; F_\mu \right) .$$

This prove the following theorem.

**Theorem 3.2.** The vector bundle $E_2 = \bigcup \Omega^q \left( \frac{1}{P_1^{\alpha_1} \cdots P_{l+1}^{\alpha_{l+1}} \cdots P_n^{\alpha_n}}; F_\mu \right)$ of rank $d_1 = \alpha_1 + \cdots + \alpha_l + n - l$ over $T_1$ is complex analytic equivalent to the direct product $T_1 \times \mathbb{C}^{d_1}$. Moreover, $q$-differentials of (6) give a base of locally holomorphic sections of this bundle over $T_1$, where $\alpha_1, \ldots, \alpha_l \geq 2$, $n \geq 1$, $0 \leq l \leq n$, and $q \in Z$.

**Remark 3.1.** For $q = 0$, $l = 0$ the collection (6) is $1, \tau_{0; P_2} = f_2, \ldots, \tau_{0; P_n} = f_n$ and is a base for the space $L \left( \frac{1}{F_1 \cdots F_n}; F_\mu \right)$, where $f_j$ are non-constant functions, and $(f_j) \geq \frac{1}{F_1 F_j}$, $j = 2, \ldots, n$, on $F_\mu$.

**Remark 3.2.** In particular, for $q = 1$ and for a fixed torus $F$ Corollary 1.1 and Theorems 2.1, 3.1, 3.2 imply classical theorems on abelian 1-differentials found in [1, 2].

**References**


Однозначные дифференциалы целого порядка на переменном торе

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В этой работе дано полное описание дивизоров элементарных дифференциалов всех родов. Получен аналог формулы Аппеля разложения однозначной функции на переменном торе. Исследованы основные типы векторных расслоений из мероморфных дифференциалов целого порядка над пространством Тейхмюллера для тора.

Ключевые слова: однозначные мероморфные дифференциалы целого порядка, дивизоры, векторные расслоения, пространство Тейхмюллера.