The Bergman and Cauchy–Szegö Kernels for Matrix Ball of the Second Type

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With the use of holomorphic automorphism of the matrix ball of the second type the validity of the integral Bergman and Cauchy–Szegö formulae is proven in this article.

Keywords: matrix ball, Bergman kernel, Cauchy–Szegö kernel, automorphism of the matrix ball.

10. Let us assume that $\mathbb{C}[m \times m]$ is the space of complex matrices of size $[m \times m]$. Direct multiplication of $n$ matrices is denoted by $\mathbb{C}^{n}[m \times m]$.

The set

$$B^{(1)}_{m,n} = \{ Z = (Z_1, ..., Z_n) \in \mathbb{C}^{n}[m \times m] : I^{(m)} - \langle Z, Z \rangle > 0 \}$$

is referred to as matrix ball of the first type (see [6]). Here $\langle Z, Z \rangle = Z_1Z_1^* + Z_2Z_2^* + ... + Z_nZ_n^*$ is the "dot" product, $I^{(m)}$ is the unit matrix of size $[m \times m]$, $Z_\nu^*$ is the conjugate transpose of matrix $Z_\nu$, $\nu = 1, 2, ..., n$, and $I - \langle Z, Z \rangle > 0$ means that a Hermitian matrix is positive definite that is all matrix eigenvalues are positive.

Matrix ball the second type $B^{(2)}_{m,n}$ has the following form (see [7]):

$$B^{(2)}_{m,n} = \{ Z = (Z_1, ..., Z_n) \in \mathbb{C}^{n}[m \times m] : I^{(m)} - \langle Z, Z \rangle > 0, \ Z_\nu^* = Z_\nu, \ \nu = 1, ..., n \} .$$

Let us denote the Shilov boundary of a matrix ball $B^{(2)}_{m,n}$ by $X^{(2)}_{m,n}$, that is,

$$X^{(2)}_{m,n} = \{ Z \in \mathbb{C}^{n}[m \times m] : \langle Z, Z \rangle = I, \ Z_\nu^* = Z_\nu, \ \nu = 1, 2, ..., n \} .$$

This domain was originally considered in [7] and a group of holomorphic automorphisms of $B^{(2)}_{m,n}$ was described. The purpose of this paper is to find kernels of the integral Bergman and Cauchy–Szegö formulae in the matrix ball of the second type. The integral Bergman formula for the matrix ball of the first type has been found in [6].

20. Let us consider a point $P = (P_1, P_2, ..., P_n) \in B^{(2)}_{m,n}$. Mapping

$$W_k = R^{-1}(I^{(m)} - \langle Z, P \rangle)^{-1} \sum_{s=1}^{n} (Z_s - P_s)G_{sk}, \ k = 1, ..., n, \ (1)$$

that transforms point $P$ into 0 is an automorphism of the matrix ball $B^{(2)}_{m,n}$ (see [7]). Here $R$ is a matrix of size $[m \times m]$ and $G$ is a block matrix of size $[m \times n]$. They satisfy the following relations

$$R'(I^{(m)} - P, P )G = I^{(m)}, \ G'(I^{(mn)} - P^*P)G = I^{(mn)}. \ (2)$$
Lemma 1. Real Jacobean $J_R$ of the mapping $W = \varphi_p(Z)$ at the point $Z = P$ is

$$
J_R \varphi_p = \left( \frac{\det(I^{(m)} - < P, P >)}{\det(I^{(m)} - < Z, P >)} \right)^\frac{m+1}{2}. 
$$

Proof. Let us find the real Jacobean $J_R$ of the mapping $W = \varphi_p(Z)$ at the point $Z = P$. It follows from (1) that

$$
dW = R^{-1}(I^{(m)} - < Z, P >)^{-1} \sum_{i=1}^n dZ_i P_i^* (I^{(m)} - < Z, P >)^{-1} \sum_{s=1}^n (Z_s - P_s) G_{sk} +
$$

$$
+ R^{-1}(I^{(m)} - < Z, P >)^{-1} \sum_{s=1}^n dz_s G_{sk}.
$$

Then applying the result of Theorem 2.1.2 from (see [2, p.37]), we find the real Jacobean of the mapping $\varphi_p$. The sign $\otimes$ means the Kronecker product of two matrices. Taking into consideration properties of the Kronecker product (see [3]) and using relation (2), we obtain

$$
\det \varphi'_p(P) = (\det R')^{\frac{m+1}{n}} (\det G')^{\frac{m+1}{n}}. 
$$

Then applying the result of Theorem 2.1.2 from (see [2, p.37]), we find the real Jacobian of the mapping $\varphi_p$. Since

$$
J_R \varphi_p = \left| \det \varphi'_p \right|^2
$$

then

$$
J_R \varphi_p(Z) = \det \frac{m+1}{n} (RR') \det \frac{m+1}{n} (G G') = \det \frac{(m+1)(n+1)}{2} (I^{(m)} - < Z, Z >). 
$$

Taking into account relations (2), we obtain

$$
\det(I^{(m)} - < W, W >) = \det(R^{-1}(I^{(m)} - < Z, P >)^{-1}) \det(I^{(m)} - < Z, Z >) \times
$$

$$
\times \det((I^{(m)} - < P, Z >)^{-1} R'^{-1}) = \frac{\det(I^{(m)} - < Z, Z >)}{\det((I^{(m)} - < Z, P >) R) \det(R' (I^{(m)} - < P, Z >))} =
$$

$$
= \frac{\det(I^{(m)} - < Z, P >) \det((I^{(m)} - < Z, P >) R) \det(R' (I^{(m)} - < P, Z >))}{\det(I^{(m)} - < Z, Z >) \det(I^{(m)} - < Z, P >) \det(I^{(m)} - < P, P >) \det(R R')}
$$

$$
= \left[ \begin{array}{c} I^{(m)} - < P, P > R^{-1} R^{-1} = (RR')^{-1} \\
\det(I^{(m)} - < P, P >) = \det(R R')^{-1} \\
\det(R R') = \det^{-1}(I^{(m)} - < P, P >) \end{array} \right].
$$
Mapping \( \psi_u = \varphi_W \circ \varphi_P \circ \varphi_Z^{-1} \) conserves 0. Therefore it is a generalized unitary mapping and the absolute value of the Jacobian determinant equals 1, i.e., \( \varphi_P = \varphi_W^{-1} \circ \psi_u \circ \varphi_Z \).

Then from relations (3) and (4) we obtain

\[
J_P \psi_P = \frac{\det(I^{(m+1)(n+1)} - <W, W>)}{\det(I^{(m+1)(n+1)} - <Z, W>)} = \left( \frac{\det(I^{(m+1)(n+1)} - <W, W>)}{\det(I^{(m+1)(n+1)} - <Z, W>)} \right)^{\frac{(m+1)(n+1)}{2}} = \left( \frac{\det(I^{(m+1)(n+1)} - <P, P>)}{\det(I^{(m+1)(n+1)} - <Z, P>)} \right)^{\frac{(m+1)(n+1)}{2}}.
\]

(5)

3°. Let us consider the normalized Lebesgue measures \( \nu \) in the ball \( B_{m,n}^{(2)} \) and \( \sigma \) on the Shilov boundary \( X_{m,n}^{(2)} \), i.e.

\[
\int_{B_{m,n}^{(2)}} d\nu(Z) = 1 \quad \text{and} \quad \int_{X_{m,n}^{(2)}} d\sigma(Z) = 1.
\]

Following the procedure given in [6] for \( B_{m,n}^{(2)} \), the Bergman kernel is defined as follows:

\[
K(Z, W) = \frac{1}{\det(I^{(m+1)(n+1)} - <Z, W>)}, \quad Z \in B_{m,n}^{(2)}.
\]

In particular, when \( n = 1 \), this kernel coincides with the Bergman kernel for the classical region of the second type (see [2]).

The Hilbert space of holomorphic functions in \( B_{m,n}^{(2)} \) that are square integrable with respect to Lebesgue measure \( d\nu \) is designated as \( H^2(B_{m,n}^{(2)}) \), i.e., \( f \in H^2(B_{m,n}^{(2)}) \) if \( f \) is a holomorphic in \( B_{m,n}^{(2)} \) function and

\[
\int_{B_{m,n}^{(2)}} |f(\zeta)|^2 d\nu(\zeta) < +\infty.
\]

\( L^2(X_{m,n}^{(2)}, d\mu) \) signifies the space of scalar functions \( f \) that are square integrable with respect to the normalized Haar measure \( d\mu \) on the Shilov boundary \( X_{m,n}^{(2)} \) of the matrix ball \( B_{m,n}^{(2)} \).

**Theorem 1.** For each function \( f \in H^1(B_{m,n}^{(2)}) \) the following relation is true

\[
f(Z) = \int_{B_{m,n}^{(2)}} f(W)K(Z, W)d\nu(W), \quad Z \in B_{m,n}^{(2)}, \ W \in X_{m,n}^{(2)}.
\]

Integral in this relation defines the orthogonal projection from space \( L^2(B_{m,n}^{(2)}) \) to space \( H^2(B_{m,n}^{(2)}) \).
Proof. Let us consider a point \( P \in B_{m,n}^{(2)} \). Let us assume first that the function \( f \in \mathcal{A}(B_{m,n}^{(2)}) \) (\( f \) is holomorphic function in \( B_{m,n}^{(2)} \) and it is continuous function on the closure \( B_{m,n}^{(2)} \)). Let us consider the following function

\[
g(Z) = \frac{K(Z, P)}{K(P, P)} f(Z).
\]

Then \( g \in \mathcal{A}(B_{m,n}^{(2)}) \) and

\[
f(P) = g(P) = (g \circ \varphi_P^{-1})(0). \tag{6}
\]

Expanding \( f \) in a series of homogeneous polynomials and integrating it over the ball, we obtain

\[
f(0) = \int_{B_{m,n}^{(2)}} f(W) d\nu(W).
\]

Taking into account this relation and relation (5) we have

\[
f(B) = \int_{B_{m,n}^{(2)}} g(\varphi_P^{-1}(W)) d\nu(W). \tag{7}
\]

After the change of variables \( \varphi_P^{-1}(W) = U \) in (7), we obtain

\[
f(P) = \int_{B_{m,n}^{(2)}} g(U) J_B \varphi_P d\nu(U) = \int_{B_{m,n}^{(2)}} f(U) K(P, U) d\nu(U).
\]

Due to the completeness of the matrix ball the space of functions \( \mathcal{A}(B_{m,n}^{(2)}) \) is dense in the space \( H^2(B_{m,n}^{(2)}) \). Then the theorem holds for functions \( f \in L^2(B_{m,n}^{(2)}) \). \( \square \)

4. Let us build the Cauchy–Szegö kernel for the matrix ball of the second type.

We define the Cauchy–Szegö kernel \( C(Z, W) \) as follows

\[
C(Z, W) = \frac{1}{\det \left( I^{(m)} - \langle Z, W \rangle \right)} , \quad Z \in B_{m,n}^{(2)}, \quad W \in X_{m,n}^{(2)}. \tag{8}
\]

At \( n = 1 \) the Cauchy–Szegö formula coincides with the Cauchy–Szegö kernel for the classical region of the second type [2].

This kernel is defined for all pairs \( (Z, W) \in C^{n}[m \times m] \times C^{n}[m \times m] \) such that the matrix

\[
I^{(m)} - \langle Z, W \rangle
\]

is not degenerate matrix. In particular, the kernel is defined for \( Z \in B_{m,n}^{(2)}, \ W \in X_{m,n}^{(2)} \).

The kernel \( C(Z, W) \) is a holomorphic function with respect to elements of the block matrix \( Z \) and it is an antiholomorphic function with respect to elements of the block matrix \( W \).

If \( f \in L^1(B_{m,n}^{(2)}) \) on \( X_{m,n}^{(2)} \) one can introduce the following integral

\[
C[f](Z) = \int_{X_{m,n}^{(2)}} C(Z, W)f(W) d\sigma(W), \quad Z \in B_{m,n}^{(2)}, \ W \in X_{m,n}^{(2)}. \tag{9}
\]

Let us designate \( C[f] \) as Cauchy integral with respect to \( f \). The operator that transforms \( f \) into \( C[f] \) we designate as Cauchy transform.

**Lemma 2.** Cauchy transform commutes with the action of the unitary group \( \psi_u \), namely,

\[
C[f \circ \psi_u] = (C[f]) \circ \psi_u, \quad f \in L^1(\sigma).
\]

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Proof. Let us show that the following equality is true

\[ C(Z, \psi_u^{-1}W) = C(\psi_u Z, W). \]  

(10)

In fact, \( UU^* = I^{(m)} \), \( VV^* = I^{(mn)} \) are unitary and block unitary matrices. Then we have

\[
C(Z, \psi_u^{-1}W) = \frac{1}{\det \left( I^{(m)} - \langle Z, \psi_u^{-1}W \rangle \right)} \quad = \quad \frac{1}{\det \left( I^{(m)} - \langle Z, U^{-1}WV^{-1} \rangle \right)} \
= \quad \frac{1}{\det \left( I^{(mn)} - Z^* \cdot U^*WV^* \right)} \quad = \quad \frac{1}{\det \left( I^{(mn)} - (UZV)^*W \right)} \
= \quad \frac{1}{\det \left( I^{(mn)} - (UZV)^* \right)} \quad = \quad C(\psi_u Z, W).
\]

Here we use the equality

\[ \det(I^{(m)} - \langle Z, W \rangle) = \det(I^{(mn)} - Z^* \cdot W), \]

which is true by virtue of Theorem 2.1.2 (see [2, p. 37]) for arbitrary \( Z = (Z_1, ..., Z_n) \) and \( W = (W_1, ..., W_n) \). Since the measure \( \sigma \) is invariant with respect to \( \psi_u \), then

\[
C[f \circ \psi_u] = \int_{X_{m,n}^{(2)}} C(Z, W)f(\psi_u W)d\sigma(W) = \int_{X_{m,n}^{(2)}} C(Z, \psi_u^{-1}W)f(W)d\sigma(W) = \int_{X_{m,n}^{(2)}} C(\psi_u Z, W)f(W)d\sigma(W) = (C[f] \circ \psi_u).
\]

\[ \square \]

Theorem 2. For each function \( f \in H^1(B_{m,n}^{(2)}) \) the following relation is true

\[ f(Z) = \int_{X_{m,n}^{(2)}} f(W)C(Z, W)d\sigma(W), Z \in B_{m,n}^{(2)}, W \in X_{m,n}^{(2)}. \]

(11)

Proof. Let us assume that \( f \in H^1(B_{m,n}^{(2)}) \) and \( Z \in B_{m,n}^{(2)} \). Let us express a point \( \zeta \in C^n[m \times m] \) as \( \zeta = (\zeta', \zeta_n) \), where \( \zeta' = (\zeta_1, ..., \zeta_{n-1}) \). By the lemma we can assume without loss of generality that \( Z_n = 0 \), i.e. \( Z = (Z', 0) \).

Let us introduce the following function

\[ g(\zeta) = C(Z, \zeta)f(\zeta), \quad \zeta \in B_{m,n}^{(2)}. \]

Because \( Z_n = 0 \) then the Cauchy–Szegö kernel in \( B_{m,n}^{(2)} \) coincides with the Bergman kernel \( B_{m,n}^{(2)} \):

\[ C(Z, \zeta) = K'(Z', \zeta). \]

Further, for any \( W \in X_{m,n}^{(2)} \) function \( g'(W, \zeta_n) \) is the holomorphic function with respect to \( \zeta_n \) in the matrix circle 5

\[ W_nW_n^* - \zeta_n\zeta_n^* > 0, \]

(12)

and it is continuous function in the closure of the circle.

Therefore, it follows from [2, c. 91] that

\[ g'(W, 0) = \int_{S_n} g'(W, W_n)d\sigma(W_n), \]

(13)
where $S_n$ is the Shilov boundary of matrix disk (12) and $d\sigma(W_n)$ is the invariant Haar measure on $S_n$. Let us integrate relation (13) over $B_m^{(2)}_{n-1}$.

According to Fubini's theorem, on the right-hand side we obtain

$$\int_{X^{(2)}_{m,n}} g\sigma(W) = C[f](Z).$$

Because $g'(W,0) = K(Z',W)f'(W,0)$ then it follows from Theorem 1 that the integral on the left-hand side of (13) is $f'(Z,0) = f(Z)$.

The theorem is proved.

\[\square\]

References


