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New Periodic Gibbs Measures for q-state Potts Model on a Cayley Tree

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In this paper under some conditions on parameters of the q-state Potts model on a Cayley tree of order k we prove existence of the periodic (non translation-invariant) Gibbs measures. Also we give a result about the number of such measures.

Keywords: Cayley tree, configuration, Potts model, Gibbs measure, periodic Gibbs measures, translation-invariant measures.

Introduction

The main problem for a given hamiltonian is the description of all corresponding limiting Gibbs measures (see f.e. [1, 3]). This problem was fully studied for the Ising model on the Cayley tree. For example, in [4] an uncountable set of extremal Gibbs measures is constructed and in [5] a necessity and sufficient condition of extremity of unordered phase for Ising model on a Cayley tree is found.

The Potts model is a generalization of the Ising model. The Potts model is not studied to the same extent as the Ising model. For example, in [6] a ferromagnetic Potts model with three-states on a second-order Cayley tree was considered and it was proved that there exists a critical temperature $T_c > 0$ such that for $T < T_c$, there are three translation-invariant and uncountably many not translation-invariant Gibbs measures. The results of [6] on the Potts model with finitely many states were generalized to a Cayley tree of an arbitrary (finite) order in [7].

It was proved [8] that the translation-invariant Gibbs measure of the antiferromagnetic Potts model with an external field is unique. In [9] the Potts model with a countable number of states and nonzero external field on a Cayley tree was considered. It is proved that this model has a unique translation-invariant Gibbs measure.

Other properties of the Potts model on a Cayley tree were studied in [10, p. 105–121]. In [11] it were showed that the Potts model (with an external field $\alpha \in R$) admits only periodic Gibbs measure of period two; it was considered the case $\alpha = 0$, and on the base of the same invariants, it was proved that all periodic Gibbs measures are neccessarily translation-invariant; it were found conditions under which the Potts model with a nonzero external field admits periodic (non translation-invariant) Gibbs measures. In [12] it was fully describe the set of translation-invariant Gibbs measures for the ferromagnetic q-state Potts model and it is proved that the number of translation-invariant measures can be up to $2^q - 1$. In [13] for q-state Potts model (with an external field $\alpha \in R$) on the Cayley tree of order $k = 3$ and $k = 4$ under some conditions on parameters it was proved existence of periodic (non translation-invariant) Gibbs measures of period two. In [14] a ferromagnetic Potts model (with zero external field $\alpha \in R$) on a Cayley tree

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tree of order \( k \geq 3 \) was studied and it was proved that there exists a critical temperature \( T_c \) such that for \( T < T_c \), there exist at least two of periodic (non translation-invariant) Gibbs measures.

In this paper under some conditions on parameters of the \( q \)-state Potts model on a Cayley tree of order \( k \geq 2 \) we shall prove existence of the periodic (non translation-invariant) Gibbs measures, and we give a lower bound for number of these measures.

1. Definitions and known facts

The Cayley tree \( \mathbb{I}^k \) of order \( k \geq 1 \) is an infinite tree, i.e., a graph without cycles, such that exactly \( k + 1 \) edges originate from each vertex. Let \( \mathbb{I}^k = (V, L, i) \), where \( V \) is the set of vertices \( \mathbb{I}^k \), \( L \) the set of edges and \( i \) is the incidence function setting each edge \( l \in L \) into correspondence with its endpoints \( x, y \in V \). If \( i(l) = \{x, y\} \), then the vertices \( x \) and \( y \) are called the nearest neighbors, denoted by \( l = \{x, y\} \). The distance \( d(x, y) \), \( x, y \in V \) on the Cayley tree is the number of edges of the shortest path from \( x \) to \( y \):

\[
d(x, y) = \min \{d | \exists x = x_0, x_1, \ldots, x_d = y \in V \text{ such that } \langle x_0, x_1 \rangle, \ldots, \langle x_{d-1}, x_d \rangle \}.
\]

For a fixed \( x^0 \in V \) we set \( W_n = \{x \in V \mid d(x, x^0) = n\} \),

\[
V_n = \{x \in V \mid d(x, x^0) \leq n\}, \quad L_n = \{l = \{x, y\} \in L \mid x, y \in V_n\}. \tag{1}
\]

It is known that there exists a one-to-one correspondence between the set of vertices \( V \) of the Cayley tree \( \mathbb{I}^k \) and the group \( G_k \) that is the free product of \( k + 1 \) cyclic groups of second order with the generators \( a_1, a_2, \ldots, a_{k+1} \).

We consider the model in which the spin variables take values in the set \( \Phi = \{1, 2, \ldots, q\} \), \( q \geq 2 \) and are located at the tree vertices. A configuration \( \sigma \) on \( V \) is then defined as a function \( x \in V \rightarrow \sigma(x) \in \Phi \); the set of all configurations coincides with \( \Omega = \Phi^V \).

The Hamiltonian of the Potts model is defined as

\[
H(\sigma) = -J \sum_{\{x, y\} \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2}
\]

where \( J \in \mathbb{R}, \{x, y\} \) are nearest neighbors and \( \delta_{ij} \) is the Kronecker symbol: \( \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \).

Define a finite-dimensional distribution of a probability measure \( \mu \) in the volume \( V_n \) as

\[
\mu_n(\sigma_n) = Z_n^{-1} \exp \left\{ -\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x} \right\}, \tag{3}
\]

where \( \beta = 1/T, \ T > 0 \) is temperature, \( Z_n^{-1} \) is the normalizing factor, \( \{h_x = (h_{1,x}, \ldots, h_{q,x}) \in \mathbb{R}^q, x \in V\} \) is a collection of vectors and

\[
H_n(\sigma_n) = -J \sum_{\{x, y\} \in L_n} \delta_{\sigma(x)\sigma(y)}
\]

is the restriction of Hamiltonian on \( V_n \).

We say that the probability distributions (3) are compatible if for all \( n \geq 1 \) and \( \sigma_{n-1} \in \Phi^{V_{n-1}} \):

\[
\sum_{\omega_n \in \Phi^{V_n}} \mu_n(\sigma_{n-1} \lor \omega_n) = \mu_{n-1}(\sigma_{n-1}). \tag{4}
\]
Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations. In this case, there exists a unique measure $\mu$ on $\Phi^V$ such that, for all $n$ and $\sigma_n \in \Phi^V$,

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such a measure is called a splitting Gibbs measure corresponding to the Hamiltonian (2) and vector-valued function $h_x, x \in V$.

The following statement describes conditions on $h_x$, guaranteeing compatibility of $\mu_n(\sigma_n)$.

**Theorem 1** (§8). The probability distributions $\mu_n(\sigma_n), n = 1, 2, \ldots$ in (3) are compatible for Potts model iff, for any $x \in V$ the following equation holds:

$$h_x = \sum_{y \in S(x)} F(h_y, \theta),$$

where $F : h = (h_1, \ldots, h_{q-1}) \in R^{q-1} \rightarrow F(h, \theta) = (F_1, \ldots, F_{q-1}) \in R^{q-1}$ is defined as

$$F_i = \ln \left( \frac{(\theta - 1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}} \right)$$

and $\theta = \exp(J\beta), S(x)$ is the set of direct successors of $x$ and $h_x = (h_{1,x}, \ldots, h_{q-1,x})$ with

$$h_{i,x} = h_{i,x} - h_{q,x}, \quad i = 1, \ldots, q - 1.$$ 

Let $\hat{G}_k$ be a subgroup of the group $G_k$.

**Definition 1.** The set of vectors $h = \{h_x, x \in G_k\}$ is said to be $\hat{G}_k$-periodic if $h_{yx} = h_x$ for all $x \in G_k, y \in \hat{G}_k$.

The $G_k$-periodic sets are said to be translation-invariant.

**Definition 2.** The measure $\mu$ is said to be $\hat{G}_k$-periodic if it corresponds to the $\hat{G}_k$-periodic set of vectors $h$.

The following theorem characterizes periodic Gibbs measures.

**Theorem 2** ([11]). Let $K$ be a normal divisor of finite index in the group $G_k$. Then for the Potts model, all $K$-periodic Gibbs measures are either $G^{(2)}_k$-periodic or translation-invariant, where $G^{(2)}_k = \{x \in G_k : \text{the length of } x \text{ is even}\}$.

## 2. Periodic Gibbs measures

We consider case $q \geq 3$, i.e. $\sigma : V \rightarrow \Phi = \{1, 2, 3, \ldots, q\}$. By Theorem 2, we have only $G^{(2)}_k$-periodic Gibbs measures corresponding to the sets of vectors $h = \{h_x \in R^{q-1} : x \in G_k\}$ of the form

$$h_x = \begin{cases} h, & \text{if } |x| \text{ is even}, \\ l, & \text{if } |x| \text{ is odd}. \end{cases}$$
Here \( h = (h_1, h_2, ..., h_{q-1}) \), \( l = (l_1, l_2, ..., l_{q-1}) \). From equality (5), we then obtain

\[
\begin{cases}
    h_i = k \ln \frac{(\theta - 1) \exp(l_i) + \sum_{j=1}^{q-1} \exp(l_j) + 1}{\sum_{j=1}^{q-1} \exp(l_j) + 1}, \\
    l_i = k \ln \frac{(\theta - 1) \exp(h_i) + \sum_{j=1}^{q-1} \exp(h_j) + 1}{\sum_{j=1}^{q-1} \exp(h_j) + 1},
\end{cases}
\]

for \( i = 1, q - 1 \).

We introduce the notations \( \exp(h_i) = x_i, \exp(l_i) = y_i \). We can then rewrite the last system of equations for \( i = 1, q - 1 \) as

\[
\begin{cases}
    x_i = \left( \frac{(\theta - 1)y_i + \sum_{j=1}^{q-1} y_j + 1}{\sum_{j=1}^{q-1} y_j + 1} \right)^k, \\
    y_i = \left( \frac{(\theta - 1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + 1} \right)^k.
\end{cases}
\]

Remark 1. 1. In the case \( q = 2 \), the Potts model coincides with the Ising model which was studied in [8].

2. In the case \( k = 2, q = 3 \) and \( J < 0 \), it was proved that all \( G_k^{(2)} \)-periodic Gibbs measures on base of invariant \( I = \{(x_1, x_2, y_1, y_2) \in R^4 : x_1 = x_2, y_1 = y_2 \} \) are translation-invariant (see [11]).

3. In the case \( k \geq 1, q = 3 \) and \( J > 0 \), it was proved that all \( G_k^{(2)} \)-periodic Gibbs measures are translation-invariant (see [11]).

For \( q \geq 3, 0 < \theta < 1, k \geq 3 \), we define

\[ I_m = \{ z = (u, v) \in R^{q-1} \times R^{q-1} : x_i = x, y_i = y, i = \overline{1, m}; x_i = y_i = 1, i = \overline{m+1, q-1} \}, \]

i.e. \( u = (x, x, ..., x, 1, 1, ..., 1) \), \( v = (y, y, ..., y, 1, 1, ..., 1) \) and

\[ I_m’ = \{ z = (u, v) \in R^{q-1} \times R^{q-1} : x_i = x, i = \overline{1, m}; x_i = 1, i = \overline{m+1, q-1 - m}; \]

\[ x_i = y, i = q - m, q - 1; y_i = y, i = \overline{1, m}; y_i = 1, i = \overline{m+1, q - 1 - m}; y_i = x, i = q - m, q - 1 \}, \]

i.e. \( u = (x, x, ..., x, 1, 1, ..., 1, y, y, ..., y) \), \( v = (y, y, ..., y, 1, 1, ..., 1, x, x, ..., x) \). Here \( 2m \leq q - 1 \).
We consider the map \( W : \mathbb{R}^{q-1} \times \mathbb{R}^{q-1} \to \mathbb{R}^{q-1} \times \mathbb{R}^{q-1} \), defined as

\[
\begin{align*}
  x_i' &= \left( \frac{(\theta - 1)y_i + \sum_{j=1}^{q-1} y_j + 1}{\sum_{j=1}^{q-1} y_j + \theta} \right)^k, \\
y_i' &= \left( \frac{(\theta - 1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + \theta} \right)^k.
\end{align*}
\]

We note that the system (6) is the equation \( z = W(z) \). Solving the system (6) is therefore equivalent to finding fixed points of the map \( z = W(z) \), where \( z = (u,v), \; z' = (u',v') \).

**Lemma 1.** Sets \( I_m \) and \( I_m' \) are invariant subsets relatively to the map \( W \).

The proof is similar to that of Lemma 2 in [11].

**The case \( I_m' \).** In the case we rewrite the system (6) as

\[
\begin{align*}
x &= \left( \frac{\theta y + (m-1)y + (q-m)}{\theta + my + (q-m-1)} \right)^k, \\
y &= \left( \frac{\theta x + (m-1)x + (q-m)}{\theta + mx + (q-m-1)} \right)^k
\end{align*}
\]

or

\[
\begin{align*}
x &= f^k(y), \\
y &= f^k(x), \text{ where } f(x) &= \left( \frac{\theta x + (m-1)x + (q-m)}{\theta + mx + (q-m-1)} \right).
\end{align*}
\]

and \( f^k(x) \) is \( k \)-power of function \( f(x) \).

**Remark 2.** Let \( \pi \in S_{q-1} \) be a permutation. We shall define the action of \( \pi \) to the vector \( x = (x_1, x_2, \ldots, x_{q-1}) \) as \( \pi(x) = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(q-1)}) \). Then \( \pi(A) = \{(\pi(x), \pi(y)) : (x,y) \in A\} \), where \( A = I_m \) or \( I_m' \) is also invariant subset relatively to the map \( W \) but in cases \( \pi(I_m) \) and \( \pi(I_m') \) corresponding system of equations coincides with (7) and (9) (see below), respectively. Therefore without loss of generality, we can consider sets \( I_m \) or \( I_m' \).

**Proposition 1.** Let \( k \geq 3, \; 3 \leq q < k + 1, \; \theta_{cr} = \frac{k - q + 1}{k + 1} < 1 \). Then system of equations (6) on \( I_m \) has at least three solutions for \( 0 < \theta < \theta_{cr} \), it has at least one solution for \( \theta = \theta_{cr} \) and it has only one solution for \( \theta > \theta_{cr} \).

**Proof.** By (8) we obtain

\[
x = g(x) = f^k(f^k(x)).
\]

We have

\[
f'(x) = \frac{(\theta - 1)(\theta + q - 1)}{(\theta + my + q - m - 1)^2};
\]

\[
g'(x) = k^2 f^{k-1}(f^k(x)) f'(f^k(x)) f^{k-1}(x) f'(x).
\]

Consequently, for \( 0 < \theta < 1 \) the function \( f(x) \) decreases monotonically and the equation \( f(x) = x \) has a unique solution \( x = 1 \) such that \( f'(1) = \frac{\theta - 1}{\theta + q - 1} \). We note that \( g(x) \) is increasing and \( x = 1 \) is a solution to \( g(x) = x \). If \( g'(1) = k^2(f'(1))^2 = \left( k \frac{\theta - 1}{\theta + q - 1} \right)^2 > 1 \) then this
solution is not unique, because in this case for \( x > 1 \) the graph of the function \( g \) lies above the bisector and the function \( g \) is bounded. Thus a critical value for \( \theta \) can be found by the equation
\[
\left( k \frac{\theta - 1}{\theta + q - 1} \right)^2 = 1 \quad \text{which for } \theta < 1 \text{ gives } \theta_{cr} = \frac{k - q + 1}{k + 1}.
\]
Hence it follows that for \( 0 < \theta < \theta_{cr} \)
the equation \( g(x) = x \) has at least three solutions \( x_0^* < x_1^* < 1 < x_2^* \), i.e. the equation \( g(x) = x \)
has at least two roots, which are distinct from roots of the equation \( f(x) = x \). For \( \theta = \theta_{cr} \)
the graph of the function \( g \) tangents to the bisector in \( x = 1 \). This means that in this condition the
equation \( g(x) = x \) has at least one solution. Besides it is clear that for \( \theta > \theta_{cr} \) the equation
\( g(x) = x \) has a unique solution \( x_1^* = 1 \), which it is solution to equation \( f(x) = x \).

**The case** \( I_m^* \). We consider the set \( I_m^* \). We rewrite the system of equations (6) on this set as
\[
\begin{align*}
\begin{cases}
x = & \left( \frac{(\theta - 1)y + my + (q - 2m - 1) + mz + 1}{\theta + mz + my + (q - 2m - 1)} \right)^k, \\
y = & \left( \frac{(\theta - 1)x + mx + (q - 2m - 1) + my + 1}{\theta + mz + my + (q - 2m - 1)} \right)^k,
\end{cases}
\end{align*}
\]
where \( \exp(h_i) = x_i, \ \exp(l_i) = y_i \).

**Remark 3.** 1. For \( m = 0 \) we obtain \( u = (1, 1, ..., 1), \ v = (1, 1, ..., 1) \), which corresponds to the
translation-invariant Gibbs measure. Thus we consider the case \( m \geq 1 \).

2. In the case \( k = 2, \ q = 3, \ m = 1 \) on \( I_m^* \) it was proved that all \( G_{k}^{(2)} \)-periodic Gibbs measures are
translation-invariant (see [11]).

In the last system substituting \( \sqrt{x} = z, \ \sqrt{y} = t \), we obtain
\[
\begin{align*}
\begin{cases}
z = & \frac{(\theta + m - 1)t^k + mz^k + q - 2m}{\theta + mz^k + mt^k + q - 2m - 1}, \\
t = & \frac{(\theta + m - 1)z^k + mt^k + q - 2m}{\theta + mz^k + mt^k + q - 2m - 1}.
\end{cases}
\end{align*}
\]
From the first equation of (10) we find \( t^k, t \):
\[
t^k = \frac{mz^{k+1} - mz^k + (\theta + q - 2m - 1)z - q + 2m}{\theta + m - 1 - mz},
\]
\[
t = \left( \frac{mz^{k+1} - mz^k + (\theta + q - 2m - 1)z - q + 2m}{\theta + m - 1 - mz} \right)^{\frac{1}{k}}
\]
and substitute to the second equation of (10). Then we obtain
\[
f(z) = [(\theta + 2m - 1)z^k - mz^{k+1} + mz + q - 2m](\theta + m - 1 - mz) -
-(mz^k + q - m - 1 + \theta)(mz^{k+1} - mz^k + (\theta + q - 2m - 1)z - q + 2m) = 0.
\]

We consider the function \( f(z) \). We note that \( f(0) = (q-2m)^k(\theta+m-1) + (q-2m)(\theta+q-m-1) > 0 \)
for \( 2m < q \). Besides \( f(1) = 0 \) and \( f(z) \rightarrow -\infty \) for \( z \rightarrow +\infty \). Consequently it is clear that if
\( f'(1) > 0 \), then the equation (11) has at least three solutions. Therefore we consider
\[
f'(1) = (k^2 - 1)s^2 - 2qs - q^2 = (k^2 - 1) \left( s - \frac{q}{k-1} \right) \left( s + \frac{q}{k+1} \right) > 0,
\]
where \( s = \theta - 1 < 0 \). Consequently, if \( s + \frac{q}{k+1} < 0 \), i.e. \( 0 < \theta < 1 - \frac{q}{k+1} = \theta_{cr} \), then \( f'(1) > 0 \).

Thus we proved the following
Proposition 2. Let $k \geq 3$, $3 \leq q < k + 1$, $\theta_{cr} = \frac{k - q + 1}{k + 1} < 1$. Then the system of equations (6) on $I_m'$:
1) for $0 < \theta < \theta_{cr}$ has at least three solutions;
2) for $\theta = \theta_{cr}$ has at least one solution;
3) for $\theta > \theta_{cr}$ has only one solution.

Remark 4. It is clear that in Propositions 1 and 2 one of measures corresponds to the solution $x_1^* = 1$ which is translation-invariant, the remaining measures are $G_k^{(2)}$-periodic (non-translation-invariant), and in case $\theta > \theta_{cr}$ the measure corresponding to the unique solution $x_1^* = 1$.

Similarly as in [12, p. 6], it is easy to show that for $0 < \theta < \theta_{cr}$ on each $I_m$ and $I_m'$, where $m = 1, 2, ..., q$, the number of $G_k^{(2)}$-periodic (non-translation-invariant) Gibbs measures is not less than $2 \cdot \binom{q}{m}$ and $2 \cdot \binom{q}{m} \cdot \binom{q-m}{m}$, respectively. Consequently, the number on $\bigcup_{m=1}^q I_m$ and $\bigcup_{m=1}^q I_m'$ is not less than
$$2 \cdot \sum_{m=1}^q \binom{q}{m} = 2^{q+1} - 2,$$ and $$2 \cdot \sum_{m=1}^{\lfloor q/2 \rfloor} \binom{q}{m} \cdot \binom{q-m}{m},$$ respectively.

Thus we have the

Theorem 3. For $k \geq 3$, $3 \leq q < k + 1$ and $0 < \theta < \theta_{cr}$ for the Potts model exist at least
$$2 \cdot \left(2^q - 1 + \sum_{m=1}^{\lfloor q/2 \rfloor} \binom{q}{m} \cdot \binom{q-m}{m}\right)$$
$G_k^{(2)}$-periodic (non translation-invariant) Gibbs measures.

Remark 5. In [12] the number and the description of all translation-invariant Gibbs measures for the Potts model were given.

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References


Новые периодические меры Гиббса для модели Поттса с $q$-состояниями на дереве Кэли

Рустамжон М. Хакимов

В данной статье изучается модель Поттса с $q$-состояниями на дереве Кэли порядка $k$ и показано существование периодических (не трансляционно-инвариантных) мер Гиббса при некоторых условиях на параметры этой модели. Кроме того, указана нижняя граница количества существующих периодических мер Гиббса.

Ключевые слова: дерево Кэли, конфигурация, модель Поттса, мера Гиббса, периодические меры, трансляционно-инвариантные меры.