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Limit Theorem for the Joint Distribution in the Q-processes

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The Q-process is considered in this paper. A link between the Q-process and the Galton-Watson branching process allowing immigration is established in the paper. Due to this link the limit theorem on the joint distribution of the population size and the total state of the Q-process is proved.

Keywords: Galton-Watson branching process; immigration; total state of the Q-process; limit theorem.

1. Preliminaries

We consider the so-called Q-processes. The Q-process is defined by the Galton-Watson process (GWP) conditioned on non-extinction of its trajectory in the distant future. We show that the Q-process may be replaced by some GWP allowing immigration (GWPI). We investigate asymptotic properties of the joint distribution of the population size and the total state in Q-processes.

1.1. On simple GWP. Let us consider a GWP. Let Z_n, $n \in \mathbb{N}_0$ ($\mathbb{N}_0 = \{0\} \cup \{\mathbb{N} = 1, 2, \ldots\}$), be the number of individuals in the nth generation defined recursively as

$$Z_0 = 1, \quad Z_n = \sum_{k=1}^{Z_{n-1}} \zeta_{n,k},$$

where independent and identically distributed (i.i.d.) random variables $\zeta_{n,k}$ denote the offspring of k-th individual in the $(n-1)$th generation. Let $\mathbf{P}\{Z_1 = k \in \mathbb{N}_0\} =: p_k$ be an offspring law of the single individual and $p_0 > 0$, $p_0 + p_1 \neq 1$. According to the branching condition the evolution law of GWP is regulated by the probability generating function (GF) $F(s) := \sum\limits_{k\in\mathbb{N}_0} p_k s^k$, $|s| < 1$ and $F_n(s) := \mathbf{E}s^{Z_n}$ is determined by the n-step iteration of $F(s)$. In this interpretation $A := F'(1) = \mathbf{E}\zeta_{nk}$ is the mean per capita number of offsprings [see, e.g., 1, pp. 1–2].

We know that when $A < 1$ and $A = 1$ the GWP is die out asymptotically. Accordingly, in these cases the properties of GWP are investigated on nonzero trajectories. In this context we recall the following theorem on joint distribution of Z_n and $Y_n := \sum\limits_{k=0}^{n-1} Z_k$. The variable Y_n denotes the total number of offsprings of single individual until time n in GWP.

Theorem 1.1 (see, e.g., [11, p. 143])**.** *Let $A = 1$ and $F''(1) < \infty$. Then two-dimensional random vector $(Z_n/\mathbf{E}[Z_n | Z_n > 0]; Y_n/\mathbf{E}[Y_n | Z_n > 0])$ weakly converges to the unique vector $(Z^*; Y^*)$ as $n \to \infty$ and the Laplace transform for $(Z^*; Y^*)$ is of the form*

$$\mathbf{E}\left[e^{-\lambda Z^* - \theta Y^*}\right] = \left[\frac{sh2\sqrt{\theta}}{2\sqrt{\theta}} + \lambda\left(\frac{sh\sqrt{\theta}}{\sqrt{\theta}}\right)^2\right]^{-1}, \quad \lambda, \theta \geqslant 0.$$

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The limit Laplace transform has been also obtained by D. Kennedy [10] in the study of the behaviour of Z_n conditioned on the event $\{Y := \lim\limits_{m\to\infty} Y_m = n\}$ as $n \to \infty$. Setting $\lambda = 0$ gives the well-known conditional limit law for $2Y_n/F''(1)n$ derived by A. Pakes [13, Theorem 4].

This theorem also contains (at $\theta = 0$) the well-known A. Yaglom's result [19]. According to that result, limit of conditional distribution

$$S_{n,\,m}(x) := \mathbf{P}\left\{\frac{2Z_n}{F''(1)n} \leqslant x \mid Z_{n+m} > 0\right\}$$

exists as $n \to \infty$ for any $m \in \mathbb{N}_0$, and $\lim\limits_{n\to\infty} S_{n,\,m}(x) = 1 - e^{-x}$, $x > 0$.

1.2. On Q-processes and main result. It was shown that the use of the conditioning of not-extinction in the distant future $Z_\infty > 0$ instead of fundamental conditioning $Z_n > 0$ gives a new limit distribution law which is distinct from classical one (conditioning of Z_n under $Z_\infty > 0$ means $\mathbf{P}\{Z_n \,|Z_\infty > 0\} = \lim\limits_{m\to\infty} \mathbf{P}\{Z_n \,|Z_{n+m} > 0\}$). T. Harris [3] pointed out that at the same conditions as in Theorem 1.1 the following result holds:

$$\lim_{n\to\infty}\lim_{m\to\infty} S_{n,m}(x) = 1 - e^{-x} - xe^{-x},\ \ x > 0.$$

Later the condition $Z_\infty > 0$ was treated by J. Lamperti and P. Ney [12], K. Athreya and P. Ney [1, pp. 56–60], A. Pakes [13, 15, 17], A. Imomov [5, 6, 8], Sh. Formanov and A. Imomov [2]. Continuous time case was discussed in [7, 9, 18].

The stochastic process $\{W_n,\ n \in \mathbb{N}_0\}$ defined by GWP under conditioning of $Z_\infty > 0$ is called the Q-process in [1, pp. 56]. In fact the Q-process $\{W_n,\ n \in \mathbb{N}_0\}$ is homogeneous Markov chain with zero state $W_0 = 1$ and it is given by transition probabilities

$$\begin{aligned} Q_{ij}^{(n)} &= \mathbf{P}\{W_{n+k} = j \mid W_k = i\} = \\ &= \mathbf{P}\{Z_{n+k} = j \mid Z_k = i,\ Z_\infty > 0\}, \end{aligned}$$

for $n, i, j, k \in \mathbb{N}$. As it was proved in [1, pp. 56–58] these probabilities are of the form

$$Q_{ij}^{(n)} = \frac{jq^{j-i}}{i\beta^n}\mathbf{P}\{Z_{n+k} = j \mid Z_k = i\}, \tag{1.1}$$

where q is an extinction probability of GWP for which $q = F(q)$ and $\beta := F'(q)$.

Further we need the GF $W_n^{(i)}(s) := \sum\limits_{j\in\mathbb{N}} Q_{ij}^{(n)} s^j$. Taking into account the branching property of GWP and (1.1), we have the following relation:

$$W_n^{(i)}(s) = \left[\frac{F_n(qs)}{q}\right]^{i-1} W_n(s), \tag{1.2}$$

where the GF $W_n(s) := W_n^{(1)}(s) = \mathbf{E}\left[s^{W_n} \,|W_0 = 1\right]$ is

$$W_n(s) = s\frac{F_n'(qs)}{\beta^n}, \quad n \in \mathbb{N}. \tag{1.3}$$

We see that the Q-process $\{W_n,\ n \in \mathbb{N}_0\}$ can be defined by GF $W(s) = W_1(s)$. Assume that $\alpha := W'(1)$ has a finite value. We find out that $\alpha = 1 + qF''(q)/\beta > 1$.

We study a limit of a joint distribution of the Q-process and its total state for the case $A = 1$ (critical case for GWP). By the total state in the Q-process we mean the variable

$$S_n := W_0 + W_1 + \cdots + W_{n-1},\ \ S_0 = 0.$$

Theorem 1.2. *Let $A = 1$. Then the random vector $(W_n/\mathbf{E}W_n\,;\ S_n/\mathbf{E}S_n)$ weakly converges to a random vector $(W^*; S^*)$ as $n \to \infty$. The vector has the following Laplace transform*

$$\mathbf{E}\left[e^{-\lambda W^* - \theta S^*}\right] = \left[ch\sqrt{\theta} + \frac{\lambda}{2}\frac{sh\sqrt{\theta}}{\sqrt{\theta}}\right]^{-2}, \quad \lambda, \theta \geqslant 0.$$

We note that the same limit Laplace transform appears in paper of A. Pakes [17] but in the different context.

Further we use the joint GF

$$J_n(s;x) := \sum_{i\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \mathbf{P}\left\{W_n = i,\ S_n = k\right\} s^i x^k, \quad (s;x) \in \mathbb{D}, \tag{1.4}$$

on the set $\mathbb{D} = \left\{|s| \leqslant 1,\ |x| \leqslant 1 :\ \sqrt{(s-1)^2 + (x-1)^2} \geqslant r > 0\right\}$.

In Section 2 we establish a link between the Q-processes and GWPI. This link allows us to find out the necessary relations for GF $J_n(s;x)$. In Section 3 we discuss several preliminary results on properties of GFs when $A = 1$. These results are used to prove the Theorem 1.2 in Section 4.

2. Q-processes as GWPI

By iterating $F(s)$, the GF (1.3) may be written as

$$W_n(s) = s\prod_{k=0}^{n-1} G\left(\frac{F_k(qs)}{q}\right), \tag{2.1}$$

with $G(s) = F'(qs)/\beta$. It is easy to see that the following random sum of random variables is comparable with GF (2.1):

$$W_0 = 1, \quad W_{n+1} = 1 + \sum_{k=1}^{W_n - 1} \xi_{n+1,\,k} + \eta_{n+1}, \tag{2.2}$$

where $\xi_{n,k}$ are i.i.d. random variables with common GF $F\,(qs)/q$ for all n and k. Variables η_n are i.i.d. random variables with $\mathbf{E}s^{\eta_n} = G(s)$. Then we can conclude that the Q-process may be replaced by the following branching process. In the beginning there is one particle. The evolution process is initiated by the stream of the immigrating particles. The emergence intensity law is described by GF $G(s)$. The immigrating particles in prospect undergo a transformation according to the GF $F\,(qs)/q$. In addition, the initial particle does not disappear and does not breed. This "immortal particle" is present throughout the evolution of the process.

Upon introducing $\overline{W}_n = W_n - 1$, relation (2.2) is written in the form

$$\overline{W}_n = \sum_{k=1}^{\overline{W}_{n-1}} \xi_{n,k} + \eta_n. \tag{2.2*}$$

One can see that the sequence $\left\{\overline{W}_n,\ n \in \mathbb{N}_0\right\}$ is nothing but the GWPI with $\overline{W}_0 = 0$ and transition probabilities $\overline{Q}_{ij}^{(n)} := \mathbf{P}\left\{\overline{W}_{n+k} = j \,\middle|\, \overline{W}_k = i\right\} = Q_{i+1,j+1}^{(n)}$. Here $\overline{Z}_{n+1} = \sum\limits_{k=1}^{\overline{Z}_n} \xi_{n,k}$ is the "internal" GWP that obeys the GF $\mathbf{E}s^{\xi_{nk}} = F\,(qs)/q$ and arrival intensity of immigranting

particles is regulated by GF $G(s)$. We refer the reader to C. Heatcote [4] and A. Pakes [14, 16] on further details regarding the GWPI with general GF $G(s)$.

It is obvious that relations similar to relations (1.2) take place for the GF $\overline{W}_n^{(i)}(s) := \sum\limits_{j\in\mathbb{N}} \overline{Q}_{ij}^{(n)} s^j$. Thus the analysis of asymptotic properties of Q-processes may be reduced to studying of corresponding properties of GWPI. Moreover the GWPI $\left\{\overline{W}_n,\ n\in\mathbb{N}_0\right\}$ may be not supercritical. *Indeed*, the mean number of the single individual offspring in "internal process" is

$$\mathbf{E}\xi_{nk} = \frac{\partial}{\partial s}\left[\frac{F(qs)}{q}\right]_{s=1} = \begin{cases} 1 & , \quad A=1, \\ \beta<1, & A\neq 1. \end{cases} \tag{2.3}$$

The variable $Y_n := \sum\limits_{k=0}^{n-1} \overline{Z}_k$ denotes the total progeny in the process $\left\{\overline{Z}_n,\, n\in\mathbb{N}_0\right\}$ up to time n. Let us define a joint GF $H_n(s;x) := \mathbf{E}s^{\overline{Z}_n}x^{\overline{Y}_n},\ (s;x)\in\mathbb{D}$. Following the reasoning given in [13], the following relations hold for the GF $H_n(s;x)$:

$$\begin{aligned} H_0(s;x) &= s, \\ H_{n+1}(s;x) &= x\frac{F\left(qH_n(s;x)\right)}{q}. \end{aligned} \tag{2.4}$$

The variable $\overline{S}_n := \sum\limits_{k=0}^{n-1} \overline{W}_k$ represents the total progeny in the GWPI, defined by relation (2.2*). We see that the joint GF $\overline{J}_n(s;x) := \mathbf{E}s^{\overline{W}_n}x^{\overline{S}_n},\ (s;x)\in\mathbb{D}$ has the form

$$\overline{J}_n(s;x) = \prod_{k=0}^{n-1} G\left(H_k(s;x)\right), \tag{2.5}$$

where GF $G(s)$ is given in (2.1) and $H_n(s;x)$ satisfies equations (2.4) (see also [17]).

Considering (2.2) and (2.2*) we see that $\overline{S}_n = S_n - n$ and, therefore, $J_n(s;x) = sx^n\overline{J}_n(s;x)$. Now using (2.4) and (2.5), we obtain the following representation for GF $J_n(s;x)$ defined by (1.4):

$$J_n(s;x) = s\prod_{k=0}^{n-1}\left[\frac{xF'\left(qH_k(s;x)\right)}{\beta}\right]. \tag{2.6}$$

3. Some discussion on generating functions in the case A = 1

From now on we consider the case $A=1$. In this case relation (2.6) becomes

$$J_n(s;x) = s\prod_{k=0}^{n-1}\left[xF'\left(H_k(s;x)\right)\right], \tag{3.1}$$

where $H_n(s;x) = \mathbf{E}s^{Z_n}x^{Y_n}$. We have

$$\left.\frac{\partial J_n(s;x)}{\partial s}\right|_{(s;x)=(1;1)} = \mathbf{E}W_n \qquad \& \qquad \left.\frac{\partial J_n(s;x)}{\partial x}\right|_{(s;x)=(1;1)} = \mathbf{E}S_n.$$

Using direct differentiation, it can be found from (3.1) that

$$\mathbf{E}W_n = (\alpha-1)n+1 \qquad \& \qquad \mathbf{E}S_n = \frac{\alpha-1}{2}n(n+1)+n, \tag{3.2}$$

where, as before, $\alpha = W'(1) = 1 + F''(1)$.

Further we use the GF $\Delta_n(s;x) := h(x) - H_n(s;x),\ n \in \mathbb{N}_0$, where $h(x) = \mathbf{E}x^Y$ is the GF of $Y = \lim\limits_{n\to\infty} Y_n$. This variable denotes the total number of particles participating in process $\{Z_n,\ n \in \mathbb{N}_0\}$ for the duration of its evolution. Since the process dies out with probability 1 in the discussed case, the variable Y always exists.

By virtue of $\mathbf{P}\{Z_n > 0\} = O\,(1/n)$ as $n \to \infty$ (see [1, p.11]) we have

$$\operatorname*{Sup}_{(s;x)\in\mathbb{D}} |\Delta_n(s;x)| \to 0, \quad n \to \infty. \tag{3.3}$$

Using arguments given in [11, p. 127, Lemma 3], we obtain

$$|\Delta_k(s;x)| \leqslant |\Delta_j(s;x)|, \tag{3.4}$$

for all $k \in \mathbb{N}_0$ and $j = 0, 1, \ldots, k$.

We also know that for $\Delta_n(s;x)$ the following asymptotic expansion holds:

$$\frac{1}{\Delta_n(s;x)} = \frac{1}{u^n(x)}\left[\frac{1}{\Delta_0(s;x)} + \frac{b(x)\,[1-u^n(x)]}{1-u(x)} + \sum_{k=1}^{n} \varepsilon_k(s;x)u^k(x)\right], \tag{3.5}$$

where

$$u(x) := xF'\,(h(x))\,, \quad b(x) := \frac{F''\,(h(x))}{2F'\,(h(x))},$$

and $\operatorname*{Sup}\limits_{(s;x)\in\mathbb{D}} |\varepsilon_n(s;x)| \leqslant \varepsilon_n \to 0,\ n \to \infty$ (see [11, p. 136].

Further we consider the behavior of $h(x)$ and $u(x)$ in a neighborhood of $x = 1$. It is known [11, p. 126] that

$$1 - h(x) \sim \sqrt{2(1-x)/F''(1)}\,, \ \ x \to 1. \tag{3.6}$$

On the other hand, we have

$$F'(h(x)) \sim 1 - F''(1)(1-h(x)), \ \ x \to 1 \tag{3.7}$$

by means of Taylor expansion. Considering relations (3.6) and (3.7) together, we obtain

$$u(x) \sim 1 - \sqrt{2F''(1)(1-x)}, \ \ x \to 1. \tag{3.8}$$

4. Proof of the Theorem 1.2

We follow the method proposed by A. Pakes [14]. Let $\Psi_n(\lambda;\theta),\ \lambda,\theta > 0$, be the Laplace transform of variable $(W_n/\mathbf{E}W_n\,;\ S_n/\mathbf{E}S_n)$. Taking into consideration (3.1) and (3.2), we obtain

$$\Psi_n(\lambda;\theta) \sim J_n(\lambda_n;\theta_n) = \lambda_n\theta_n^n \prod_{k=0}^{n-1} F'\,(H_k(\lambda_n;\theta_n)), \ \ n \to \infty, \tag{4.1}$$

where $\lambda_n = exp\{-\lambda/(\alpha-1)n\}$, $\theta_n = exp\{-2\theta/(\alpha-1)n^2\}$. We see that the term $\lambda_n\theta_n^n$ on the right-hand side of (4.1) tends to unity as $n \to \infty$. It is ensured that $A_{kn}(\lambda;\theta) := F'\,(H_k(\lambda_n;\theta_n))$ does not decrease with k for a fixed n and $\lambda > 0,\ \theta > 0$. Then using the inequality $\ln(1-x) \geqslant$

$-x - x^2/(1-x)$, $0 \leqslant x < 1$, we obtain

$$\begin{aligned} \ln \prod_{k=0}^{n-1} A_{kn}(\lambda;\theta) &= \sum_{k=0}^{n-1} \ln\{1-(1-A_{kn}(\lambda;\theta))\} = \\ &= -\sum_{k=0}^{n-1}(1-A_{kn}(\lambda;\theta)) + \rho_n^{(1)}(\lambda;\theta) = \\ &= I_n(\lambda;\theta) + \rho_n^{(1)}(\lambda;\theta), \end{aligned} \tag{4.2}$$

where

$$I_n(\lambda;\theta) = -\sum_{k=0}^{n-1}(1-A_{kn}(\lambda;\theta)), \tag{4.3}$$

and

$$\begin{aligned} 0 \geqslant \rho_n^{(1)}(\lambda;\theta) &\geqslant -\sum_{k=0}^{n-1}\frac{[1-A_{kn}(\lambda;\theta)]^2}{A_{kn}(\lambda;\theta)} \geqslant \\ &\geqslant \frac{1-A_{0n}(\lambda;\theta)}{A_{0n}(\lambda;\theta)} I_n(\lambda;\theta). \end{aligned} \tag{4.4}$$

It is easy to see that $1 - A_{0n}(\lambda;\theta) = 1 - F'(\lambda_n) \to 0$ as $n \to \infty$. Then $\rho_n^{(1)}(\lambda;\theta) \to 0$ if $I_n(\lambda;\theta)$ has a finite limit as $n \to \infty$.

By Taylor expansion we have

$$F'(t) = F'(t_0) - F''(t_0)(t_0 - t) + (t_0 - t)g(t_0;t), \tag{4.5}$$

where $g(t_0;t) = (t_0 - t)F'''(\tau)/2$ with $t < \tau < t_0$. Setting $t = H_k(\lambda_n;\theta_n)$ and $t_0 = h(\theta_n)$ in (4.5) and using the GF $\Delta_k(s;x)$, equation (4.3) becomes

$$I_n(\lambda;\theta) = -\left[1 - F'(h(\theta_n))\right] n - F''(h(\theta_n)) \sum_{k=0}^{n-1} \Delta_k(\lambda_n;\theta_n) + \rho_n^{(2)}(\lambda;\theta), \tag{4.6}$$

where

$$0 \leqslant \rho_n^{(2)}(\lambda;\theta) = \sum_{k=0}^{n-1} \Delta_k(\lambda_n;\theta_n) g_{kn}(\lambda;\theta) \leqslant \Delta_0(\lambda_n;\theta_n) \sum_{k=0}^{n-1} g_{kn}(\lambda;\theta). \tag{4.7}$$

Here we use inequality (3.4) and relation $g_{kn}(\lambda;\theta) := g\left(h(\theta_n); H_k(\lambda_n;\theta_n)\right)$. Using relation (3.6) we get

$$\Delta_0(\lambda_n;\theta_n) \sim \frac{\lambda - 2\sqrt{\theta}}{F''(1)n}, \quad n \to \infty. \tag{4.8}$$

Considering (3.3), we see that $g_{kn}(\lambda;\theta) \to 0$ as $k \to \infty$ for all $n \in \mathbb{N}_0$. Hence the arithmetical mean of these expressions $\dfrac{1}{n}\sum\limits_{k=0}^{n-1} g_{kn}(\lambda;\theta) \to 0$ as $n \to \infty$. Then it follows from (4.7) and (4.8) that $\rho_n^{(2)}(\lambda;\theta)$ approaches zero as $n \to \infty$. Therefore, it follows from (3.6), (3.7) and (4.6) that

$$I_n(\lambda;\theta) = -2\sqrt{\theta} - F''(1) \sum_{k=0}^{n-1} \Delta_k(\lambda_n;\theta_n) + o(1), \quad n \to \infty. \tag{4.9}$$

Consider now the sum on the right-hand side of (4.9). From (3.5) we have

$$\Delta_k(\lambda_n;\theta_n) = \frac{u^k(\theta_n)}{\frac{1}{\Delta_0(\lambda_n;\theta_n)} + \frac{b(\theta_n)[1-u^k(\theta_n)]}{1-u(\theta_n)} + \Sigma_{kn}(\lambda;\theta)}, \tag{4.10}$$

where $\Sigma_{kn}(\lambda;\theta) := \sum_{k=1}^{n} \varepsilon_k(\lambda_n;\theta_n)u^k(\theta_n)$. Since $|\varepsilon_k(\lambda_n;\theta_n)| \leqslant \varepsilon_k \to 0$, $k \to \infty$, and $|u(x)| = |xF'(h(x))| \leqslant F'(1) = 1$, then

$$\frac{1}{n}\Sigma_{kn}(\lambda;\theta) = o(1), \quad n \to \infty. \tag{4.11}$$

Using relations (3.6)–(3.8), it is easy to find that as $n \to \infty$

$$1 - u(\theta_n) = \frac{2\sqrt{\theta}}{n}(1 + o(1)), \tag{4.12}$$

$$b(\theta_n) = \frac{F''(1)}{2}(1 + o(1)). \tag{4.13}$$

Taking into account equations (4.8), (4.10)–(4.13) we see that

$$\left|\Delta_k(\lambda_n;\theta_n) - \frac{4\sqrt{\theta}}{F''(1)n}\frac{\mu e^{-2k\sqrt{\theta}/n}}{\left(1 - e^{-2k\sqrt{\theta}/n}\right)\mu + 1}\right| = O\left(\frac{1}{n^2}\right),$$

as $n \to \infty$, where $\mu = \left(\lambda - 2\sqrt{\theta}\right)\Big/4\sqrt{\theta}$. Then the second term in expression (4.9) can be transformed to

$$F''(1)\sum_{k=0}^{n-1}\Delta_k(\lambda_n;\theta_n) = \sum_{k=0}^{n-1}\frac{4\mu\sqrt{\theta}e^{-2k\sqrt{\theta}/n}}{\left(1 - e^{-2k\sqrt{\theta}/n}\right)\mu + 1}\frac{1}{n} + o(1).$$

The sum on the right-hand side of the last equation can be recognized as the upper (if $\mu+1 < 0$) or the lower (if $\mu + 1 > 0$) Darboux sum of the Riemann integral

$$\int_0^1 \frac{4\mu\sqrt{\theta}e^{-2\sqrt{\theta}x}}{\left(1 - e^{-2\sqrt{\theta}x}\right)\mu + 1}dx = 2\ln\left[\left(1 - e^{-2\sqrt{\theta}}\right)\mu + 1\right].$$

Then we finally obtain

$$I_n(\lambda;\theta) = -2\sqrt{\theta} - 2\ln\left[\frac{\lambda - 2\sqrt{\theta}}{4\sqrt{\theta}}\left(1 - e^{-2\sqrt{\theta}}\right) + 1\right] + o(1), \quad n \to \infty. \tag{4.14}$$

After considering relations (4.1), (4.2) and (4.14), we complete the proof of the theorem.

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Одна предельная теорема для совместных распределений в Q-процессах

Азам А. Имомов

Мы рассматриваем Q-процессы. Устанавливается глубокий связь между Q-процессами и ветвящимися процессами Гальтона-Ватсона с иммиграцией. Доказывается предельная теорема для совместных распределений состояний и общих состояний в Q-процессе.

Ключевые слова: ветвящиеся процессы Гальтона-Ватсона; иммиграция; Q-процессы; общее состояние Q-процесса; предельная теорема.