DOI: 10.17516/1997-1397-2020-13-1-48-57
УДК 517.956.6; 517.44

Fundamental Solutions for a Class of Multidimensional Elliptic Equations with Several Singular Coefficients

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Received 13.09.2019, received in revised form 14.10.2019, accepted 04.12.2019

Abstract. The main result of the present paper is the construction of fundamental solutions for a class of multidimensional elliptic equations with several singular coefficients. These fundamental solutions are directly connected with multiple hypergeometric functions and the decomposition formula is required for their investigation which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. In this paper, such a formula is proved instead of a previously existing recurrence formula.The order of singularity and other properties of the fundamental solutions that are necessary for solving boundary value problems for degenerate second-order elliptic equations are determined.

Keywords: multidimensional elliptic equation with several singular coefficients; fundamental solutions; decomposition formula.

Citation: T.G.Ergashev, Fundamental Solutions for a Class of Multidimensional Elliptic Equations with Several Singular Coefficients, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 48–57.
DOI: 10.17516/1997-1397-2020-13-1-48-57.

Introduction

It is known that fundamental solutions have an essential role in studying partial differential equations. Formulation and solving of many local and non-local boundary value problems are based on these solutions. Moreover, fundamental solutions appear as potentials, for instance, as simple-layer and double-layer potentials in the theory of potentials.

The explicit form of fundamental solutions gives a possibility to study the considered equation in detail. For example, in the works of Barros-Neto and Gelfand [1–3] fundamental solutions for Tricomi operator, relative to an arbitrary point in the plane were explicitly calculated. In this direction we would like to note the works [4, 5], where three-dimensional fundamental solutions for elliptic equations were found. In the works [6–8] , fundamental solutions for a class of multidimensional degenerate elliptic equations with spectral parameter were constructed. The found solutions can be applied to solving some boundary value problems [9–15]. We also mention papers [16, 17] which are devoted to the study of partial differential equations with the singular coefficients and their solutions.

Let us consider the generalized Helmholtz equation with a several singular coefficients

$$L^m_{(\alpha)}(u) := \sum_{i=1}^{m} u_{x_i x_i} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} u_{x_j} = 0 \tag{1}$$

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in the domain $R_m^{n+} := \{(x_1, \ldots, x_m) : x_1 > 0, \ldots, x_n > 0\}$, where m is a dimension of the Euchlidean space, n is a number of the singular coefficients of equation (1); $m \geqslant 2$, $0 < n \leqslant m$; α_j are real constants and $0 < 2\alpha_j < 1$, $j = 1, \ldots, n$; $(\alpha) = (\alpha_1, \ldots, \alpha_n)$.

Various modifications of the equation (1) in the two- and three-dimensional cases were considered in many papers [4, 18–27].

Fundamental solutions for elliptic equations with singular coefficients are directly connected with hypergeometric functions. Therefore, basic properties such as decomposition formulas, integral representations, formulas of analytical continuation, formulas of differentiation for hypergeometric functions are necessary for studying fundamental solutions.

Since the aforementioned properties of hypergeometric functions of Gauss, Appell, Kummer were known [28], results on investigations of elliptic equations with one or two singular coefficients were successful. In the paper [4] when finding and studying the fundamental solutions of equation (1) for $m = 3$, an important role was played the decomposition formula of Hasanov and Srivastava [29, 30], however, the recurrence of this formula did not allow further advancement in the direction of increasing the number of singular coefficients.

In the present paper we construct all fundamental solutions for equation (1) in an explicit form and we prove a new formula for the expansion of several Lauricella hypergeometric functions by simple Gauss, with which it is possible to reveal that the found hypergeometric functions have a singularity of order $1/r^{m-2}$ at $r \to 0$. In the present paper, we assume that $m > 2$ and $0 < n \leqslant m$.

The plan of this paper is as follows. In Section 1 we briefly give some preliminary information, which will be used later. We transform the recurrence decomposition formula of Hasanov and Srivastava [29] to the form convenient for further research. Also some constructive formulas for the operator L are given. In Section 2 we describe the method of finding fundamental solutions for the considered equation and in Section 3 we show what order of singularity the found solutions will have.

1. Preliminaries

Below we give definition of Pochhammer symbol and some formulas for Gauss hypergeometric functions of one and two variables, Lauricella hypergeometric functions of three and more variables, which will be used in the next section.

A symbol $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shified factorial, since $(1)_l = l!$ $(l \in N \cup \{0\};\ N := \{1, 2, 3, \ldots\})$, which is defined (for $\kappa, \nu \in C$), in terms of the familiar Gamma function, by

$$(\kappa)_\nu := \frac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} = \begin{cases} 1 & (\nu = 0;\ \kappa \in C\backslash\{0\}), \\ \kappa(\kappa+1)\ldots(\kappa+l-1) & (\nu = l \in N;\ \kappa \in C), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the Γ-quotient exists.

A function

$$F\left[\begin{array}{c} a, b; \\ c; \end{array} x\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} x^k, \quad |x| < 1$$

is known as the Gauss hypergeometric function and an equality

$$F\left[\begin{array}{c} a, b; \\ c; \end{array} 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c \neq 0, -1, -2, \ldots, \mathrm{Re}(c-a-b) > 0 \tag{2}$$

holds [31, Ch.II,2.1(14)]. Moreover, the following autotransformer formula [31, Ch.II,2.1(22)]

$$F\left[\begin{matrix} a, b; \\ c; \end{matrix}\ x\right] = (1-x)^{-b} F\left[\begin{matrix} c-a, b; \\ c; \end{matrix}\ \frac{x}{x-1}\right] \tag{3}$$

is valid.

The hypergeometric function of n variables has a form [28, Ch.VII] (see also [32, Ch.1,1.4(1)])

$$F_A^{(n)}\left[\begin{matrix} a, b_1, \ldots, b_n; \\ c_1, \ldots, c_n; \end{matrix}\ x_1, \ldots, x_n\right] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{m_1!\ldots m_n!(c_1)_{m_1}\ldots(c_n)_{m_n}} x_1^{m_1}\ldots x_n^{m_n}, \tag{4}$$

where $|x_1| + \cdots + |x_n| < 1$, $n \in \mathrm{N}$.

For a given multivariable function, it is useful to fund a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables.

In the case of two variables for the function

$$F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix}\ x, y\right] = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b_1)_i(b_2)_j}{i!j!(c_1)_i(c_2)_j} x^i y^j$$

was known expansion formula [33]

$$F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix}\ x, y\right] = \sum_{k=0}^{\infty} \frac{(a)_k(b_1)_k(b_2)_k}{k!(c_1)_k(c_2)_k} x^k y^k F\left[\begin{matrix} a+k, b_1+k; \\ c_1+k; \end{matrix}\ x\right] F\left[\begin{matrix} a+k, b_2+k; \\ c_2+k; \end{matrix}\ y\right]. \tag{5}$$

Following the works [33, 34] Hasanov and Srivastava [29] found following decomposition formula for the Lauricella function of three variables

$$\begin{aligned} F_A^{(3)}\left[\begin{matrix} a, b_1, b_2, b_3; \\ c_1, c_2, c_3; \end{matrix}\ x, y, z\right] &= \sum_{i,j,k=0}^{\infty} \frac{(a)_{i+j+k}(b_1)_{j+k}(b_2)_{i+k}(b_3)_{i+j}}{i!j!k!(c_1)_{j+k}(c_2)_{i+k}(c_3)_{i+j}} \times \\ &\times x^{j+k} y^{i+k} z^{i+j} F\left[\begin{matrix} a+j+k, b_1+j+k; \\ c_1+j+k; \end{matrix}\ x\right] \times \\ &\times F\left[\begin{matrix} a+i+j+k, b_2+i+k; \\ c_2+i+k; \end{matrix}\ y\right] F\left[\begin{matrix} a+i+j+k, b_3+i+j; \\ c_3+i+j; \end{matrix}\ z\right] \end{aligned} \tag{6}$$

and they proved that for all $n \in N\backslash\{1\}$ is true the recurrence formula [29]

$$\begin{aligned} F_A^{(n)}\left[\begin{matrix} a, b_1, \ldots, b_n; \\ c_1, \ldots, c_n; \end{matrix}\ x_1, \ldots, x_n\right] &= \sum_{m_2,\ldots,m_n=0}^{\infty} \frac{(a)_{m_2+\cdots+m_n}(b_1)_{m_2+\cdots+m_n}(b_2)_{m_2}\ldots(b_n)_{m_n}}{m_2!\ldots m_n!(c_1)_{m_2+\cdots+m_n}(c_2)_{m_2}\ldots(c_n)_{m_n}} \times \\ &\times x_1^{m_2+\cdots+m_n} x_2^{m_2}\ldots x_n^{m_n} F\left[\begin{matrix} a+m_2+\cdots+m_n, b_1+m_2+\cdots+m_n; \\ c_1+m_2+\cdots+m_n; \end{matrix}\ x_1\right] \times \\ &\times F_A^{(n-1)}\left[\begin{matrix} a+m_2+\cdots+m_n, b_2+m_2, \ldots, b_n+m_n; \\ c_2+m_2, \ldots, c_n+m_n; \end{matrix}\ x_2, \ldots, x_n\right]. \end{aligned} \tag{7}$$

Further study of the properties of the Lauricella function (4) showed that the formula (7) can be reduced to a more convenient form.

Lemma 1. *The following formula holds true at $n \in N$*

$$F_A^{(n)}\left[\begin{array}{c} a, b_1, \dots, b_n; \\ c_1, \dots, c_n; \end{array} x_1, \dots, x_n\right] = \sum_{\substack{m_{i,j}=0 \\ (2\leqslant i\leqslant j\leqslant n)}}^{\infty} \frac{(a)_{N_2(n,n)}}{\underset{(2\leqslant i\leqslant j\leqslant n)}{m_{2,2}!m_{2,3}!\dots m_{i,j}!\dots m_{n,n}!}} \times$$
$$\times \prod_{k=1}^{n} \frac{(b_k)_{M_2(k,n)}}{(c_k)_{M_2(k,n)}} x_k^{M_2(k,n)} F\left[\begin{array}{c} a+N_2(k,n), b_k+M_2(k,n); \\ c_k+M_2(k,n); \end{array} x_k\right], \tag{8}$$

where

$$M_l(k,n) = \sum_{i=l}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i}, \quad N_l(k,n) = \sum_{i=l}^{k+1}\sum_{j=i}^{n} m_{i,j}, \quad l \in N.$$

Proof. We carry out the proof by the method mathematical induction.

The equality (8) in the case $n = 1$ is obvious.

Let $n = 2$. Since $M_2(1,2) = M_2(2,2) = N_2(1,2) = N_2(2,2) = m_{2,2}$, we obtain the formula (5).

For the sake of interest, we will check the formula (8) in yet another value of n.

Let $n = 3$. In this case

$$M_2(1,3) = m_{2,2} + m_{2,3}, \quad M_2(2,3) = m_{2,2} + m_{3,3}, \quad M_2(3,3) = m_{2,3} + m_{3,3},$$
$$N_2(1,3) = m_{2,2} + m_{2,3}, \quad N_2(2,3) = N_2(3,3) = m_{2,2} + m_{2,3} + m_{3,3}.$$

For brevity, making the substitutions $m_{2,2} := i,\ m_{2,3} := j,\ m_{3,3} := k$, we obtain the formula (6).

So the formula (8) works for $n = 1$, $n = 2$ and $n = 3$.

Now we assume that for $n = s$ equality (8) holds; that is, that

$$F_A^{(s)}\left[\begin{array}{c} a, b_1, \dots, b_s; \\ c_1, \dots, c_s; \end{array} x_1, \dots, x_s\right] = \sum_{\substack{m_{i,j}=0 \\ (2\leqslant i\leqslant j\leqslant s)}}^{\infty} \frac{(a)_{N_2(s,s)}}{\underset{(2\leqslant i\leqslant j\leqslant s)}{m_{2,2}!m_{2,3}!\dots m_{i,j}!\dots m_{s,s}!}} \times$$
$$\times \prod_{k=1}^{s} \frac{(b_k)_{M_2(k,s)}}{(c_k)_{M_2(k,s)}} x_k^{M_2(k,s)} F\left[\begin{array}{c} a+N_2(k,s), b_k+M_2(k,s); \\ c_k+M_2(k,s); \end{array} x_k\right]. \tag{9}$$

Let $n = s+1$. We prove that following formula

$$F_A^{(s+1)}\left[\begin{array}{c} a, b_1, \dots, b_{s+1}; \\ c_1, \dots, c_{s+1}; \end{array} x_1, \dots, x_{s+1}\right] = \sum_{\substack{m_{i,j}=0 \\ (2\leqslant i\leqslant j\leqslant s+1)}}^{\infty} \frac{(a)_{N_2(s+1,s+1)}}{\underset{(2\leqslant i\leqslant j\leqslant s+1)}{m_{2,2}!m_{2,3}!\dots m_{i,j}!\dots m_{s+1,s+1}!}} \times$$
$$\times \prod_{k=1}^{s+1} \frac{(b_k)_{M_2(k,s+1)}}{(c_k)_{M_2(k,s+1)}} x_k^{M_2(k,s+1)} F\left[\begin{array}{c} a+N_2(k,s+1), b_k+M_2(k,s+1); \\ c_k+M_2(k,s+1); \end{array} x_k\right] \tag{10}$$

is valid.

We write the Hasanov-Srivastava's formula (7) in the form

$$F_A^{(s+1)}\left[\begin{array}{c} a, b_1, \dots, b_{s+1}; \\ c_1, \dots, c_{s+1}; \end{array} x_1, \dots, x_{s+1}\right] =$$
$$= \sum_{m_{2,2},\dots,m_{2,s+1}=0}^{\infty} \frac{(a)_{N_2(1,s+1)}(b_1)_{M_2(1,s+1)}(b_2)_{m_{2,2}}\dots(b_{s+1})_{m_{2,s+1}}}{m_{2,2}!\dots m_{2,s+1}!(c_1)_{M_2(1,s+1)}(c_2)_{m_{2,2}}\dots(c_{s+1})_{m_{2,s+1}}} \times$$
$$\times\ x_1^{M_2(1,s+1)} x_2^{m_{2,2}} \dots x_{s+1}^{m_{2,s+1}} F\left[\begin{array}{c} a+N_2(1,s+1), b_1+M_2(1,s+1); \\ c_1+M_2(1,s+1); \end{array} x_1\right] \times \tag{11}$$
$$\times\ F_A^{(s)}\left[\begin{array}{c} a+N_2(1,s+1), b_2+m_{2,2}, \dots, b_{s+1}+m_{2,s+1}; \\ c_2+m_{2,2}, \dots, c_{s+1}+m_{2,s+1}; \end{array} x_2, \dots, x_{s+1}\right].$$

By virtue of the formula (9) we have

$$F_A^{(s)}\left[\begin{matrix} a+N_2(1,s+1), b_2+m_{2,2},\dots,b_{s+1}+m_{2,s+1}; \\ c_2+m_{2,2},\dots,c_{s+1}+m_{2,s+1}; \end{matrix} x_2,\dots,x_{s+1}\right]=$$

$$=\sum_{\substack{m_{i,j}=0\\(3\leqslant i\leqslant j\leqslant s+1)}}^{\infty}\frac{(a+N_2(1,s+1))_{N_3(s+1,s+1)}}{\underset{(3\leqslant i\leqslant j\leqslant s+1)}{m_{3,3}!m_{3,4}!\dots m_{i,j}!\dots m_{s+1,s+1}!}}\prod_{k=2}^{s+1}\frac{(b_k+m_{2,k})_{M_3(k,s+1)}}{(c_k+m_{2,k})_{M_3(k,s+1)}}x_k^{M_3(k,s+1)}\times \tag{12}$$

$$\times\ F\left[\begin{matrix} a+N_2(1,s+1)+N_3(k,s+1), b_k+m_{2,k}+M_3(k,s+1); \\ c_k+m_{2,k}+M_3(k,s+1); \end{matrix} x_k\right].$$

Substituting from (12) into (11) we obtain

$$F_A^{(s+1)}[a,b_1,\dots,b_{s+1};c_1,\dots,c_{s+1};x_1,\dots,x_{s+1}]=$$

$$=\sum_{\substack{m_{i,j}=0\\(2\leqslant i\leqslant j\leqslant s+1)}}^{\infty}\frac{(a)_{N_2(1,s+1)+N_3(s+1,s+1)}}{\underset{(2\leqslant i\leqslant j\leqslant s+1)}{m_{2,2}!m_{2,3}!\dots m_{i,j}!\dots m_{s+1,s+1}!}}\prod_{k=1}^{s+1}\frac{(b_k)_{m_{2,k}+M_3(k,s+1)}}{(c_k)_{m_{2,k}+M_3(k,s+1)}}x_k^{m_{2,k}+M_3(k,s+1)}\times$$

$$\times\ F\left[\begin{matrix} a+N_2(1,s+1)+N_3(k,s+1), b_k+m_{2,k}+M_3(k,s+1); \\ c_k+m_{2,k}+M_3(k,s+1); \end{matrix} x_k\right].$$

Further, by virtue of the following obvious equalities

$$N_2(1,s+1)+N_3(k,s+1)=N_2(k,s+1),\ 1\leqslant k\leqslant s+1,\ s\in N,$$

$$m_{2,k}+M_3(k,s+1)=M_2(k,s+1),\ 1\leqslant k\leqslant s+1,\ s\in N,$$

we finally find the equality (10). The lemma is proved. □

2. Fundamental solutions

Consider equation (1) in R_m^{n+}. Let $x:=(x_1,\dots,x_m)$ be any point and $\xi:=(\xi_1,\dots,\xi_m)$ be any fixed point of R_m^{n+}. We search for a solution of (1) as follows:

$$u(x,\xi)=P(r)w(\sigma), \tag{13}$$

where

$$\sigma=(\sigma_1,\dots,\sigma_n)\,,\quad \tilde{\alpha}_0=\alpha_1+\dots+\alpha_n-1+\frac{m}{2},$$

$$P(r)=\left(r^2\right)^{-\tilde{\alpha}_0},\quad r^2=\sum_{i=1}^{m}(x_i-\xi_i)^2,$$

$$r_k^2=(x_k+\xi_k)^2+\sum_{i=1,i\neq k}^{m}(x_i-\xi_i)^2,\quad \sigma_k=\frac{r^2-r_k^2}{r^2},\quad k=1,2,\dots,n.$$

We calculate all necessary derivatives and substitute them into equation (1) :

$$\sum_{k=1}^{n}A_k\omega_{\sigma_k\sigma_k}+\sum_{k=1}^{n}\sum_{l=k+1}^{n}B_{k,l}\omega_{\sigma_k\sigma_l}+\sum_{k=1}^{n}C_k\omega_{\sigma_k}+D\omega=0, \tag{14}$$

where

$$A_k = P\sum_{i=1}^{m}\left(\frac{\partial \sigma_k}{\partial x_i}\right)^2, \quad B_{k,l} = 2P\sum_{i=1}^{m}\frac{\partial \sigma_k}{\partial x_i}\frac{\partial \sigma_l}{\partial x_i}, \quad k\neq l,\ k=1,\ldots,n,$$

$$C_k = P\sum_{i=1}^{m}\frac{\partial^2 \sigma_k}{\partial x_i^2} + 2\sum_{i=1}^{m}\frac{\partial P}{\partial x_i}\frac{\partial \sigma_k}{\partial x_i} + 2P\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\frac{\partial \sigma_k}{\partial x_j},$$

$$D = \sum_{i=1}^{m}\frac{\partial^2 P}{\partial x_i^2} + 2P\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\frac{\partial P}{\partial x_j}.$$

After several evaluations we find

$$A_k = -\frac{4P(r)}{r^2}\frac{x_k}{\xi_k}\sigma_k(1-\sigma_k), \tag{15}$$

$$B_{k,l} = \frac{4P(r)}{r^2}\left(\frac{\xi_k}{x_k}+\frac{\xi_l}{x_l}\right)\sigma_k\sigma_l, \quad k\neq l, \quad l=1,\ldots,n, \tag{16}$$

$$C_k = -\frac{4P(r)}{r^2}\left\{-\sigma_k\sum_{j=1}^{n}\frac{\xi_j}{x_j}\alpha_j + \frac{\xi_k}{x_k}[2\alpha_k - \tilde{\alpha}_0\sigma_k]\right\}, \tag{17}$$

$$D = \frac{4\tilde{\alpha}_0 P(r)}{r^2}\sum_{j=1}^{n}\frac{\xi_j}{x_j}\alpha_j. \tag{18}$$

Substituting equalities (15)–(18) into (14) we obtain the following system of hypergeometric equations of Lauricella [28], which has 2^n linearly-independent solutions. Considering those solutions, from (13) we obtain 2^n fundamental solutions of equation (1):

$$1\{\,F_A^{(n)}\left[\begin{matrix} a, b_1,\ldots,b_n; \\ c_1,\ldots,c_n; \end{matrix}\ \sigma\right], \tag{19}$$

$$C_n^1\left\{\begin{matrix} (x_1\xi_1)^{1-c_1}F_A^{(n)}\left[\begin{matrix} a+1-c_1, b_1+1-c_1, b_2,\ldots,b_n; \\ 2-c_1, c_2,\ldots,c_n; \end{matrix}\ \sigma\right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (x_n\xi_n)^{1-c_n}F_A^{(n)}\left[\begin{matrix} a+1-c_n, b_1,\ldots,b_{n-1}, b_n+1-c_n; \\ c_1,\ldots,c_{n-1}, 2-c_n; \end{matrix}\ \sigma\right], \end{matrix}\right. \tag{20}$$

$$C_n^2\left\{\begin{matrix} (x_1\xi_1)^{1-c_1}(x_2\xi_2)^{1-c_2}F_A^{(n)}\left[\begin{matrix} a+2-c_1-c_2, b_1+1-c_1, b_2+1-c_2, b_3,...,b_n; \\ 2-c_1, 2-c_2, c_3,\ldots,c_n; \end{matrix}\ \sigma\right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (x_1\xi_1)^{1-c_1}(x_n\xi_n)^{1-c_n}F_A^{(n)}\left[\begin{matrix} a+2-c_1-c_n, b_1+1-c_1, b_2,\ldots,b_{n-1}, b_n+1-c_n; \\ 2-c_1, c_2,\ldots,c_{n-1}, 2-c_n; \end{matrix}\ \sigma\right], \\ (x_2\xi_2)^{1-c_2}(x_3\xi_3)^{1-c_3}F_A^{(n)}\left[\begin{matrix} a+2-c_2-c_3, b_1, b_2+1-c_2, b_3+1-c_3, b_4,\ldots,b_n; \\ c_1, 2-c_2, 2-c_3, c_4,\ldots,c_n; \end{matrix}\ \sigma\right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{(x_{n-1}\xi_{n-1})^{1-c_{n-1}}}{(x_n\xi_n)^{c_n-1}}F_A^{(n)}\left[\begin{matrix} a+2-c_{n-1}-c_n, b_1,\ldots,b_{n-2}, b_{n-1}+1-c_{n-1}, b_n+1-c_n; \\ c_1,\ldots,c_{n-2}, 2-c_{n-1}, 2-c_n; \end{matrix}\ \sigma\right], \end{matrix}\right. \tag{21}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$1\left\{(x_1\xi_1)^{1-c_1}\cdot\ldots\cdot(x_n\xi_n)^{1-c_n}F_A^{(n)}\left[\begin{matrix} a+n-c_1-\cdots-c_n, b_1+1-c_1,\ldots,b_n+1-c_n; \\ 2-c_1,\ldots,2-c_n; \end{matrix}\ \sigma\right],\right.$$

where

$$a=\tilde{\alpha}_0,\ b_i=\alpha_i,\ c_i=2\alpha_i,\ 1\leqslant i\leqslant n;\quad C_n^k=\frac{n!}{k!(n-k)!},\ 0\leqslant k\leqslant n.$$

It is easy to see that in (19) there is one function, in (20) there are $C_n^1=n$ functions, in (21) there are $C_n^2=n(n-1)/2$ functions and so on, and therefore

$$1+C_n^1+C_n^2+\cdots+C_n^{n-1}+1=2^n.$$

Taking into account the symmetry property of the Lauricella function $F_A^{(n)}$ with respect to the parameters $b_1,\ldots,b_n,c_1,\ldots,c_n$, we can reduced the quantity of the fundamental solutions that are necessary in the study of boundary value problems: from each of the systems (19), (20), (21) and so on we take only one fundamental solution. Consequently, all $n+1$ (non-symmetric) fundamental solutions of equation (1) can be written in the form which is a convenient for further investigation:

$$q_0(x,\xi)=\gamma_0 r^{-2\tilde{\alpha}_0}F_A^{(n)}\left[\begin{matrix}\tilde{\alpha}_0,\alpha_1,\ldots,\alpha_n; \\ 2\alpha_1,\ldots,2\alpha_n;\end{matrix}\ \sigma\right], \tag{22}$$

$$q_k(x,\xi)=\gamma_k\prod_{i=1}^{k}(x_i\xi_i)^{1-2\alpha_i}\cdot r^{-2\tilde{\alpha}_k}F_A^{(n)}\left[\begin{matrix}\tilde{\alpha}_k,1-\alpha_1,\ldots,1-\alpha_k,\alpha_{k+1},\ldots,\alpha_n; \\ 2-2\alpha_1,\ldots,2-2\alpha_k,2\alpha_{k+1},\ldots,2\alpha_n;\end{matrix}\ \sigma\right],\ k=\overline{1,n}, \tag{23}$$

where

$$\tilde{\alpha}_k=\frac{m}{2}+k-1-\alpha_1-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_n,\quad k=\overline{1,n},$$

$$\gamma_k=2^{2\tilde{\alpha}_k-m}\frac{\Gamma(\tilde{\alpha}_k)}{\pi^{m/2}}\prod_{i=k+1}^{n}\frac{\Gamma(\alpha_i)}{\Gamma(2\alpha_i)}\prod_{j=1}^{k}\frac{\Gamma(1-\alpha_j)}{\Gamma(2-2\alpha_j)},\quad k=\overline{0,n}.$$

3. Singularity properties of fundamental solutions

Let us show that the fundamental solutions (22) and (23) have a singularity at $r=0$.

We choose a solution $q_0(x,\xi)$ and we use the expansion for the hypergeometric function of Lauricella (8). As a result, a solution defined by (22) can be written as follows

$$\begin{aligned} q_0(x,\xi)=\gamma_0 r^{-2\tilde{\alpha}_0}\sum_{\substack{m_{i,j}=0\\(2\leqslant i\leqslant j\leqslant n)}}^{\infty}\frac{(\tilde{\alpha}_0)_{N_2(n,n)}}{\underset{(2\leqslant i\leqslant j\leqslant n)}{m_{2,2}!m_{2,3}!\ldots m_{i,j}!\ldots m_{n,n}!}}\times \\ \times\prod_{k=1}^{n}\frac{(\alpha_k)_{M_2(k,n)}}{(2\alpha_k)_{M_2(k,n)}}\left(1-\frac{r_k^2}{r^2}\right)^{M_2(k,n)}F\left[\begin{matrix}\tilde{\alpha}_0+N_2(k,n),\alpha_k+M_2(k,n); \\ 2\alpha_k+M_2(k,n);\end{matrix}\ 1-\frac{r_k^2}{r^2}\right]. \end{aligned} \tag{24}$$

By virtue of formula (3) we rewrite (24) as

$$q_0(x,\xi)=\frac{\gamma_0}{r^{m-2}}\prod_{k=1}^{n}r_k^{-2\alpha_k}\cdot f_0\left(r^2,r_1^2,\ldots,r_n^2\right),$$

where

$$f_0\left(r^2,r_1^2,\ldots,r_n^2\right)=\sum_{\substack{m_{i,j}=0\\(2\leqslant i\leqslant j\leqslant n)}}^{\infty}\frac{(\tilde{\alpha}_0)_{N_2(n,n)}}{\underset{(2\leqslant i\leqslant j\leqslant n)}{m_{2,2}!m_{2,3}!\cdots m_{i,j}!\cdots m_{n,n}!}}\times$$

$$\times \prod_{k=1}^{n} \frac{(\alpha_k)_{M_2(k,n)}}{(2\alpha_k)_{M_2(k,n)}} \left(\frac{r^2}{r_k^2} - 1\right)^{M_2(k,n)} F\left[\begin{matrix} 2\alpha_k - \tilde{\alpha}_0 + M_2(k,n) - N_2(k,n), \alpha_k + M_2(k,n); \\ 2\alpha_k + M_2(k,n); \end{matrix} 1 - \frac{r^2}{r_k^2}\right].$$

Below we show that $f_0\left(r^2, r_1^2, \ldots, r_n^2\right)$ will be constant at $r \to 0$.

For this aim we use an equality (2) and following inequality

$$N_2(k,n) - M_2(k,n) := \sum_{i=2}^{k} \left(\sum_{j=i}^{n} m_{i,j} - m_{i,k} \right) \geqslant 0, \quad 1 \leqslant k \leqslant n \leqslant m.$$

Then we get

$$\lim_{r \to 0} f_0\left(r^2, r_1^2, \ldots, r_n^2\right) = \frac{1}{\Gamma^n(\tilde{\alpha}_0)} \prod_{k=1}^{n} \frac{\Gamma\left(2\alpha_k\right) \Gamma\left(\tilde{\alpha}_0 - \alpha_k\right)}{\Gamma\left(\alpha_k\right)}. \tag{25}$$

Expressions (24) and (25) give us the possibility to conclude that the solution $q_0(x, \xi)$ reduces to infinity of the order r^{2-m} at $r \to 0$. Similarly it is possible to be convinced that solutions $q_k(x; \xi)$, $k = 1, 2, \ldots, n$ also reduce to infinity of the order r^{2-m} when $r \to 0$.

It can be directly checked that constructed functions (22) and (23) possess following properties

$$\left(x_j^{2\alpha_j} \frac{\partial q_0\left(x, \xi\right)}{\partial x_j} \right)\Bigg|_{x_j=0} = 0, \quad q_n\left(x, \xi\right)\big|_{x_j=0} = 0, \quad 1 \leqslant j \leqslant n,$$

$$q_k\left(x, \xi\right)\big|_{x_j=0} = 0, \quad 1 \leqslant j \leqslant k, \quad \left(x_j^{2\alpha_j} \frac{\partial q_k\left(x, \xi\right)}{\partial x_j} \right)\Bigg|_{x_j=0} = 0, \quad k+1 \leqslant j \leqslant n, \quad 1 \leqslant k \leqslant n-1.$$

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Фундаментальные решения многомерного эллиптического уравнения с несколькими сингулярными коэффициентами

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Аннотация. Основным результатом настоящей работы является построение фундаментальных решений для одного класса эллиптических уравнений с несколькими сингулярными коэффициентами. Поскольку эти решения напрямую связаны с гипергеометрическими функциями многих переменных Лауричелла, то для изучения свойств найденных фундаментальных решений требуется найти формулу разложения, которая выражала бы многомерную гипергеометрическую функцию в виде суммы произведений нескольких более простых гипергеометрических функций с меньшим числом переменных. В этой работе такая формула доказана вместо ранее существовавшей рекуррентной формулы и определен порядок особенности фундаментальных решений.

Ключевые слова: многомерное эллиптическое уравнение с несколькими сингулярными коэффициентами, фундаментальные решения, формула разложения.