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ORIGINAL ARTICLE

Inverse Problems for the Stationary and Pseudoparabolic Equations of Diffusion

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The identification of an unknown coefficient in the lower term of pseudoparabolic differential equation of diffusion $(u+\eta Mu)_t+Mu+ku=f$ and elliptic second order differential equation $Mu+ku=f$ with the Dirichlet boundary condition is considered. The identification of k is based on an integral boundary data. The local existence and uniqueness of generalized strong solutions for the inverse problems are proved. The stability estimates are exposed.

Keywords: inverse problems for PDE; boundary value problems; second-order elliptic equations; pseudoparabolic equation; existence and uniqueness theorems

AMS Subject Classifications: 35A01, 35B30, 35G16, 35J25, 35R30

1. Introduction

In this paper we discuss inverse problems for the pseudoparabolic diffusion equation

$$(u + \eta Mu)_t + Mu + ku = f \tag{1.1}$$

and the stationary equation associated with (1.1). Here M is an elliptic linear differential operator of the second order in the space variables. We establish the existence, uniqueness and stability of the strong solution of the inverse problems for (1.1) and the associated stationary equation with an unknown coefficient k under the Dirichlet boundary condition and the additional integral boundary data akin to the conditions of overdetermination considered in [1–4]. An exact statement of the problems will be given below. Following the idea of [1–4] based on the method of [5] we prove the existence of the solution by reducing the inverse problem to an operator equation of the second type for the unknown coefficient. We show that the operator of this equation is a contraction on a set constructed with the use of the comparison theorems for elliptic and pseudoparabolic equations. The contractibility of the operator provides the uniqueness and stability (continuous dependence on the input data) of the solution.

Applications of such problems deal with the recovery of unknown parameters indicating physical properties of a medium. In particular, the lowest coefficient k specifies, for instance, the catabolism of contaminants due to chemical reactions [6] or the absorption (also known as potential) in the diffusion and acoustics problems [7].

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The study of inverse problems for pseudoparabolic equations goes back to 1980s. The first result [8] refers to the inverse problems of determining a source function f of equation

$$(u + L_1 u)_t + L_2 u = f \tag{1.2}$$

in case $L_1 = L_2$ where L_1 and L_2 are the linear differential operators of the second order in spacial variables. We should mention also the results in [9, 10] concerning with coefficient inverse problems for the linear equation (1.1). In [10], the uniqueness theorem is obtained and an algorithm of determining the coefficients of L_2 is constructed. In [9], the solvability is established for two inverse problems of recovering the unknown coefficients in terms u (the lowest term of $L_2 u$) and u_t of (1.2). In [11], an inverse problem of recovering time-depending right-hand side and coefficients of (1.2) is considered. The values of the solution at separate points are employed as overdetermination conditions. The existence and uniqueness theorems are proven for this problem and the stability estimates of the solution are exposed.

Theoretical and numerical aspects of inverse problems for the stationary equation

$$L_2 u = f. \tag{1.3}$$

are discussed in [6, 12–16] (see also the references given there). In [12–14], the method of the Carleman estimates was designed for proofs of uniqueness theorems for coefficient inverse problems with unknown coefficients depending on all spatial variables for (1.3) and other PDEs with single measurement data. The work [6] is devoted to the problem on the determination of the lowest coefficient of the linear equation (1.3) by additional mesurements in a subset of the domain. The work [16] discusses two types of additional information (overdetermination): the trace of the solution on a manifold of lower dimension inside the domain and the normal derivative of the solution on a portion of the boundary. In [15], the coefficient is recovered by the integral information on a manifold inside of the domain. The solvability of the problems and the uniqueness of the solution are examined.

The paper is organized as follows. Section 2 presents the formulation of the inverse problems and certain preliminary results concerning the direct problems for elliptic and pseudoparabolic equations. In Section 3 we prove the existence, uniqueness and the stability of the solution to the stationary inverse problem for the diffusion equation. Section 4 discusses the same metters regarding the nonstationary inverse problem for the equation of pseudoparabolic type.

2. The statement of the problems and preliminary results

Let Ω be a bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega \in C^2$, $\overline{\Omega}$ be the closure of Ω. T is an arbitrary real number, $Q_T = \Omega \times (0, T)$ with the lateral surface $S_T = (0, T) \times \partial\Omega$, $\overline{Q}_T$ is the closure of Q_T and the pair (t, x) is a point of Q_T.

From now on we keep the notations: $(\cdot, \cdot)_R$ is the inner product of $\mathbf{R}^n$; $\|\cdot\|$ and $(\cdot, \cdot)$ are the norm and the inner product of $L^2(\Omega)$, respectively; $\|\cdot\|_j$ is the norm of $W_2^j(\Omega)$, $j = 1, 2$; and $\langle \cdot, \cdot \rangle_1$ is the duality relation between $\overset{\circ}{W}{}_2^j(\Omega)$ and $W_2^{-j}(\Omega)$; $\|\cdot\|_{p/2}$ is the norm of $W_2^{p/2}(\partial\Omega)$, $p = 1, 3$.

We introduce a linear differential operator $M = -\text{div}(\mathcal{M}(x)\nabla) + m(x)I$ where $\mathcal{M}(x) \equiv (m_{ij}(x))$ is a matrix of functions $m_{ij}(x),\ i, j = 1, 2, \ldots, n$; I – the identity operator. We also keep the notation

$$\langle Mv_1, v_2 \rangle_M = \int_\Omega ((\mathcal{M}(x)\nabla v_1, \nabla v_2)_R + m(x)v_1v_2)dx$$

for $v_1, v_2 \in W_2^1(\Omega)$ and assume that the following conditions are fulfilled.

I. $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i, j, l = 1, 2, \dots, n$, and $m(x)$ are bounded in Ω. Operator M is elliptic, that is, there exist positive constants m_0 and m_1 such that for all $v \in W_2^1(\Omega)$

$$m_0\|v\|_1^2 \le \langle Mv, v \rangle_M \le m_1\|v\|_1^2. \tag{2.1}$$

II. M is a selfadjoint operator, that is, $m_{ij}(x) = m_{ji}(x)$, $i, j = 1, 2, \dots, n$ for $x \in \Omega$.

In this paper we are studying the inverse problems of recovering unknown coefficients in the lowest terms of the diffusion equations. The first one is the inverse problem for the pseudoparabolic equation.

Problem 1. For given functions $f(t, x)$, $U_0(x)$, $\beta(t, x)$, $\omega(t, x)$, $\varphi(t)$ and a constant η find the pair of unknown functions $\{u(t, x), k(t)\}$ satisfying the equation

$$(u + \eta Mu)_t + Mu + k(t)u = f, \tag{2.2}$$

the initial data

$$(u + \eta Mu)|_{t=0} = U_0(x), \tag{2.3}$$

the boundary condition

$$u|_{\partial\Omega} = \beta(t, x) \tag{2.4}$$

and the condition of overdetermination

$$\int_{\partial\Omega} \Big\{\eta \frac{\partial u_t}{\partial \overline{N}} + \frac{\partial u}{\partial \overline{N}}\Big\}\omega(t, x)ds = \varphi(t). \tag{2.5}$$

Here $\frac{\partial}{\partial \overline{N}} = (\mathcal{M}(x)\nabla, \mathbf{n})$, $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$.

The second inverse problem corresponds to Problem 1 in the case of the steady-state process.

Problem 2. For given functions $f(x)$, $\beta(x)$, $\omega(x)$ and a constant μ find the pair of function $u(t, x)$ and constant k satisfying the equation

$$Mu + ku = f, \tag{2.6}$$

the boundary condition

$$u|_{\partial\Omega} = \beta(x) \tag{2.7}$$

and the condition of overdetermination

$$\int_{\partial\Omega} \frac{\partial u}{\partial \overline{N}}\omega(x)ds = \mu. \tag{2.8}$$

If $\omega \equiv 1$, then the integral conditions (2.8) and (2.5) means a given flux of a liquid through the surface $\partial\Omega$, for instance, the total discharge of a liquid through the surface of the ground. Similar nonlocal conditions were applied to control problems in [17] and inverse problems for elliptic and pseudoparabolic equations in [1, 4].

The existence and uniqueness results for Problems 1 and 2 rely upon two lemmas for the direct problems (2.2)–(2.4) and (2.6)–(2.7) with the known coefficient k. The first lemma covers the comparison theorems for the elliptic partial differential equations.

LEMMA 2.1 *Let* $u_1, u_2 \in W_2^2(\Omega)$ *be the solutions of the problems* $Mu_j + k_j u_j = f_j$, $u_j \mid_{\partial\Omega} = \beta_j$, $j = 1, 2$. *If* $0 \le k_1 \le k_2$, $\beta_1 \ge \beta_2 \ge 0$ *and* $f_1 \ge f_2 \ge 0$, *then* $u_1 \ge u_2 \ge 0$ *for almost all* $x \in \overline{\Omega}$.

Proof. By the maximum principle for the elliptic equations, $u_i \ge 0$, $i = 1, 2$, for almost all $x \in \overline{\Omega}$. The difference of the solutions $u_1 - u_2$ satisfies equation

$$M(u_1 - u_2) + k_1(u_1 - u_2) = (k_2 - k_1)u_2 + f_1 - f_2 \tag{2.9}$$

and the boundary condition $(u_1 - u_2) \mid_{\partial\Omega} = \beta_1 - \beta_2$. In the hypotheses of the lemma the right side of (2.9) is nonnegative and $\beta_1 - \beta_2 \ge 0$. Hence, by the maximum principle for the elliptic equations, $u_1 - u_2 \ge 0$ for almost all $x \in \overline{\Omega}$. ■

LEMMA 2.2 *Let* $v_j \in C^1([0, T]; W_2^2(\Omega)$ *be the solutions of the problems*

$$\begin{cases} v_{jt} + \eta M v_{jt} + M v_j + k_j(t) v_j = f_j, \\ (v_j + \eta M v_j)|_{t=0} = U_j(x), \quad v_j|_{\partial\Omega} = \beta_j(t, x), \quad j = 1, 2. \end{cases} \tag{2.10}$$

If $k_2(t) \le k_1(t) \le \frac{1}{\eta}$ *on* $[0, T]$, $0 \le f_1 \le f_2$ *almost everywhere in* $\overline{Q}_T$, $0 \le U_1(x) \le U_2(x)$ *for almost all* $x \in \Omega$ *and* $0 \le \beta_1(t, x) \le \beta_2(t, x)$ *for almost all* $(t, x) \in S_T$, *then* $0 \le v_1 \le v_2$ *for almost all* $(t, x) \in \overline{Q}_T$.

Proof. By (2.10), the function $\bar{v} = v_2 - v_1$ is the solution of the problem

$$\begin{cases} \bar{v}_t + \eta M \bar{v}_t + M\bar{v} + k_2(t)\bar{v} = (k_1 - k_2)v_1 + f_2 - f_1, \\ (\bar{v}_i + \eta M\bar{v})|_{t=0} = U_2 - U_1, \quad \bar{v}|_{\partial\Omega} = \beta_2 - \beta_1. \end{cases} \tag{2.11}$$

By Theorem 2.2 [4], $v_j \ge 0, j = 1, 2$, almost everywhere in $\overline{Q}_T$. Hence, the right term of (2.11) is nonnegative. Therefore by Theorem 2.2 [4], $\bar{v} \ge 0$. ■

3. The stationary inverse problem

We begin with the existence and uniqueness theorem for Problem 2. By a solution of this problem is meant the pair involving a function $u \in W_2^2(\Omega)$ and a positive real number k which satisfy (2.6)–(2.8). To formulate the theorem we introduce functions a, a^σ and b as the solutions of the problems

$$Ma = f(x), \qquad a \mid_{\partial\Omega} = \beta(x); \tag{3.1}$$

$$Ma^\sigma + \sigma a^\sigma = f(x), \quad a^\sigma \mid_{\partial\Omega} = \beta(x); \tag{3.2}$$

$$Mb = 0, \qquad b \mid_{\partial\Omega} = \omega(x) \tag{3.3}$$

where $\sigma > 0$ is a real number.

THEOREM 3.1 *Let $\partial\Omega \in C^2$ and the assumptions I, II are fulfilled. Suppose also that*

(i) $f(x) \in L^2(\Omega)$, $\beta(x), \omega(x) \in W_2^{3/2}(\partial\Omega)$;

(ii) $f(x) \geq 0$ *almost everywhere in* Ω; $\beta(x) \geq 0$, $\omega(x) \geq 0$ *for almost all* $x \in \partial\Omega$ *and there is a smooth piece* Γ *of the boundary* $\partial\Omega$ *and a constant* $\delta > 0$ *such that* $\beta \geq \delta$ *and* $\omega \geq \delta$ *almost everywhere on* Γ;

$$0 \leq \mu - \Psi \leq \frac{m_0(a,b)^2}{4\|a\|\|b\|} \tag{3.4}$$

where $\Psi = \langle Ma, b \rangle_M - (f, b)$.

Then Problem 2 has a solution $\{u, k\}$. *If*

$$0 \leq \mu - \Psi < \frac{m_0(a,b)^2}{4\|a\|\|b\|}, \tag{3.5}$$

then the solution is unique. Moreover, the estimates

$$a^\sigma \leq u \leq a, \qquad 0 \leq k \leq \sigma, \qquad \|u\|_2 \leq C\|a\| + \|a\|_2 \tag{3.6}$$

hold with some $\sigma > 0$ *and a constant* C *dependent on* $\text{mes}\Omega$, σ, m_0 *and* m_1.

Proof. Following the idea of [1–4] and the method of [5], we reduce Problem 2 to an equivalent inverse problem with a nonlinear operator equation for $k(t)$. From (2.6)–(2.8) it follows that the function $w \equiv a - u$ and the constant k obeys the relations

$$Mw + kw = ka, \tag{3.7}$$

$$w \mid_{\partial\Omega} = 0, \tag{3.8}$$

$$\int_{\partial\Omega} \frac{\partial w}{\partial \overline{N}} \omega ds = -\mu + \int_{\partial\Omega} \frac{\partial a}{\partial \overline{N}} \omega ds = \Psi - \mu. \tag{3.9}$$

In view of (3.8) and (3.9) multiplying (3.7) by b in terms of the inner product of $L_2(\Omega)$ and integration by parts twice yields

$$k(u,b) = \mu - \Psi. \tag{3.10}$$

Let the operator $A : R_+ \to R$ (R_+ is the set of all nonnegative real numbers) maps every $y \in R_+$ into the real number Ay by the rule

$$Ay = (\mu - \Psi)(u_y, b)^{-1}, \tag{3.11}$$

where u_y is the solution of the direct problem (2.6)–(2.7) with $k = y$. It can be shown that Problem 2 is solvable if and only if the operator A has a fixed point, i. e. the operator equation $k = Ak$ has a solution.

The operator A maps a set $[0, \sigma]$ into itself and continuous on the set for some $\sigma > 0$. Indeed, Lemma 2.1 implies that $b^\eta \geq 0$ and for all $0 \leq y \leq \sigma$

$$a^\sigma \leq u_y \leq a. \tag{3.12}$$

Therefore $Ay \geq (\mu - \Psi)(a, b)^{-1} \geq 0$. On the other hand, multiplying the difference of (3.1) and (3.2) by $a - a^\sigma$ in terms of the inner product of $L^2(\Omega)$, integrating by parts and estimating the left-hand side of the result with the help of (2.1) gives

$$\|a - a^\sigma\| \leq \|a - a^\sigma\|_1 \leq \sigma m_0^{-1}\|a\|.$$

This estimate and (3.12) allows to obtain the lower bound of (u_y, b) in (3.11).

$$(u_y, b) \geq (a^\sigma, b) = (a, b) - (a - a^\sigma, b) \geq (a, b) - \sigma m_0^{-1}\|a\|\|b\| > 0 \tag{3.13}$$

when

$$\sigma < m_0(a, b)(\|a\|\|b\|)^{-1}. \tag{3.14}$$

In view of (3.11), (3.13)

$$Ay \leq \frac{m_0(\mu - \Psi)}{m_0(a, b) - \sigma\|a\|\|b\|}.$$

Hence, the inequality $Ay \leq \sigma$ holds for all $\sigma > 0$ such that

$$-\|a\|\|b\|m_0^{-1}\sigma^2 + (a, b)\sigma - (\mu - \Psi) \geq 0. \tag{3.15}$$

The last relation is possible because in the hypotheses of the theorem

$$D \equiv (a, b)^2 - 4(\mu - \Psi)m_0^{-1}\|a\|\|b\| \geq 0. \tag{3.16}$$

(3.15) is valid for σ obeying the inequality

$$\frac{m_0((a, b) - \sqrt{D})}{2\|a\|\|b\|} \leq \sigma \leq \frac{m_0((a, b) + \sqrt{D})}{2\|a\|\|b\|}.$$

Thus, the operator A maps the segment $[0, \sigma]$ into itself.

We are now in a position to obtain the estimate of u_y in $W_2^2(\Omega)$ provided that $y \in [0, \sigma]$. Let $w_y = a - u_y$. This function satisfies (3.7)–(3.9) with $k = y$. Multiplying (3.7) for $k = y$ by w_y in terms of the inner product of $L^2(\Omega)$ and integration by parts in the first summand yields

$$(Mw_y, w_y) + y\|w_y\|^2 = \langle Mw_y, w_y\rangle_M + y\|w_y\|^2 = y(a, w_y). \tag{3.17}$$

In view of (3.12) and the definition of w_y we have $y|(a, w_y)| \leq \sigma\|a\|^2$. This inequality and (3.17) implies by the ellipticity of M that

$$\|w_y\|_1 \leq \left(\frac{\sigma}{m_0}\right)^{1/2}\|a\| \tag{3.18}$$

whence

$$\|u_y\|_1 \leq \left(\frac{\sigma}{m_0}\right)^{1/2}\|a\| + \|a\|_1. \tag{3.19}$$

Furthermore, by [18, Chapter 3], the direct problem (2.6), (2.7) has a unique solution $w_y \in W_2^2(\Omega)$ for all $y \geq 0$. Consequently, (3.7) is fulfilled almost everywhere in Ω and $Mw_y \in L_2(\Omega)$. Multiplying (3.7) with $k = y$ by Mw_y in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the second summand one can obtain the equality

$$\|Mw_y\|^2 + y\langle w_y, Mw_y\rangle_M = y(a, Mw_y). \tag{3.20}$$

By (2.1), the second term of (3.20) is nonnegative and

$$y \mid (a, Mw_y) \mid \leq \sigma\|a\| \, \|Mw_y\| \leq \frac{1}{2}\sigma^2\|a\|^2 + \frac{1}{2}\|Mw_y\|^2. \tag{3.21}$$

From (3.20)–(3.21) it follows that

$$\|Mw_y\| \leq \sigma\|a\|. \tag{3.22}$$

In view of the definition of w_y the inequalities (3.18), (3.22) and [18, Chapter 2]

$$\|v\|_2 \leq C_M(\|Mv\| + \|v\|), \tag{3.23}$$

for $v \in \overset{\circ}{W}{}_2^1(\Omega) \bigcap W_2^2(\Omega)$ with the constant C_M depending on M and mesΩ imply the estimate

$$\|u_y\|_2 \leq C_M(\sigma + (\sigma m_0^{-1})^{1/2})\|a\| + \|a\|_2. \tag{3.24}$$

Now we can show that the operator A is continuous on $[0, \sigma]$. Let y_1, $y_2 \in [0, \sigma]$ and u_{y_1}, u_{y_2} be the solutions of the problem (3.7), (3.8) with $k = y_1$ and $k = y_2$, respectively. By the definition of the operator A, (3.12), (3.13),

$$|Ay_1 - Ay_2| \leq \frac{\|u_{y_2} - u_{y_1}\| \; \|b\| \; (\mu - \Psi)}{(a^\sigma, b)^2} \leq \frac{m_0\|u_{y_1} - u_{y_2}\| \; \|b\| \; (\mu - \Psi)}{(m_0(a, b) - \sigma \; \|a\| \; \|b\|)^2}. \tag{3.25}$$

On the other hand, multiplying the difference of equations (2.6) for $k = y_1$ and $k = y_2$ by $u_{y_1} - u_{y_2}$ in terms of the inner product of $L^2(\Omega)$ and integration by parts in the first summand of the resulting equality gives

$$\langle M(u_{y_1} - u_{y_2}), u_{y_1} - u_{y_2}\rangle_M + y_1 \, \|u_{y_1} - u_{y_2}\|^2 = (y_2 - y_1)(u_{y_2}, u_{y_1} - u_{y_2}). \tag{3.26}$$

The right term of (3.26) is estimated with the use of (3.12) as

$$|(y_2 - y_1)(u_{y_2}, u_{y_1} - u_{y_2})| \leq \frac{1}{2m_0}|y_2 - y_1|^2 \; \|a\|^2 + \frac{m_0}{2}\|u_{y_1} - u_{y_2}\|_1^2. \tag{3.27}$$

By (2.1) and the nonnegativity of y_1, from (3.26)–(3.27) we obtain the relation

$$\|u_{y_1} - u_{y_2}\|_1 \leq m_0^{-1}\|a\| \, |y_2 - y_1|. \tag{3.28}$$

Joining (3.25) with

$$\sigma = \frac{m_0((a, b) - \sqrt{D})}{2\|a\| \; \|b\|} \equiv \sigma_0, \tag{3.29}$$

and (3.28) we are led to the inequality

$$|Ay_1 - Ay_2| \le \frac{4\|a\|\,\|b\|(\mu - \Psi)}{m_0((a,b)+\sqrt{D})^2}|y_1 - y_2| \tag{3.30}$$

which implies the continuity of A. Thus, by Brower's theorem, the operator A has a fixed point $k^* \in (0, k_1)$ and the pair $\{u^*, k^*\}$ where the function u^* satisfies (2.6)–(2.7) with $k = k^*$ gives a solution of the problem (2.6)–(2.8).

It remains to prove that the solution of the problem (2.6)–(2.8) is unique under the assumption (3.5). In this case the operator A is a contraction on $[0, \sigma_0]$ because A satisfies (3.30) with

$$q = \frac{4\,\|a\|\,\|b\|(\mu - \Psi)}{m_0((a,b)+\sqrt{D})^2} < \frac{(a,b)^2}{((a,b)+\sqrt{D})^2} < 1.$$

Let (u', k') and (u'', k'') be two solutions of the problem (2.6)–(2.8). Then k', k'' are the fixed poins of the operator A. By (3.30),

$$|k' - k''| = |Ak' - Ak''| \le q|k' - k''|$$

whence $k' - k'' = 0$. This in turn implies $u' - u'' = 0$ in view of (3.28). ■

Under the assumption (3.5) the solution $\{u, k\}$ depends continuously on the input data of Problem 2.

Theorem 3.2 *Let the hypotheses of Theorem 3.1 be fulfilled and $\{u_j, k_j\}$ be the unique solution of Problem 2 where $f = f_j$, $\beta = \beta_j$, $\omega = \omega_j$ and $\mu = \mu_j$, $j = 1, 2$. Then the estimate*

$$\|u_1 - u_2\|_2 + |k_1 - k_2| \le K\big(|\mu_1 - \mu_2| + \|f_1 - f_2\| + \|\beta_1 - \beta_2\|_{3/2} + \|\omega_1 - \omega_2\|_{1/2}\big) \tag{3.31}$$

holds with a positive constant K.

Proof. Let a_j, b_j be the solutions of the problems (3.1) and (3.3) where $f = f_j$, $\beta = \beta_j$, $\omega = \omega_j$, $j = 1, 2$. Repeating the arguments led to (3.10), one can show that k_j is the solution of the operator equation $y = A_j y$ where $A_j y$ is defined by (3.11) for every $y \in [0, \sigma_j]$ and σ_j is given by (3.16) and (3.29) with $\mu = \mu_j$, $f = f_j$, $a = a_j$, $b = b_j$, $j = 1, 2$. Estimating the right-hand side of the difference $k_1 - k_2 = A_1 k_1 - A_2 k_2$ in the absolute value with the use of (3.10) and (3.13) we come to the inequality

$$|k_1 - k_2| \le K_1\big(|\mu_1 - \mu_2| + |\Psi_1 - \Psi_2| + \|b_1 - b_2\|_1\big) + \frac{k_2 m_0 \|b_1\|\|u_1 - u_2\|}{m_0(a_1, b_1) - \sigma_1\|a_1\|\|b_1\|}. \tag{3.32}$$

Here the positive constant K_1 depends on m_0, mesΩ, μ_j, $|\Psi_j|$, $\|a_j\|_1$, $\|b_j\|_1$, $j = 1, 2$.

On the other hand, the difference $u = u_1 - u_2$ satisfies (2.6)–(2.7) where $k = k_1$, $f = (k_2 - k_1)u_2 + f_1 - f_2$ and $\beta = \beta_1 - \beta_2$. By the use of (3.6), (3.19), (3.24) for u_j and k_j with $a = a_j$ and $\sigma = \sigma_j$, $j = 1, 2$, one can obtain the estimates

$$\|u_1 - u_2\|_1 \le m_0^{-1}(\sigma_1\|a_1 - a_2\| + |k_1 - k_2|\|a_1\|) + \|a_1 - a_2\|_1, \tag{3.33}$$

$$\|u_1 - u_2\|_2 \le C_M(m_0 + 1)m_0^{-1}(\sigma_1\|a_1 - a_2\| + |k_1 - k_2|\,\|a_1\|) + \|a_1 - a_2\|_2, \tag{3.34}$$

much as (3.19) and (3.24) are proved. Without loss of generality we can suppose that $k_1 \geq k_2$. Then (3.32) and (3.33) with $\sigma_1 = \sigma_0$ (see (3.29)) yield

$$|k_1 - k_2| \leq K_2\Big[|\mu_1 - \mu_2| + |\Psi_1 - \Psi_2| + \|b_1 - b_2\|_1\Big] \tag{3.35}$$

where K_2 depends on K_1, m_0, σ_1, $\|a_1\|$. Taking into account the definition of Ψ_j, $j = 1, 2$, and joining (3.34)–(3.35) and the inequalities [19, Chapter 2]

$$\|a_1 - a_2\|_j \leq C_2(\|f_1 - f_2\| + \|\beta_1 - \beta_2\|_{j-1/2}), \quad j = 1, 2,$$
$$\|b_1 - b_2\|_1 \leq C_1\|\omega_1 - \omega_2\|_{1/2}, \tag{3.36}$$

where the constants $C_i > 0$, $i = 1, 2$, depend on m_0, m_1 and mesΩ, we are led to the estimate (3.31). ■

Remark 3.1 The set of the input data fitting the conditions (3.4) and (3.5) is not empty. For instance, if $M = -\Delta$ (Δ is the Laplacian), $\beta \equiv const > 0$, $\omega \equiv const > 0$ and $f = 0$, then by the strong maximum principle for the Laplace equation $a \equiv \beta$, $b \equiv \omega$ in $\overline{\Omega}$ and $\Psi = 0$. In this case $m_0(a, b)^2\big(\|a\|\|b\|\big)^{-1} = \beta\omega\text{mes}\Omega$. Therefore the inequality (3.5) is valid for any $0 \leq \mu < \frac{1}{4}\beta\omega\text{mes}\Omega$.

4. The inverse problem for pseudoparabolic equation

In this section we prove the existence and uniqueness theorem for Problem 1. For the sake of convenience we keep the same notations f, β and ω in Problems 1, 2 and suppose that from now on these functions depend on t and x. We define the functions $a^\gamma(t, x)$, $a^\eta(t, x)$ and $b^\eta(t, x)$ as the solutions of the following problems.

$$a^\gamma_t + \eta M a^\gamma_t + M a^\gamma - \gamma a^\gamma = f, \quad (a^\gamma + \eta M a^\gamma)|_{t=0} = U_0, \quad a^\gamma|_{\partial\Omega} = \beta; \tag{4.1}$$
$$a^\eta_t + \eta M a^\eta_t + M a^\eta + \eta^{-1} a^\eta = f, \quad (a^\eta + \eta M a^\eta)|_{t=0} = U_0, \quad a^\eta|_{\partial\Omega} = \beta; \tag{4.2}$$
$$b^\eta + \eta M b^\eta = 0, \qquad b^\eta|_{\partial\Omega} = \omega, \tag{4.3}$$

where $\gamma > 0$ is a real number. We also introduce the notation

$$\Psi_{a,b}(t) = (a^\eta_t, b^\eta) + \eta\langle M a^\eta_t, b^\eta\rangle_M + \langle M a^\eta, b^\eta\rangle_M - (f, b^\eta). \tag{4.4}$$

By a solution of Problem 1 is meant the pair of functions $\{u(t, x), k(t)\}$ such that $u(t, x) \in C^1([0, T]; W^2_2(\Omega)), k(t) \in C([0, T])$; the pair $\{u(t, x), k(t)\}$ satisfies equation (2.2) almost everywhere in Q_T, the initial and boundary data (2.3), (2.4) for almost all $(t, x) \in S_T \bigcup \overline{\Omega}_0$ ($\Omega_0 = \{(0, x),\, x \in \Omega\}$) and the condition of overdetermination (2.5) in (0,T).

The existence and uniqueness of the solution to Problem 1 is established by the following theorem.

Theorem 4.1 *Let the assumptions I-II are fulfilled, $\eta > 0$, $\partial\Omega \in C^2$ and*

(iii) *$f \in C([0, T]); L^2(\Omega))$, $U_0 \in L^2(\Omega)$, $\beta, \omega \in W^{3/2}_2(\partial\Omega)$, $\varphi(t) \in C([0, T])$;*
(iv) *$f \geq 0$ almost everywhere in Q_T, $U_0 \geq 0$ for almost all $x \in \Omega$, $\beta \geq 0$, $\omega \geq 0$ for almost all $(t, x) \in S_T$; there exist α_0, $\Phi_0 \in \mathbf{R}$, $\alpha_0 > 0$ such that for all*

$t \in [0, T]$

$$(a^\eta, b^\eta) \geq \alpha_0, \tag{4.5}$$

$$\Phi_0 \leq \Phi(t) \equiv \varphi(t) - \Psi_{a,b}(t) \leq \eta^{-1}(a^\eta, b^\eta). \tag{4.6}$$

Then there exists a unique solution $\{u, k\}$ of Problem 1. Moreover, the estimates

$$0 \leq a^\eta \leq u \leq a^\gamma, \qquad -\gamma \leq k(t) \leq \eta^{-1}, \qquad \|u\|_2 + \|u_t\|_2 \leq C_3 \tag{4.7}$$

hold with constants $\gamma \geq 0$ and $C_3 > 0$ for almost all $(t, x) \in Q_T$.

Proof. We reduce Problem 1 to an equivalent inverse problem with a nonlinear operator equation for $k(t)$ in much the same way as in the proof of Theorem 3.1. To do this we multiply (2.2) by b^η in terms of the inner product of $L^2(\Omega)$ and integrate by parts in the second and third terms of the resulting equality. By (2.3)–(2.5), this yields

$$(u_t, b^\eta) - \varphi(t) + \int_{\partial\Omega} \big(\eta\beta_t + \beta\big)\frac{\partial b^\eta}{\partial \overline{N}} ds + (k(t)u, b^\eta) + (\eta u_t + u, Mb^\eta) = (f, b^\eta),$$

whence in view of (4.3), (4.4), (4.6) and the fact that

$$\int_{\partial\Omega} \big(\eta\beta_t + \beta\big)\frac{\partial b^\eta}{\partial \overline{N}} ds = \Psi_{a,b}(t) + \eta^{-1}(a^\eta, b^\eta) + (f, b) \tag{4.8}$$

the relation

$$k(t)(u, b^\eta) = \varphi(t) - \Psi_{a,b}(t) - \eta^{-1}(a^\eta, b^\eta) + \eta^{-1}(u, b^\eta) \tag{4.9}$$

comes for all $t \in [0, T]$.

Let the operator B maps every element $z(t) \in C_\gamma([0, T]) = \{z \in C([0, T]), -\gamma \leq z \leq \eta^{-1}\}$ into a function $Bz \in C([0, T])$ by the rule

$$Bz = \frac{\varphi(t) - \Psi_{a,b}(t) - \eta^{-1}(a^\eta, b^\eta) + \eta^{-1}(u_z, b^\eta)}{(u_z, b^\eta)}, \tag{4.10}$$

where u_z is the solution of the direct problem (2.2)–(2.4) with $k(t) = z(t)$. The element Bz is meaningful for every $z \in C_\gamma([0, T])$. Indeed, the direct problem (2.2)–(2.4) with $k = z$ has a unique solution $u_z \in C^1([0, T]); W_2^2(\Omega))$ for every $z \in C_\gamma([0, T])$ by Theorem 2.1 [4]. Moreover, from Lemma 2.2 it follows that for $z(t) \in C_\gamma([0, T])$, the solution u_z fulfils the inequality

$$0 \leq a^\eta \leq u_z \leq a^\gamma \tag{4.11}$$

almost everywhere in Q_T, which implies $(u_z, b) > 0$ on $[0, T]$.

Problem 1 is solvable if and only if the operator equation

$$z = Bz \tag{4.12}$$

has a solution in $C([0, T])$. Really, the deduction of equation (4.10) shows that if $\{u_z, z\}$ is a solution of Problem 1, then z is a fixed point of the operator B

by (4.10). On the other hand, let z^* is a solution of equation (4.12) and u^* is a solution of (2.2)–(2.4) with $k(t) = z^*(t)$. In view of (4.8), (4.10), (4.12) multiplying (2.2) by b^η in terms of the inner product of $L^2(\Omega)$ and integration by parts twice in the second and third summands implies that the pair $\{u^*(t,x), z^*(t)\}$ obeys the cobdition of overdetermination and is the solution of Problem 1.

Let us find such $\gamma > 0$ that the operator B maps $C_\gamma([0,T])$ into itself. In view of (4.6) $Bz \le \eta^{-1}$ for every $z(t) \in C([0,T])$ and $z(t) \le \eta^{-1}$. By (4.6), (4.10) and (4.11),

$$Bz \ge \frac{\min\{0,\Phi_0\} - \eta^{-1}(a^\eta, b^\eta)}{(a^\eta, b^\eta)} = -\Big(\frac{1}{\eta} - \frac{\min\{0,\Phi_0\}}{(a^\eta, b^\eta)}\Big).$$

Thus, the operator B maps $C_\gamma([0,T])$ with

$$\gamma = \frac{1}{\eta} - \frac{\min\{0,\Phi_0\}}{(a^\eta, b^\eta)}, \tag{4.13}$$

into itself and the inequality

$$-\gamma \le Bz \le \eta^{-1} \tag{4.14}$$

is valid. Moreover, the operator B is a contraction on $C_\gamma([0,T])$. Really, let $z_1, z_2 \in C_\gamma([0,T])$ and u_1, u_2 are the solutions to the problem (2.2)–(2.4) with $k(t) = z_1(t)$ and $k(t) = z_2(t)$, respectively. Multiplying the difference of equations (2.2) with $k = z_1$ and $k = z_2$ by $\bar u = u_1 - u_2$ in terms of the inner product of $L^2(\Omega)$, integrating by parts in the first and second summands and estimating the right side of the resulting relation, one can show that

$$\frac{d}{dt}\Big[\|\bar u\|^2 + \eta\langle M\bar u, \bar u\rangle_1\Big] \le \Big(2\gamma + 1\Big)\Big(\|\bar u\|^2 + \eta\langle M\bar u, \bar u\rangle_1\Big) + |\bar z|^2 \max_{t\in[0,T]} \|a^\eta\|^2$$

in view of (4.11) where $\bar z = k_1 - k_2$. Acording to the Gronwall lemma, this inequality implies the estimate

$$\|\bar u\|^2 + \eta\langle M\bar u, \bar u\rangle_1 \le e^{2\gamma+1} \max_{t\in[0,T]} \|a^\eta\| \int_0^t |\bar z|^2 d\tau \equiv C_4 \int_0^t |\bar z|^2 d\tau. \tag{4.15}$$

On the other hand, by the definition of B, (4.11), (4.13) and (4.14)

$$|Bz_1 - Bz_2| \le \frac{\gamma\|b^\eta\|}{(a^\eta, b^\eta)}\|u_1 - u_2\|.$$

Joining (4.15) and the last inequality we obtain

$$|Bz_1 - Bz_2| \le C_5\Big(\int_0^t |\bar z|^2 d\tau\Big)^{1/2} \tag{4.16}$$

where the constant C_5 depends on C_4, γ, $\max_{t\in[0,T]}\|b^\eta\|$ and $\min_{t\in[0,T]}(a^\eta, b^\eta)$. Let us define the norm

$$|z|_\nu = \max_{t\in[0,T]}\{e^{-\nu t}|z|\} \tag{4.17}$$

in $C([0,T])$ with a constant $\nu > 0$. By (4.16),

$$|Bz_1 - Bz_2|_\nu \le C_5 \max_{t\in[0,T]} \{e^{-\nu t} Big(\int_0^t e^{2\nu\tau}\, e^{-2\nu\tau} |\bar{z}|^2 d\tau \Big)^{1/2} \} \le \frac{C_5}{(2\nu)^{1/2}} |\bar{z}|_\nu,$$

from which we conclude that the operator B is a contraction on $C_\gamma([0,T])$ in terms of the norm $|\cdot|_\nu$ with $\nu > \frac{1}{2}C_5^2$. Then according to the principle of contraction mappings, the operator B has a unique fixed point $k^*(t) \in C_\gamma([0,T])$. The pair of the functions $k^*(t)$ and u^* satisfying (2.2)–(2.4) with $k = k^*(t)$ gives the solution of Problem 1. The uniqueness of this solution follows from the contractibility of B and the estimate (4.15).

It remains to obtain the estimates of u and u_t in $W_2^2(\Omega)$. We multiply the difference of equations (2.2) and (4.2)

$$(u - a^\eta)_t + \eta M(u - a^\eta)_t + M(u - a^\eta) = -k(t)u + \frac{1}{\eta} a^\eta \tag{4.18}$$

by Mw^η where $w^\eta \equiv u - a^\eta$ in terms of the inner product of $L^2(\Omega)$ and integrate by parts in the first summand. This yields

$$\frac{1}{2}\frac{d}{dt}\langle w^\eta, Mw^\eta\rangle_M + \frac{\eta}{2}\frac{d}{dt}\|Mw^\eta\|^2 + \|Mw^\eta\|^2 = \big(\eta^{-1}a^\eta - k(t)u, Mw^\eta\big). \tag{4.19}$$

By (4.7),

$$|(\eta^{-1}a^\eta - k(t)u, Mw^\eta)| \le \frac{1}{2}\big(\gamma\|a^\gamma\| + \eta^{-1}\|a^\eta\|\big)^2 + \frac{1}{2}\|Mw^\eta\|^2. \tag{4.20}$$

Integrating (4.19) with respect to t from 0 to τ, $0 < \tau \le T$, and taking into account (4.20) we are led to the inequality

$$\langle w^\eta, Mw^\eta\rangle_M + \eta\|Mw^\eta\|^2 \le \int_0^\tau \big(\gamma\|a^\gamma\| + \eta^{-1}\|a^\eta\|\big)^2 dt, \tag{4.21}$$

which implies by (3.23) that for every $t \in (0,T)$

$$\|u\|_2 \le \frac{C_M}{\eta^{1/2}}\Big[\Big(\int_0^t \big(\gamma\|a^\gamma\| + \eta^{-1}\|a^\eta\|\big)^2 d\tau\Big)^{1/2} + \|a^\gamma\|\Big] + \|a^\gamma\|_2. \tag{4.22}$$

Turning back to equation (4.18) we can get the estimate for u_t. In view of (4.7), (4.21) we have

$$\|(I + \eta M)w_t^\eta\| \le (\eta^{-1}T^{1/2} + 1)(\eta^{-1}\|a^\eta\| + \gamma\|a^\gamma\|) \equiv C_6. \tag{4.23}$$

Furthermore, multiplying (4.18) by w_t^η in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the second summand of the left-hand side of the resulting equality give

$$\|w_t^\eta\|^2 + \eta\langle Mw_t^\eta, w_t^\eta\rangle_1 = \big(-Mw^\eta - k(t)u + \eta^{-1}a^\eta, w_t^\eta\big),$$

from which and (2.1), (4.7), (4.21) it follows that

$$\|w_t^\eta\|^2 + 2\eta m_0\|w_t^\eta\|_1^2 \le C_6^2. \tag{4.24}$$

Since the operator $I + \eta M$ is elliptic, the inequality [18, Chapter 2]

$$\|v\|_2 \leq \tilde{C}_M(\|(I+\eta M)v\| + \|v\|) \tag{4.25}$$

is valid for all $v \in \overset{\circ}{W}{}_2^1(\Omega) \bigcap W_2^2(\Omega)$. Here the constant $\tilde{C}_M$ depends on M, η and $\partial\Omega$. This inequality, (4.23) and (4.24) imply that

$$\|u_t\|_2 \leq 2\tilde{C}_M C_6 + \|a_t^\eta\|_2. \tag{4.26}$$

Joining (4.23) and (4.26) we obtain the last estimate (4.7). ■

Under the hypotheses of Theorem 4.1 the constructed solution $\{u(t,x), k(t)\}$ depends continuously on the input data of Problem 1.

Theorem 4.2 *Let the hypotheses of Theorem 4.1 be fulfilled and $\{u_j(t,x), k_j(t)\}$ be the unique solution of Problem 1 where $f = f_j(t,x)$, $U_0 = U_0^j(x)$, $\beta = \beta_j(t,x)$, $\omega = \omega_j(t,x)$ and $\varphi = \varphi_j(t)$, $j = 1, 2$. Then the estimates*

$$\|k_1 - k_2\|_{C([0,T])} \leq K_3\big\{ \max_{t\in[0,T]} \big[|\varphi_1 - \varphi_2| + \|f_1 - f_2\| + \|\beta_1 - \beta_2\|_{1/2}$$
$$+\|\omega_1 - \omega_2\|_{1/2}\big] + \|U_0^1 - U_0^2\|\big\}, \tag{4.27}$$
$$\|u_1 - u_2\|_{C^1([0,T];W_2^2(\Omega))} \leq K_4\big\{ \max_{t\in[0,T]} \big[|\varphi_1 - \varphi_2| + \|f_1 - f_2\|\big] + \|U_0^1 - U_0^2\|$$
$$+\|\beta_1 - \beta_2\|_{C^1([0,T];W_2^{3/2}(\partial\Omega))} + \|\omega_1 - \omega_2\|_{C^1([0,T];W_2^{1/2}(\partial\Omega))}\big\} \tag{4.28}$$

hold with certain positive constants K_3 and K_4.

Proof. Let a_j^η, a_j^γ and b_j^η be the solutions of the problems (4.1)–(4.3) where $f = f_j$, $U_0 = U_0^j$ $\beta = \beta_j$, $\omega = \omega_j$, $j = 1, 2$. Repeating the arguments led to (4.12) one can show that $k_j(t)$ is the solution of the operator equation $z = B_j z$ where $B_j z$ is defined by (4.10) for every $z \in C_{\gamma_j}([0,T])$ and γ_j is given by (4.13) with $a^\eta = a_j^\eta$, $b^\eta = b_j^\eta$, $j = 1, 2$. In view of (4.5), (4.7) and (4.9) the difference $\tilde{k} = k_1 - k_2$ satisfies the inequality where $\Psi_{a,b}^j(t)$ is defined by (4.4) with $a^\eta = a_j^\eta$, $b^\eta = b_j^\eta$ and $f = f_j$, $j = 1, 2$. From the last inequality, (4.5) and (4.7) it follows that

$$|\tilde{k}| \leq C_7\big[|\varphi_1 - \varphi_2| + |\Psi_{a,b}^2 - \Psi_{a,b}^1| + \|\tilde{a}\| + \|\tilde{b}\|\big] + \alpha_0^{-1}(\gamma_2 + \eta^{-1})\|\tilde{u}\|\|b_1^\eta\|. \tag{4.29}$$

Here $\Psi_{a,b}^j(t)$ is defined by (4.4) with $a^\eta = a_j^\eta$, $b^\eta = b_j^\eta$ and $f = f_j$, $j = 1, 2$, $\tilde{a} = a_1^\eta - a_2^\eta$, $\tilde{b} = b_1^\eta - b_2^\eta$, $\tilde{u} = u_1 - u_2$, the positive constant C_7 depends on η, γ, α_0, $\max_{t\in[0,T]}\{\|a_j^\eta\|, \|a_j^\gamma\|, \|b_j^\eta\|\}$.

On the other hand, the functions $\tilde{w} = \tilde{u} - \tilde{a}$ and $\tilde{k}$ satisfy the relations

$$\tilde{w}_t + \eta M\tilde{w}_t + M\tilde{w} + k_1(t)\tilde{w} = -(k_1(t) - \eta^{-1})\tilde{a} - \tilde{k}u_2, \tag{4.30}$$
$$(\tilde{w} + \eta M\tilde{w})|_{t=0} = 0, \qquad \tilde{w}|_{\partial\Omega} = 0.$$

Multiplying (4.30) by $\tilde{w}$ in terms of the inner product of $L^2(\Omega)$, integrating by parts in the left part and estimating the right-hand side of the resulting equality with the help of (4.7) we conclude by the Gronwall lemma that

$$\|\tilde{u}\| \leq \left[\int_0^t \big((\eta^{-1}+\gamma_1)^2\|\tilde{a}\|^2 + |\tilde{k}|^2\|a_2^\gamma\|^2\big)e^{2(1+\gamma_1)(t-\tau)}d\tau\right]^{1/2} + \|\tilde{a}\|. \qquad (4.31)$$

This inequality and (4.29) imply that

$$|\tilde{k}|_\nu \leq C_8\big\{|\varphi_1-\varphi_2|_\nu + |\Psi^2_{a,b}-\Psi^1_{a,b}|_\nu + \max_{t\in[0,T]}[(\|\tilde{a}\|+\|\tilde{b}\|)e^{-\nu t}] + (2\nu)^{-1/2}|\tilde{k}|_\nu\big\}$$

where $|\cdot|_\nu$ is the norm (4.17) and the positive constant C_8 depends on C_7, η, α_0, T, γ_j, $\max_{t\in[0,T]}\{\|a_j^\eta\|, \|a_j^\gamma\|, \|b_j^\eta\|\}$, $j=1,2$. Choosing $\nu=\nu_1\equiv 2C_8^2$ we are led to the estimate

$$\|\tilde{k}\|_{C([0,T])} \leq 2C_8e^{\nu_1 T}\max_{t\in[0,T]}\big\{|\varphi_1-\varphi_2| + |\Psi^2_{a,b}-\Psi^1_{a,b}| + \|\tilde{a}\| + \|b_1^\eta - b_2^\eta\|\big\} \quad (4.32)$$

and in view of (4.31) we have

$$\max_{t\in[0,T]}\|\tilde{u}\| \leq C_9\max_{t\in[0,T]}\big\{|\varphi_1-\varphi_2| + |\Psi^2_{a,b}-\Psi^1_{a,b}| + \|\tilde{a}\| + \|b_1^\eta - b_2^\eta\|\big\} \qquad (4.33)$$

where C_9 depends on C_8, η, α_0, T, ν_1, γ_j, $\max_{t\in[0,T]}\|a_j^\gamma\|$, $j=1,2$.

The inequalities (4.32) and (4.33) allows to get the appropriate estimates for u_1-u_2 in $C^1([0,T];W_2^2(\Omega))$. By the use of (4.7), (4.22) for u_j and k_j with $a=a_j$ and $\gamma=\gamma_j$, $j=1,2$, one can obtain the estimates

$$\|\tilde{u}\|_2 \leq C_M\big[\gamma_1\big(\|\tilde{a}\|_{L^2(Q_T)} + \|\tilde{u}\|_{L^2(Q_T)}\big) + \|\tilde{k}\|\,\|a_2^\gamma\|_{L^2(Q_T)}\big] + \|\tilde{a}\|_2, \qquad (4.34)$$

$$\|\tilde{u}_t\| \leq (\eta^{-1}(T+\eta))^{1/2}\big[\gamma_1\|\tilde{u}\| + \|\tilde{k}\|\,\|a_2^\gamma\|_{C([0,T];L^2(\Omega))}\big] + \|\tilde{a}_t\|, \qquad (4.35)$$

much as (4.22) and (4.24) are proved. From (4.30) (4.34) and the assumption I it follows that

$$\|(I+\eta M)\tilde{w}_t\| \leq \gamma_1\|\tilde{u}\| + |\tilde{k}|\|a_2^\gamma\| + m_2(\|\tilde{a}\|_2 + \|\tilde{u}\|_2) \qquad (4.36)$$

where the positive constant m_2 depends on n, $\|m_{il}\|_{L^\infty(\Omega)}$ and $\|(m_{il})_{x_i}\|_{L^\infty(\Omega)}$ $i,l=1,2,\ldots,n$. By (4.25) and (4.36),

$$\|\tilde{u}_t\|_2 \leq \tilde{C}_M\big[\gamma_1\|\tilde{u}\| + |\tilde{k}|\|a_2^\gamma\| + m_2(\|\tilde{a}\|_2 + \|\tilde{u}\|_2) + \|\tilde{a}_t\| + \|\tilde{u}_t\|\big] + \|\tilde{a}_t\|_2.$$

Joining the last relation, (3.36), (4.32)–(4.35), the definition of $\Psi^j_{a,b}$, $j=1,2$, and the inequalities

$$\|\tilde{a}\|_q \leq C_{10}(\|f_1-f_2\| + \|\tilde{\beta}\|_{q-1/2} + \|U_0^1 - U_0^2\|),$$
$$\|\tilde{a}_t\|_q \leq C_{11}(\|f_1-f_2\| + \|\tilde{\beta}\|_{q-1/2} + \|\tilde{\beta}_t\|_{q-1/2} + \|U_0^1 - U_0^2\|), \quad q=1,2,$$

for all $t\in[0,T]$ where $\tilde{\beta}\equiv\beta_1-\beta_2$, the constants C_{10}, C_{11} depend on m_0, m_1 and mesΩ, we come to the estimates (4.27), (4.28). ∎

Remark 4.1 The set of the input data fitting the conditions of Theorem 4.1 is not empty. Let $\Omega = (0, \pi)$, $\eta > 0$, $M = -\frac{\partial^2}{\partial x^2}$, $\beta(t, x) \equiv const \geq 0$, $\omega(t, x) \equiv const > 0$, $f \equiv 0$ and $U_0 = \beta + (1 + \eta)\mathrm{sin}x \geq 0$. In this case $a^\eta = e^{-t/\eta}(\beta + \mathrm{sin}x)$, $b^\eta = \omega(\sinh\frac{\pi - x}{\sqrt{\eta}} + \sinh\frac{x}{\sqrt{\eta}})/\sinh\frac{\pi}{\sqrt{\eta}}$. Then $(a^\eta, b^\eta) \geq 2\eta\omega e^{-t/\eta}(1 + \eta)^{-1} > 0$, that is (4.5) is fulfilled. Since $\Psi_{a,b}(t) = -\eta^{-1}(a^\eta, b^\eta)$, the condition (4.6) is valid for any nonpositive $\varphi(t) \in C([0, T])$.

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