

# Two-dimensional stationary flow of two immiscible fluids in a cylinder taking into account the internal energy of the interface

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**Abstract.** In this paper, the problem of a two-dimensional stationary flow of two immiscible viscous heat-conducting fluids in a cylinder is solved. The fluids have a common movable non-deformable interface. The cylinder has a solid outer wall. At the same time the mass forces are absent. The total energy condition at the interface is taken into account. The temperature in liquids is distributed in a quadratic law, which is consistent with the velocity field of the Himenz type. From a mathematical point of view, this initial-boundary value problem is nonlinear and inverse with respect to pressure gradients along the cylinder axis. The modified Galerkin method is used to solve the problem. The effect of the Marangoni number on the fluids flow was investigated.

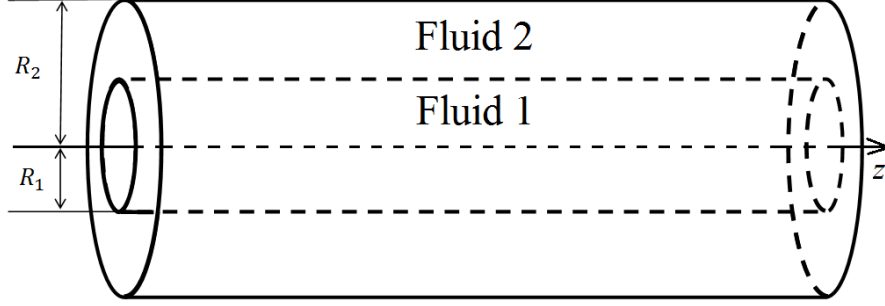
## 1. Introduction

The modeling of convective flows is an important problem in both theoretical and applied terms. The convective flows of two or more fluid media contacting through the interface play an important role, for example, in nanotechnology, the nuclear industry, as well as when cooling devices in microelectronics. The analysis of such flows leads to the study of conjugate problems with complex boundary conditions on the interfaces, where, in particular, the heat fluxes are not equal to each other, since the change in the interfacial energy is taken into account [1]. For ordinary fluids at room temperature, the effect of changes in the internal energy of the interfacial surface on the formation of heat fluxes, temperature fields and velocities in its vicinity is insignificant relative to its viscous friction and heat transfer [2]. Therefore, the class of problems associated with these phenomena remains unattended by most researchers. However, for thin layers or for fluids with reduced viscosity (for example, for some cryogenic fluids, such as liquid CO<sub>2</sub>) it is necessary to take into account the energy spent on the deformation of the interfaces [3], [4]. This effect will appear even in simple cases of a flat interface, when the velocity field depends on two spatial variables.

## 2. Statement of the problem

We consider the conjugate stationary nonlinear boundary value problem describing a two-dimensional two-layer motion of viscous heat-conducting fluids in a cylinder with a solid side surface  $r = R_2 = \text{const}$  (the variable  $r$  is radius cylinder). The fluids form a common interface  $r = R_1 = \text{const}$  (Fig. 1). The mass forces are absent. The velocity, pressure and temperature fields of equations systems describing the axisymmetric stationary motion of a viscous heat-conducting fluid are sought as

$$u_j = u_j(r), \quad w_j = v_j(r)z, \quad p_j = p_j(r, z), \quad \theta_j = \theta_j(r, z), \quad (1)$$



**Figure 1.** The scheme of solution field.

where the index  $j = 1, 2$  is the number of the fluid,  $u_j(r, z)$  and  $w_j(r, z)$  are projections of the velocity vector on the axis  $r$  and  $z$  of the cylindrical coordinate system,  $p_j(r, z, t)$  are pressure that satisfies the ratio

$$\frac{1}{\rho_j} p_j(r, z) = d_j(r) - \frac{f_j}{2} z^2, \quad d_j = \nu \left( u_{jr} + \frac{1}{r} u_j \right) - \frac{1}{2} u_j^2 + d_{j0}, \quad d_{j0} = \text{const.} \quad (2)$$

The temperature field is searched in the following form

$$\theta_j(r, z) = a_j(r) z^2 + b_j(r). \quad (3)$$

So the temperature is extreme at the point  $z = 0$ . It has the maximum at  $a(r) < 0$  and the minimum at  $a(r) > 0$ .

Thus we have the following the equation system for functions  $u_j(r)$ ,  $a_j(r)$ ,  $b_j(r)$

$$\begin{aligned} u_j v_{jr} + v_j^2 &= \nu_j \left( v_{jrr} + \frac{1}{r} v_{jr} \right) + f_j, \\ u_{jr} + \frac{1}{r} u_j + v_j &= 0. \end{aligned} \quad (4)$$

$$\begin{aligned} 2v_j a_j + u_j a_{jr} &= \chi_j \left( a_{jrr} + \frac{1}{r} a_{jr} \right), \\ u_j b_{jr} &= \chi_j \left( b_{jrr} + \frac{1}{r} b_{jr} \right) + 2\chi_j a_j, \end{aligned} \quad (5)$$

where  $\nu_j$  are kinematic viscosity coefficients,  $\chi_j$  are coefficients of thermal diffusivity.

The following conditions are met on a solid wall  $r = R_2$

$$u_2(R_2) = 0, \quad v_2(R_2) = 0, \quad (6)$$

$$a_2(R_2) = \alpha, \quad b_2(R_2) = \beta, \quad (7)$$

with given constants  $\alpha, \beta$ .

At the interface  $r = R_1$  taking into account the dependence of the surface tension coefficient on temperature ( $\sigma(\Theta) = \sigma^0 - \varkappa(\Theta - \Theta^0)$ ,  $\varkappa = -d\sigma/d\Theta = \text{const}$ ) and (3) we obtain the following conditions [6]

$$u_1(R_1) = u_2(R_1) = 0, \quad v_1(R_1) = v_2(R_1), \quad (8)$$

$$\mu_2 v_{2r}(R_1) - \mu_1 v_{1r}(R_1) = -2\varkappa a_1(R_1), \quad (9)$$

$$a_1(R_1, t) = a_2(R_1), \quad k_2 a_{2r}(R_1) - k_1 a_{1r}(R_1) = \varkappa a_1(R_1) v_1(R_1), \quad (10)$$

$$b_1(R_1, t) = b_2(R_1), \quad k_2 b_{2r}(R_1) - k_1 b_{1r}(R_1) = \varkappa b_1(R_1) v_1(R_1), \quad (11)$$

where  $\mu_j = \rho_j \nu_j$  are dynamic viscosity coefficients,  $\rho_j$  are densities,  $k_j$  are coefficients conductivity,

Note that the problem is non-linear and inverse, since, along with  $v_j(r)$ ,  $a_j(r)$ ,  $b_j(r)$  the pressure gradients along the layers  $f_j$  are the searched constant. If we exclude the function  $u_j(r)$  from the second equations (4) taking into account the conditions of sticking on the walls, we will obtain the conjugate boundary value problem for finding functions  $v_j(r)$ ,  $a_j(r)$ . With known  $u_j(r)$ ,  $a_j(r)$  the problem for functions  $b_j(r)$  will separated. The functions  $d_j(r)$  are calculated by the second formula from (2).

We introduce dimensionless functions and parameters

$$V_j(\xi) = \frac{R_1^2}{M \chi_1} v_j(r), \quad A_j(\xi) = \frac{a_j(r)}{\alpha}, \quad F_j = \frac{R_1^4}{M \chi_1^2} f_j, \quad M = \frac{\alpha \alpha R_1^3}{\mu_2 \chi_1},$$

$$\xi = \frac{r}{R_1}, \quad \text{Pr}_j = \frac{\nu_j}{\chi_j}, \quad \chi = \frac{\chi_1}{\chi_2}, \quad \mu = \frac{\mu_1}{\mu_2}, \quad k = \frac{k_1}{k_2}, \quad \gamma = \frac{R_2}{R_1},$$

where M is the Marangoni number,  $\text{Pr}_j$  are Prandtl numbers. Then in dimensionless variables the nonlinear conjugate inverse boundary value problem will take the form

$$\text{Pr}_1 \left( V_{1\xi\xi} + \frac{1}{\xi} V_{1\xi} \right) + \frac{M}{\xi} V_{1\xi} \int_0^\xi x V_1(x) dx - M V_1^2 + F_1 = 0, \quad (12)$$

$$A_{1\xi\xi} + \frac{1}{\xi} A_{1\xi} + \frac{M}{\xi} A_{1\xi} \int_0^\xi x V_1(x) dx - 2M V_1 A_1 = 0, \quad 0 < \xi \leq 1;$$

$$\frac{\text{Pr}_2}{\chi} \left( V_{2\xi\xi} + \frac{1}{\xi} V_{2\xi} \right) - \frac{M}{\xi} V_{2\xi} \int_\xi^\gamma x V_2(x) dx - M V_2^2 + F_2 = 0, \quad (13)$$

$$\frac{1}{\chi} \left( A_{2\xi\xi} + \frac{1}{\xi} A_{2\xi} \right) - \frac{M}{\xi} A_{2\xi} \int_\xi^\gamma x V_2(x) dx - 2M V_2 A_2 = 0, \quad 1 \leq \xi \leq \gamma,$$

$$V_2(\gamma) = 0, \quad A_2(\gamma) = 1, \quad (14)$$

$$\int_0^1 x V_1(x) dx = 0, \quad \int_1^\gamma x V_2(x) dx = 0, \quad (15)$$

$$V_{2\xi}(1) - \mu V_{1\xi}(1) = -2A_1(1), \quad (16)$$

$$V_1(1) = V_2(1), \quad |V_1(0)| < \infty, \quad (17)$$

$$A_{2\xi}(1) - k A_{1\xi}(1) = E A_1(1) V_1(1), \quad (18)$$

$$A_1(1) = A_2(1), \quad |A_1(0)| < \infty, \quad (19)$$

where  $E = \alpha^2 \alpha R_1^2 / \mu_2 k_2$  is parameter which determines the influence of internal interfacial energy on the dynamics of the fluids motion inside the layers. The integral override conditions (15) allow finding unknown constants (pressure gradients along the layers)  $F_j$ ,  $j = 1, 2$ .

### 3. Results of numerical calculations

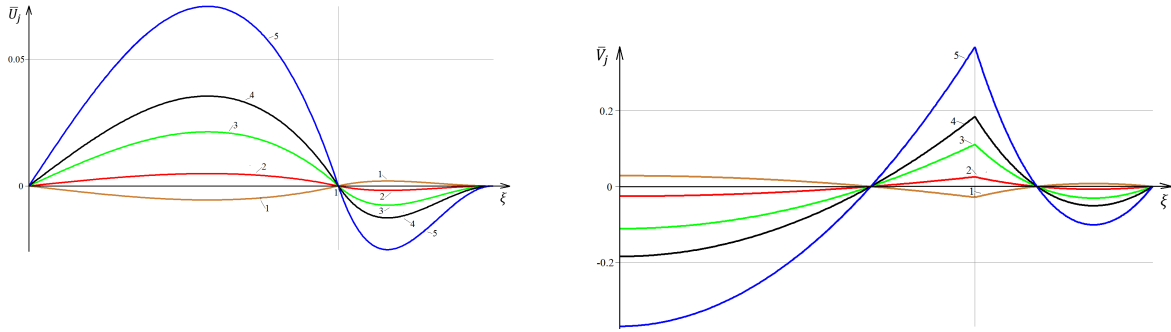
To solve the problem (12)-(19), a modified Galerkin method was applied. This method differs from the Galerkin method in that the approximate solution does not necessarily have to satisfy the boundary conditions [7]. The sought functions  $V_j(\xi)$  and  $A_j(\xi)$  are searched for in the form of an expansion in basic functions, which have the form of the shifted Legendre polynomials. The approximate solutions are substituted into the equations, and their discrepancy are calculated. Further, the requirement of the orthogonality of the residual to the basis functions is advanced.

The result the closed system of nonlinear algebraic equations for the unknown expansion coefficients and pressure gradients along the layers is obtained. To solve the obtained system, the Newton method was used, where the results obtained in solving the model problem [8] were used as the initial approximation.

For the calculations, water H<sub>2</sub>O ( $0 \leq r \leq 1$ ) and saturated liquid CO<sub>2</sub> ( $1 \leq r \leq \gamma$ ) at the temperature of 30°C were used as working media. Their parameters are given in the following order "H<sub>2</sub>O; CO<sub>2</sub>":  $\rho = \{995.6; 598\}$  kg/m<sup>3</sup>,  $\nu = \{0.8012 \cdot 10^{-6}; 0.08 \cdot 10^{-6}\}$  m<sup>2</sup>/s,  $\chi = \{1.4741 \cdot 10^{-7}; 2.7875 \cdot 10^{-9}\}$  m<sup>2</sup>/s,  $k = \{0.6133; 0.07\}$  W/(m·K),  $\alpha = 2.11 \cdot 10^{-4}$  N/(m·K),  $\sigma_0 = 71.2 \cdot 10^{-3}$  N/m. Also the parameters are set  $n = 10$ ,  $\gamma = 1.5$ ,  $R_1 = 10^{-9}$  m,  $M = 9$ . It corresponds to the temperature 300.81 K (the temperature value at the critical point for saturated liquid CO<sub>2</sub> is 304.15 ),  $E \approx 4$ . Four different values of dimensionless constants  $F_1$ ,  $F_2$  were found. It is  $\{F_1^1 = -0.88658; F_2^1 = -0.2294\}$ ,  $\{F_1^2 = 2.04503; F_2^2 = 0.52953\}$ ,  $\{F_1^3 = -9.18797; F_2^3 = -2.37428\}$ ,  $\{F_1^4 = 97.13247; F_2^4 = 26.83752\}$  (the superscript denotes the number of the solution). In this case, the difference between the values obtained for  $n = 10$  and  $n = 17$  is of the order  $10^{-14}$ ,  $10^{-12}$ ,  $10^{-8}$ ,  $10^{-4}$  for  $F_j^1$ ,  $F_j^2$ ,  $F_j^3$ ,  $F_j^4$  respectively. This suggests a good convergence of the tau method in solving the problem.

It should be noted that with a decrease in the Marangoni number for a fixed value of the parameter  $E \approx 4$  the solutions obtained  $F_j^1$  tends at the exact solution of the creeping flow problem with a Himenz-type velocity field (1)  $\{F_1^{01} = -1.1164715; F_2^{01} = -0.28893927\}$  and  $\{F_1^{02} = 27.931; F_2^{02} = 7.228\}$ . At  $M = 10^{-3}$  we have  $\{F_1^1 = -1.11641882; F_2^1 = -0.2889256\}$  and  $\{F_1^1 = 27.883; F_2^1 = 7.216\}$ .

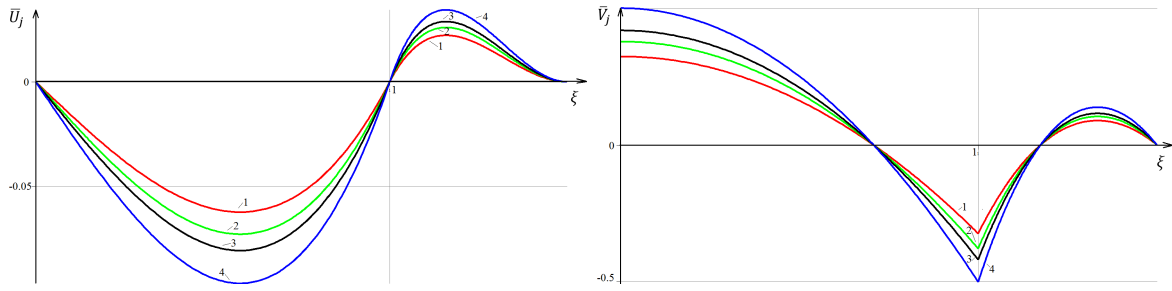
The velocity profiles  $\bar{U}_j(\xi)$  and the functions  $\bar{V}_j(\xi)$  for the 1st solution for different values of the parameter  $M = \{-1; 1; 5; 9; 20\}$  are obtained Fig. 2, 3. Here  $\bar{V}_j(\xi) = MV_j(\xi)$  and on the interval  $0 \leq \xi \leq 1$  the velocity profile  $\bar{U}_1(\xi)$  and the function  $\bar{V}_1(\xi)$  are shown and on the interval  $1 \leq \xi \leq \gamma$  the velocity profile  $\bar{U}_2(\xi)$  and the function  $\bar{V}_2(\xi)$  are shown. The case when  $M < 0$  means that  $\alpha < 0$  and the temperature on the tube wall has the maximum value at the point  $z = 0$ . From the Fig. 2, 3 we can see, that with an increase in the Marangoni number when  $M > 0$  the velocity profile values  $\bar{U}_j(\xi)$  and functions  $\bar{V}_j(\xi)$  increase. The nature of the parameter influence  $M$  on these functions will not change when considering other solutions. This dependence for the second solution is shown in the Fig. 4, 5.



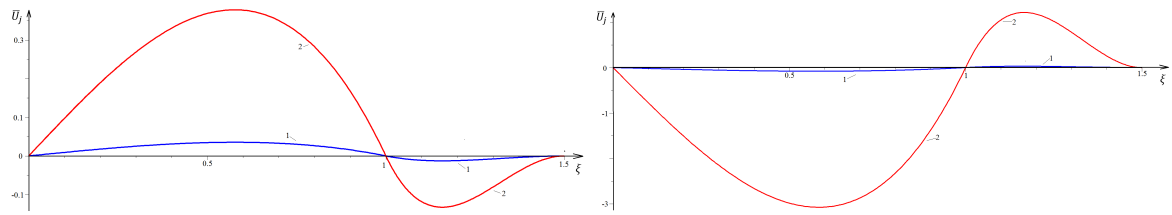
**Figure 2.** The effect of the Marangoni number on the value of the velocity profiles  $\bar{U}_j(\xi)$  for the first solution. 1 –  $M = -1$ , 2 –  $M = 1$ , 3 –  $M = 5$ , 4 –  $M = 9$ , 5 –  $M = 20$ . **Figure 3.** The effect of the Marangoni number on the value of the velocity profiles  $\bar{V}_j(\xi)$  for the first solution. 1 –  $M = -1$ , 2 –  $M = 1$ , 3 –  $M = 5$ , 4 –  $M = 9$ , 5 –  $M = 20$ .

Note that the view figure of  $\bar{U}_j(\xi)$  for the third and fourth solutions will correspond to the form of the same graphs, built for the first and second solutions (Fig. 6, 7). At the same time, as you can see, with each solution the flow of fluids becomes more intense.

Thus, the effect of the Marangoni number on the axially symmetric two-layer stationary thermocapillary flow in a cylinder has been investigated taking into account the total energy condition at the interface.



**Figure 4.** The effect of the Marangoni number on the value of the velocity profiles  $\bar{U}_j(\xi)$  for the first solution. 1 –  $M = 1$ , 2 –  $M = 5$ , 3 –  $M = 9$ , 4 –  $M = 20$ . **Figure 5.** The effect of the Marangoni number on the value of the velocity profiles  $\bar{V}_j(\xi)$  for the second solution. 1 –  $M = 1$ , 2 –  $M = 5$ , 3 –  $M = 9$ , 4 –  $M = 20$ .



**Figure 6.** The velocity profiles  $\bar{U}_j(\xi)$  for: 1 – the first solution, 2 – the third solution. **Figure 7.** The velocity profiles  $\bar{U}_j(\xi)$  for: 1 – the second solution, 2 – the fourth solution.

## References

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