An identity for generating functions is proved in this paper. A novel method to compute the number of restricted lattice paths is developed on the basis of this identity. The method employs a difference equation with non-constant coefficients. Dyck paths, Schröder paths, Motzkins path and other paths are computed to illustrate this method.

Keywords: difference equation, generating function, lattice path.

For $z = (z_1, ..., z_N)$ we introduce the ring of polynomials $\mathbb{C}[z]$, the field of rational functions $\mathbb{C}(z)$ and $\mathbb{C}[[z]]$ - the ring of formal power series in $z_1, ..., z_N$, where $z^x = z_1^x_1 \cdots z_N^x_N$. If $f$ is a function on $\mathbb{Z}^N$ it is identified as a function on $\mathbb{Z}^N$ by setting it equal to zero on the complement $\mathbb{Z}^N \setminus P$.

A linear finite difference equation is an equation of the form

$$\sum_{y \in S} c_y(x)f(x-y) = g(x), \quad x \in \mathbb{Z}^N, \quad (1)$$

where $S \subset \mathbb{Z}^N$ is finite, $c_y : \mathbb{Z}^N \rightarrow \mathbb{C}$ are a set of coefficient functions and $g : \mathbb{Z}^N \rightarrow \mathbb{C}$. A solution of (1) is a function $f : \mathbb{Z}^N \rightarrow \mathbb{C}$ that satisfies the equation.

Let $\ell \geq 0$. A lattice path with length $\ell$ is a finite sequence $p(0), p(1), \ldots, p(\ell)$ of points in $\mathbb{Z}^N$, and its steps are the set of lattice vectors $\{0\} \cup \{p(k)-p(k-1) : k = 1, \ldots, \ell\}$. Specific classes of lattice paths arise by imposing some conditions on the paths: the steps are in a specified $S \subset \mathbb{Z}^N$, the points are in a specified $P \subset \mathbb{Z}^N$, the length $L$ is fixed, and the points are distinct (non intersecting paths).

In the context of lattice path counting problems the function $f : \mathbb{Z}^N \rightarrow \mathbb{Z}_2$ that counts the number $f(x)$ of paths in a specified class for which $p(0) = 0$ and $p(L) = x$ is computed (the condition $p(0) = 0$ does not result in a loss of generality).

In the context of restricted lattice class problems function $f$ is computed for a class of paths whose points belong to a specified subset $P \subset \mathbb{Z}^N$. Clearly $0 \in P$ otherwise there are no paths in $P$ that start at 0. If the possible set of steps $S \subset \mathbb{Z}^N$ then counting function $f$ has support in $P$, $f(0) = 1$, and $f$ satisfies the linear homogeneous difference equation

$$f(x) = \sum_{y \in S} \chi_P(x)f(x-y), \quad (2)$$

where $\chi_P$ is the characteristic function of $P$. We recall that $\chi_P(x) = 1$ if $x \in P$ and $\chi_P(x) = 0$ if $x \notin P$.

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In this paper function \( f \) is computed for selected classes of lattice paths. For these classes \( S \subseteq \mathbb{Z}^N_x \) so \( f \) is supported on \( \mathbb{R}^N_x \), and therefore \( f \) is uniquely represented by its generating function \( F(z) \in \mathbb{C}[[z]] \). It is defined by

\[
F(z) = \sum_{x \in \mathbb{Z}^N_x} f(x) z^x. \tag{3}
\]

The proposed method employs a difference equation with non-constant coefficients for \( f(x) \) to compute its generating function \( F(z) \). This method is illustrated by counting Dyck paths, Schröder paths, Motzkin paths and more general paths. For these cases the terms in \( F \) correspond to the terms with non-constant coefficients in the difference equation for \( f(x) \) are the generating functions of a diagonal subsequence of \( f(x) \).

Let \( P(z) = \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \) be a polynomial in \( z \in \mathbb{C}^N \) and \( x, m, \alpha \in \mathbb{Z}^N_x \). The inequality \( 0 \leq \alpha \leq m \) means that \( 0 \leq \alpha_j \leq m_j \) for all \( j = 1, \ldots, N \). Let us introduce \( F_\alpha(z) = \sum_{x \geq m} f(x) z^x \) and \( \Phi_\alpha(z) = F(z) - F_\alpha(z) \), where the inequality \( x \not\geq \alpha \) means that for at least one \( j_0 \in \{1, \ldots, N\} \) the inequality \( x_{j_0} < \alpha_{j_0} \) holds.

Let \( \delta_j \) be a shift operator over \( j^{th} \) variable: \( \delta_j f(x) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_N) \). Then \( \delta^\alpha = \delta_1^{\alpha_1} \circ \cdots \circ \delta_N^{\alpha_N} \) and \( P(\delta) = \sum_{0 \leq \alpha \leq m} c_\alpha \delta^\alpha \) is a polynomial difference operator with constant coefficients.

First a general identity for the generating functions is derived. Let us note that this theorem generalize the identity for generating functions given in [1].

**Theorem 1.** For any \( F(z) \in \mathbb{C}[[z]] \) the identity

\[
P(z) F(z) - \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \Phi_{m-\alpha}(z) = \sum_{x \geq m} P(\delta^{-1}) f(x) z^x \tag{4}
\]

holds, where \( I = (1, \ldots, 1) \).

**Proof.** Let \( F(z) = \sum_{x \geq 0} f(x) z^x \) and \( P(z) = \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \). Let us consider the product

\[
P(z) F(z) = \left( \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \right) \left( \sum_{x \geq 0} f(x) z^x \right) = \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \left( \sum_{x \geq m-\alpha} f(x) z^x + \sum_{x \not\geq m-\alpha} f(x) z^x \right) =
\]

\[
= \sum_{0 \leq \alpha \leq m} c_\alpha \sum_{x \geq m-\alpha} f(x) z^{x+\alpha} + \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \sum_{x \not\geq m-\alpha} f(x) z^x =
\]

\[
= \sum_{0 \leq \alpha \leq m} c_\alpha \sum_{x \geq m} f(x) z^x - \alpha z^\alpha \sum_{x \not\geq m-\alpha} f(x) z^x =
\]

\[
= \sum_{x \geq m} \left( \sum_{0 \leq \alpha \leq m} c_\alpha f(x - \alpha) \right) z^x + \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \sum_{x \not\geq m-\alpha} f(x) z^x.
\]

Since \( \sum_{0 \leq \alpha \leq m} c_\alpha f(x - \alpha) = P(\delta^{-1}) f(x) \) and \( \Phi_{m-\alpha}(z) = \sum_{x \not\geq m-\alpha} f(x) z^x \) we obtain (4). \( \square \)

Identity (4) implies that for any initial data \( \phi(x), x \not\geq m, x \geq 0 \) and any function \( g(x), x \geq m \) the equation \( P(\delta^{-1}) f(x) = g(x) \) has a unique solution \( f(x) \) that satisfies initial data: \( f(x) = \phi(x), x \geq 0, x \not\geq m \) (see [2, 3, 4]). If \( G(z) = \sum_{x \geq m} g(x) z^x \) then identity (4) gives

\[
F(z) = \frac{1}{P(z)} \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \Phi_{m-\alpha}(z) + \frac{G(z)}{P(z)}.
\]
Let $\Delta = \{e^1, \ldots, e^N\}$, where vector $e^j = (0, \ldots, 0, 1, 0, \ldots, 0)$ contains one on the $j^{th}$ place for $j = 1, \ldots, N$. Let $f(x)$ be the number of paths from the origin to the point $x \in \mathbb{Z}_+^N$.

**Corollary.** If $f(x)$ is the number of lattice paths from the origin to $x \in \mathbb{Z}_+^N$ using steps from the set $\Delta$ then its generating function $F(z)$ is

$$F(z) = \frac{1}{1 - z_1 - \cdots - z_N}.$$  

**Proof.** Let us note that function $f(x)$ satisfies the basic recurrence relation $f(x) = f(x-e^1) + \cdots + f(x-e^N)$ which implies that the right side of identity (4) is equal to 0.

Let us write identity (4) for the two dimensional case:

$$(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_2)F(0, z_2) - (1 - z_1)F(z_1, 0) + F(0, 0) = 0.$$  

Since $f(x_1, 0) = f(0, x_2) = 1$ for all non-negative integers $x_1$ and $x_2$ we obtain

$$F(0, z_2) = \frac{1}{1 - z_2}, \quad F(z_1, 0) = \frac{1}{1 - z_1}, \quad F(0, 0) = 1.$$  

Then $F(z_1, z_2) = \frac{1}{1 - z_1 - z_2}$.

For $N = 3$ we have

$$(1 - z_1 - z_2 - z_3)F(z_1, z_2, z_3) - (1 - z_1 - z_2)F(z_1, z_2, 0) - (1 - z_1 - z_3)F(z_1, 0, z_3) -$$

$$- (1 - z_2 - z_3)F(0, z_2, z_3) + (1 - z_1)F(z_1, 0, 0) + (1 - z_2)F(0, z_2, 0) +$$

$$+ (1 - z_3)F(0, 0, z_3) - F(0, 0, 0) = 0.$$  

Considering the three dimensional case, we obtain

$$F(z_1, z_2, z_3) = \frac{1}{1 - z_1 - z_2 - z_3}.$$  

Repeating this process, one can obtain the generating function for any $N > 1$. 

Let us demonstrate another way of using identity (4) for the two-dimensional case which is useful for some lattice path problems.

Let us introduce $F(z_1, z_2) = \sum_{(x_1, x_2) > (0, 0)} f(x_1, x_2)z_1^{x_1}z_2^{x_2}$ and $F_{p,q}(z_1, z_2) = \sum_{k=1}^{\infty} f(pk, qk)z_1^p z_2^q$, where $(p, q) \in \mathbb{Z}_+^2$.

Let us assume that function $f(x, y) = \varphi(x, y)$, $(x, y) \not\in (p, q)$, $(x, y) \geq 0$ satisfies the difference equation

$$P(\delta^{-1}_1, \delta^{-1}_2)f(x, y) = g(x, y),$$

where

$$g(x, y) = \left\{ \begin{array} {ll} f(x, y), & \text{if } x = pk, y = qk, k \geq 1 \\ 0, & \text{otherwise} \end{array} \right.$$  

Then $F_{p,q}(z_1, z_2) = \sum_{(x,y) > (p,q)} g(x, y)z_1^x z_2^y$ and identity (4) becomes

$$P(z_1, z_2)F(z_1, z_2) = \sum_{0 \leq \alpha_1, \alpha_2 \leq \beta \leq \delta} c_{\alpha_1, \alpha_2}z_1^{\alpha_1}z_2^{\alpha_2}P_{\beta - \alpha_1, \beta - \alpha_2}(z_1, z_2) = F_{p,q}(z_1, z_2).$$
If \( z_1 = z_1(t), \) \( z_2 = z_2(t) \) is a solution of system \[
\begin{align*}
\frac{d^2}{dt^2}z_1^2 = t \\
F(z_1, z_2) = 0
\end{align*}
\] then function \( F_{p,q} \) is

\[
F_{p,q}(z_1(t), z_2(t)) = \sum_{0 \leq \alpha_1, \alpha_2 \leq p} c_{\alpha_1, \alpha_2} z_1^{\alpha_1}(t) z_2^{\alpha_2}(t) \Phi_{p-\alpha_1, q-\alpha_2}(z_1(t), z_2(t)).
\]

Let us consider examples of some well-known lattice paths: Dyck, Motzkin and Schröder paths (see [2, 5, 6, 7, 8, 9, 10]). A linear transformation is used to map the mentioned above lattice paths to the lattice path in \( \mathbb{Z}^2 \). It allows one to use methods for finding generating functions [3] and [11]. However, to study lattice paths on or over a rational slope linear difference equations with non-constant coefficients are used to put restrictions on them.

Dyck paths start at the origin and stay on or above the main diagonal \( y = x \) (see [2, 6, 12]). They use steps \( e_1 = (1,0) \) and \( e_2 = (0,1) \). Let \( f(x, y) \) be the number of paths going from \((0,0)\) to \((x,y)\). The number of paths \( f(x, y) \) satisfies the difference equation

\[
f(x, y) - f(x-1, y) - f(x, y-1) = -\delta_0(x - y - 1)f(x - 1, y),
\]

where \( \delta_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} \) is the Kronecker symbol. The initial data are

\[
f(x, 0) = 0, \quad x = 1, 2, \ldots, \quad f(0, y) = 1, \quad y = 0, 1, 2, \ldots.
\]

Let \( F_{11}(t) \) be a diagonal power series of \( F(z_1, z_2) \)

\[
F_{11}(t) = \sum_{k=1}^{\infty} f(k,k)t^k.
\]

**Proposition 1.** Let \( F(z_1, z_2) \) be the generating function of the solution of equation (5). Then the series \( F(z_1, z_2) \) satisfies the following functional equation

\[
(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k,k)(z_1z_2)^k.
\]

If \( f(x, y) \) satisfies initial conditions (6) then we obtain a diagonal power series

\[
F_{11}(t) = \sum_{k=1}^{\infty} f(k,k)t^k = \frac{1 - 2t - \sqrt{1 - 4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \cdots.
\]

**Proof.** For \( N = 2 \) and \( P(z_1, z_2) = 1 - z_1 - z_2 \) we have \( c_{00} = 1, \) \( c_{10} = c_{01} = -1, \) \( c_{11} = 0, \) \( m = (1,1), \) \( \Phi_{1,1}(z_1, z_2) = F(z_1, 0) + F(0, z_2) - F(0, 0), \) \( \Phi_{1,0}(z_1, z_2) = F(0, z_2), \) \( \Phi_{0,1}(z_1, z_2) = F(z_1, 0), \) \( \Phi_{0,0}(z_1, z_2) = 0. \)

Then by theorem we obtain

\[
(1 - z_1 - z_2)F(z_1, z_2) - c_{00}\Phi_{1,1}(z_1, z_2) - c_{10}\Phi_{0,1}(z_1, z_2) - c_{01}\Phi_{1,0}(z_1, z_2) = \sum_{x \geq 1} (1 - \delta_1^{-1} - \delta_2^{-1})f(x, y)z_1^xz_2^y.
\]

Using difference equation (5), we have

\[
(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k,k)(z_1z_2)^k,
\]
where \( F(0, z_2) = \sum_{y \geq 0} f(0, y) z_2^y = \frac{1}{1 - z_2}, \) \( F(z_1, 0) = \sum_{x \geq 0} f(x, 0) z_1^x = 1, \) \( F(0, 0) = f(0, 0) = 1. \)

Let \( P(z_1, z_2) = 1 - z_1 - z_2 = 0. \) Then we obtain
\[
-1 + \frac{1}{z_1} = \sum_{k=1}^{\infty} f(k, k)(z_1(1 - z_1))^k.
\]

Let us introduce \( t = z_1(1 - z_1). \) Then \( z_1 = \frac{1 - t + \sqrt{1 - 6t + t^2}}{2}. \) After expansion of \( \frac{1}{z_1} \) we obtain (10).

\[\sum_{k=1}^{\infty} f(k, k)t^k = -1 + \frac{1 - \sqrt{1 - 4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \ldots.\]

It proves the proposition.

The coefficients of series (7) represent the Catalan numbers \( f(k, k) = \frac{1}{k - 1} \left(\begin{array}{c} 2k \\ k \end{array}\right), \) \( k \geq 1. \)

Schröder paths start at the origin and stay on or above the main diagonal \( y = x \) (see [2]) using steps \((1, 0), (0, 1), (1, 1). \) Let \( f(x, y) \) be the number of paths going from \((0, 0)\) to \((x, y)\). The number of paths \( f(x, y) \) satisfies the difference equation
\[f(x, y) - f(x - 1, y) - f(x, y - 1) - f(x - 1, y - 1) = -\delta_0(x - y - 1)f(x - 1, y)\] (8)

with the initial data
\[f(x, 0) = 0, \quad x = 1, 2, \ldots, \quad f(0, y) = 1, \quad y = 0, 1, 2, \ldots.\] (9)

**Proposition 2.** Let \( F(z_1, z_2) \) be the generating function of the solution of equation (8). Then the series \( F(z_1, z_2) \) satisfies the following functional equation
\[(1 - z_1 - z_2 - z_1z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k, k)(z_1z_2)^k.\]

If \( f(x, y) \) satisfies initial conditions (9) then we obtain a diagonal power series
\[F_{11}(t) = \sum_{k=1}^{\infty} f(k, k)t^k = \frac{1 - 3t - \sqrt{1 - 6t + t^2}}{2t} = 2t + 6t^2 + 22t^3 + 90t^4 + \ldots.\] (10)

**Proof.** Using theorem and difference equation (8), we have
\[(1 - z_1 - z_2 - z_1z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k, k)(z_1z_2)^k,\]

where \( F(0, z_2) = \sum_{y \geq 0} f(0, y) z_2^y = \frac{1}{1 - z_2}, \) \( F(z_1, 0) = \sum_{x \geq 0} f(x, 0) z_1^x = 1, \) \( F(0, 0) = f(0, 0) = 1. \)

Let \( P(z_1, z_2) = 1 - z_1 - z_2 - z_1z_2 = 0 \), we obtain
\[-1 + \frac{1}{z_1} = \sum_{k=1}^{\infty} f(k, k) \left( \frac{z_1(1 - z_1)}{1 + z_1} \right)^k.\]

Let us introduce \( t = \frac{z_1(1 - z_1)}{1 + z_1}. \) Then \( z_1 = \frac{1 - t + \sqrt{1 - 6t + t^2}}{2}. \) After expansion of \( \frac{1}{z_1} \) we obtain (10). \( \square \)
Coefficients of series (10) coincide with the numbers of the Schröder paths ending on the main diagonal $y = x$.

**Motzkin paths** start at the origin and stay on or above the main diagonal $y = x$ (see [8]) using steps $(2, 0), (0, 2), (1, 1)$. Let $f(x, y)$ be the number of paths going from $(0, 0)$ to $(x, y)$. The number of paths $f(x, y)$ satisfies the difference equation

$$f(x, y) - f(x - 2, y) - f(x, y - 2) - f(x - 1, y - 1) = -((\delta_0(x - y) + \delta_0(x - y - 2)) f(x - 2, y)$$

with the initial data

$$f(x, 0) = 0, \quad x = 1, 2, 3, \ldots, \quad f(0, y) = \frac{1 + (-1)^y}{2}, \quad y = 0, 1, 2, \ldots,$$

$$f(x, 1) = 0, \quad x = 2, 3, 4, \ldots, \quad f(1, y) = \frac{(1 - (-1)^y(y + 1)}{4}, \quad y = 1, 2, 3, \ldots$$

**Proposition 3.** Let $F(z_1, z_2)$ be the generating function of the solution of equation (11). Then the series $F(z_1, z_2)$ satisfies the following functional equation

$$(1 - z_1^2 - z_2^2 - z_1z_2) F(z_1, z_2) - (1 - z_1^2) F(z_1, 0) - (1 - z_2^2) F(0, z_2) + F(0, 0)(1 - z_1z_2) - \Phi_1,0(z_1, z_2)(1 - z_2^2) - \Phi_0,1(z_1, z_2)(1 - z_1^2) + f(1, 1) z_1z_2 =$$

$$= -z_1^2 \sum_{k \geq 2} f(k, k)(z_1z_2)^k,$$

where $F(z_1, 0) = 1$, $F(0, z_2) = \frac{1}{1 - z_2^2}$, $F(0, 0) = 1$, $\Phi_1,0(z_1, z_2) = z_1z_2$, $\Phi_0,1(z_1, z_2) = \frac{z_1^2 - 1}{(1 - z_2^2)^2}$, $f(1, 1) = 1$.

Let $P(z_1, z_2) = 1 - z_1^2 - z_2^2 - z_1z_2 = 0$. Then we obtain

$$-z_1z_2 - 1 + \frac{1}{z_1^2} = \sum_{k=2}^{\infty} f(k, k) \left( \frac{z_1 (\sqrt{4 - 3z_1^2} - z_1)}{2} \right)^k.$$

Let us introduce $t = z_1z_2$. Then $z_1^2 = \frac{1 - t + \sqrt{1 - 2t - 3t^2}}{2}$. After expansion of $\frac{1}{z_1^2}$ we obtain

$$F_{11}(t) = \sum_{k=2}^{\infty} f(k, k)t^k = -1 - t + \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2} = 2t^2 + 4t^3 + 9t^4 + 21t^5 + \cdots \quad (13)$$

**Proof.** Using theorem and difference equation (11), we have

$$(1 - z_1^2 - z_2^2 - z_1z_2) F(z_1, z_2) - (1 - z_1^2) F(z_1, 0) - (1 - z_2^2) F(0, z_2) + F(0, 0)(1 - z_1z_2) - \Phi_1,0(z_1, z_2)(1 - z_2^2) - \Phi_0,1(z_1, z_2)(1 - z_1^2) + f(1, 1) z_1z_2 =$$

$$= -z_1^2 \sum_{k \geq 2} f(k, k)(z_1z_2)^k,$$

where $F(z_1, 0) = 1$, $F(0, z_2) = \frac{1}{1 - z_2^2}$, $F(0, 0) = 1$, $\Phi_1,0(z_1, z_2) = z_1z_2$, $\Phi_0,1(z_1, z_2) = \frac{z_1^2 - 1}{(1 - z_2^2)^2}$, $f(1, 1) = 1$.

Let $P(z_1, z_2) = 1 - z_1^2 - z_2^2 - z_1z_2 = 0$. Then we obtain

$$-z_1z_2 - 1 + \frac{1}{z_1^2} = \sum_{k=2}^{\infty} f(k, k) \left( \frac{z_1 (\sqrt{4 - 3z_1^2} - z_1)}{2} \right)^k.$$

Let us introduce $t = z_1z_2$. Then $z_1^2 = \frac{1 - t + \sqrt{1 - 2t - 3t^2}}{2}$. After expansion of $\frac{1}{z_1^2}$ we obtain

$$(13).$$

Coefficients of series (13) coincides with the numbers of the Motzkin paths ending on the main diagonal $y = x$.

Let us consider some generalization of the problem of enumerating lattice paths with steps $(1, 0), (0, 1), (r, r)$ which start at the origin and stay on or above the main diagonal $y = x$. Let $f(x, y)$ be the number of paths going from $(0, 0)$ to $(x, y)$. The number of paths $f(x, y)$ satisfies the difference equation

$$f(x, y) - f(x - 1, y) - f(x, y - 1) - f(x - r, y - r) = -\delta_0(x - y - 1) f(x - 1, y)$$

(14)
with some initial data

\[ f(x, y) = \varphi(x, y), \quad (x, y) \geq (0, 0), \quad (x, y) \neq (r, r). \]  

**Proposition 4.** Let \( F(z_1, z_2) \) be the generating function of the solution of equation (14). Then the series \( F(z_1, z_2) \) satisfies the following functional equation

\[
(1 - z_1 - z_2 - z_1^r z_2^r)F(z_1, z_2) - \Phi_{r, r}(z_1, z_2) + z_1 \Phi_{r-1, r}(z_1, z_2) + z_2 \Phi_{r-1, r}(z_1, z_2) = -z_1 \sum_{k \geq r} f(k, k)(z_1 z_2)^k.
\]

**Proof.** Using theorem and difference equation (14), we have

\[
(1 - z_1 - z_2 - z_1^r z_2^r)F(z_1, z_2) - \Phi_{r, r}(z_1, z_2) + z_1 \Phi_{r-1, r}(z_1, z_2) + z_2 \Phi_{r-1, r}(z_1, z_2) = -z_1 \sum_{k \geq r} f(k, k)(z_1 z_2)^k,
\]

where

\[
\Phi_{r, r}(z_1, z_2) = \Phi_{r-1, r-1}(z_1, z_2) + \Phi_{0, r-1}(z_1, z_2) + \Phi_{r-1, 0}(z_1, z_2) - f(r-1, r-1)(z_1 z_2)^{r-1},
\]

\[
\Phi_{r-1, r}(z_1, z_2) = \Phi_{r-1, r-1}(z_1, z_2) + \Phi_{0, r-1}(z_1, z_2),
\]

\[
\Phi_{r-1, r-1}(z_1, z_2) = \sum_{t=0}^{r-2} \Phi_{t, 0}(z_1, z_2),
\]

\[
\Phi_{0, r-1}(z_1, z_2) = \sum_{x=r-1}^{\infty} f(x, r-1)z_1^x z_2^{r-1}, \quad \Phi_{r-1, 0}(z_1, z_2) = \sum_{y=r-1}^{\infty} f(r-1, y)z_1^{r-1} z_2^y.
\]

Let \( P(z_1, z_2) = 1 - z_1 - z_2 - z_1^r z_2^r = 0 \). Then we obtain

\[
\Phi_{r-1, r-1}(z_1, z_2) + \Phi_{0, r-1}(z_1, z_2) + \Phi_{r-1, 0}(z_1, z_2) - f(r-1, r-1)(z_1 z_2)^{r-1} - z_1(\Phi_{r-1, r-1}(z_1, z_2) +
\]

\[
+ \Phi_{0, r-1}(z_1, z_2)) - z_2(\Phi_{r-1, r-1}(z_1, z_2) + \Phi_{r-1, 0}(z_1, z_2)) = z_1 \sum_{k \geq r} f(k, k)(z_1 z_2)^k.
\]

Since \( f(x, y) = 0 \) below the diagonal, we obtain

\[
(1 - z_1 - z_2)\Phi_{r-1, r-1}(z_1, z_2) + (1 - z_2)\Phi_{r-1, 0}(z_1, z_2) = z_1 \sum_{k \geq r} f(k, k)(z_1 z_2)^k.
\]

Let us introduce \( t = z_1 z_2 \) implies \( z_1 = \frac{1 - t^r + \sqrt{1 - 4t - 2t^r + t^{2r}}}{2} \). Then

\[
\frac{1}{z_1}((1 - z_1 - z_2)\Phi_{r-1, r-1}(z_1, z_2) + (1 - z_2)\Phi_{r-1, 0}(z_1, z_2)) = \sum_{k \geq r} f(k, k) t^k. \tag{16}
\]

For \( r = 2 \) we have the initial data \( f(x, 0) = 0, \ x = 1, 2, 3, \ldots, \ f(0, y) = 1, \ y = 0, 1, 2, \ldots, \ f(x, 1) = 0, \ x = 2, 3, 4, \ldots, \ f(1, y) = y, \ y = 1, 2, 3, \ldots. \)

Using (16), we obtain

\[-1 + \frac{1}{z_1} = \sum_{k=1}^{\infty} f(k, k)(z_1 z_2)^k.\]
Let us introduce $t = z_1 z_2$. Then $z_1 = \frac{1 - t^2 + \sqrt{1 - 4t^2 - 2t^4 + t^4}}{2}$. After expansion of $\frac{1}{z_1}$, we obtain

$$-1 + \frac{1}{z_1} = -1 + \frac{1 - t^2 - \sqrt{1 - 4t^2 - 2t^4 + t^4}}{2t} = t + 3t^2 + 8t^3 + 25t^4 + 83t^5 + \cdots.$$  \hspace{1cm} (17)

It proves the proposition. \hfill \Box

Coefficients of series (17) coincides with the numbers of other paths ending on the main diagonal $y = x$.

References


Разностные уравнения и производящие функции в некоторых задачах о решеточных путях

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В работе доказано тождество для производящих функций, на основе которого разработан метод вычисления числа путей на целочисленной решетке с ограничениями. Данный метод использует разностные уравнения с переменными коэффициентами. В качестве примеров вычислены производящие функции для путей Дика, Моцкина и Шрёdera.

Ключевые слова: разностные уравнения, производящие функции, решеточные пути.