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The Discrete Analog of the Newton-Leibniz Formula in the Problem of Summation over Simplex Lattice Points

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Definition of the discrete primitive function is introduced in the problem of summation over simplex lattice points. The discrete analog of the Newton-Leibniz formula is found.

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Introduction

The problem of summation of functions of a discrete argument is one of the classical problems of the calculus of finite differences. For example, the sum of the sequence of power of natural numbers was computed by Bernoulli (1713), and his studies gave rise the development of several branches of combinatorial analysis. Euler (1733) and Maclaurin (1738) independently found that required sum is expressed in terms of derivatives and integral of a given function. It was firstly proved by Jacobi (1834) (see [1, 2]).

The problem of summation of functions implies the computation of the following sum

$$S(x) = \sum_{t=0}^x \varphi(t) \quad (1)$$

with a variable upper limit x for a given function $\varphi(t)$. Euler proposed a method which reduces the problem to solving the difference equation

$$f(x+1) - f(x) = \varphi(x),$$

where $f(x)$ is an unknown function.

In this case, sum (1) is expressed in terms of the values of the function $f(x)$ at points 0 and $x+1$ of the segment $[0, x+1]$

$$S(x) = f(x+1) - f(0). \quad (2)$$

Function $f(x)$ is called a *discrete primitive function of the function $\varphi(x)$* and formula (2) is called *discrete analog of the Newton-Leibniz formula*.

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Many works in different branches of the mathematics are devoted to various aspects of the summation problem and its applications, among the relatively recent works note [3–5].

The summation problem in several variables can be formulated in various ways. Summation of polynomial of several variables over integer points of the rational convex polytope with variable faces was studied [6, 7]. The multidimensional analogue of the Euler-Maclaurin formula was obtained.

Summation of arbitrary functions over integer points of a rational parallelotop was studied [8–10]. Euler approach based on definitions of discrete primitive function and discrete analog of the Newton-Leibniz formula was used to solve the problem.

1. Formulation of the main result

Let us introduce following notations and definitions.

Vectors a^1, \dots, a^n have integer coordinates $a^j = (a_1^j, \dots, a_m^j)$, $a_i^j \in \mathbb{Z}$, where \mathbb{Z} is the set of integer numbers.

Columns of matrix A contains coordinates of vectors a^j :

$$A = \begin{pmatrix} a_1^1 & \dots & a_1^n \\ \dots & \dots & \dots \\ a_m^1 & \dots & a_m^n \end{pmatrix}.$$

Let cone $K = \{x \in \mathbb{R}^n : x = \lambda_1 a^1 + \dots + \lambda_n a^n, \lambda_j \geq 0\}$ generated by vectors a^1, \dots, a^n be a pointed cone, i.e., it doesn't contain lines. In this case the system of linear equations $At = x$ has a finite number of integer non-negative solutions for any $x \in K$ and for any function $\varphi(t) = \varphi(t_1, \dots, t_n)$ we can correctly define the following function

$$V_A(x, \varphi) = \sum_{At=x, t \in \mathbb{Z}_{\geq}^n} \varphi(t). \tag{3}$$

If $\varphi(t) \equiv 1$ then $V_A(x; 1)$ is a number of representatives of the vector x by vectors a^1, \dots, a^n : $x = t_1 a^1 + \dots + t_n a^n$, i.e., number of vector partitions. In the case $\varphi(t) = e^{\langle t, p \rangle}$ function $V_A(x; e^{\langle t, p \rangle})$ is called vector partition function (see, for example, [7]). *Function $V_A(x; \varphi)$ is a vector partition function with weight $\varphi(t)$.* One can say that the problem of finding vector partition function (3) is generalization of the problem of summation of function φ . Indeed, if $m = 1$, $n = 2$, $A = (1, 1)$ and $\varphi(t_1, t_2) = \varphi(t_1)$ then $V_A(x; \varphi) = \sum_{t_1+t_2=x} \varphi(t_1) = \sum_{t_1=0}^x \varphi(t_1) = S(x)$.

If $A = (1, 1, \dots, 1)$ then for the vector partition function associated with the weight function $\varphi(t_1, \dots, t_n)$ we obtain $V_A(x; \varphi) = \sum_{\|t\|=x} \varphi(t)$, where $\|t\| = t_1 + \dots + t_n$. Let us denote the left part of this equality by $S(x)$ and formulate the following problem

Find the sum of values of function $\varphi(t_1, \dots, t_n)$ over integer points of the simplex $\{t \in \mathbb{R}_{\geq}^n : \|t\| = x\}$:

$$S(x) = \sum_{\|t\|=x, t \in \mathbb{Z}_{\geq}^n} \varphi(t). \tag{4}$$

Let us note that the case $\varphi(t) = \psi(\|t\|)$, where $\psi(\tau)$ is a function of one variable was considered [10].

Let us introduce linear shift operators on j -th variable: $\delta_j f(t) = f(t_1, \dots, t_j + 1, \dots, t_n)$, $\delta = (\delta_1, \dots, \delta_n)$, $\delta^\alpha = \delta_1^{\alpha_1} \dots \delta_n^{\alpha_n}$. The polynomial difference operator $W(\delta) = \sum_{\alpha} c_{\alpha} \delta^{\alpha}$ acts on $f(t)$ in the following way $W(\delta) f(t) = \sum_{\alpha} c_{\alpha} f(t + \alpha)$.

Let us consider polynomial difference operator

$$W(\delta) = \prod_{1 \leq i < j \leq n} (\delta_i - \delta_j) = \begin{vmatrix} \delta_1^{n-1} & \dots & 1 \\ \delta_2^{n-1} & \dots & 1 \\ \dots & \dots & \dots \\ \delta_n^{n-1} & \dots & 1 \end{vmatrix}. \tag{5}$$

Using formula (5), for $n=2$ and $n=3$ we obtain $W(\delta_1, \delta_2) = \delta_1 - \delta_2$ and $W(\delta_1, \delta_2, \delta_3) = (\delta_1 - \delta_2)(\delta_1 - \delta_3)(\delta_2 - \delta_3)$, respectively.

A discrete primitive function $f(t)$ of the function $\varphi(t)$, $t \in \mathbb{Z}_{\geq}^n$ is a solution of the difference equation

$$W(\delta) f(t) = \varphi(t), t \in \mathbb{Z}_{+}^n. \tag{6}$$

Let π_j be a projection operator on the hyperplane $t_j = 0$:

$$\pi_j f(t) = f(t_1, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_n).$$

Let us define for $n \geq 2$ the Newton-Leibniz operator W_{NL} as

$$W_{NL}(\delta, \pi) = \prod_{1 \leq i < j \leq n} (\delta_i \pi_j - \delta_j \pi_i) = \begin{vmatrix} \delta_1^{n-1} & \delta_1^{n-2} \pi_1 & \dots & \pi_1^{n-1} \\ \delta_2^{n-1} & \delta_2^{n-2} \pi_2 & \dots & \pi_2^{n-1} \\ \dots & \dots & \dots & \dots \\ \delta_n^{n-1} & \delta_n^{n-2} \pi_n & \dots & \pi_n^{n-1} \end{vmatrix}.$$

Note that for $n=2$

$$W_{NL}(\delta, \pi) = (\delta_1 \pi_2 - \delta_2 \pi_1)$$

and if $f(t_1, t_2)$ is a discrete primitive function of the function $\varphi(t_1, t_2)$ then

$$\begin{aligned} S(x) &= \sum_{t_1+t_2=x} \varphi(t_1, t_2) = \sum_{t_1+t_2=x} (f(t_1+1, t_2) - f(t_1, t_2+1)) = \\ &= f(x+1, 0) - f(0, x+1) = W_{NL}(\delta, \pi) f(x, x). \end{aligned}$$

In the general case problem of summation (4) can be solved with the use of the following theorem.

Theorem 1. *If $f(t)$ is a discrete primitive function of the function $\varphi(t)$ then the following discrete analog of the Newton-Leibniz formula for the vector partition function $S(x)$ associated with φ holds true*

$$S(x) = \sum_{\|t\|=x} \varphi(t) = W_{NL}(\delta, \pi) f(t_1, t_2, \dots, t_n) |_{t_1=t_2=\dots=t_n=x}. \tag{7}$$

2. Preliminary results

To prove the theorem we need some additional results, particularly, the formula for the sum of the geometric progression

$$s_q(x) = \sum_{t_1+\dots+t_n=x, t \in \mathbb{Z}_{\geq}^n} q^t,$$

where $q = (q_1, \dots, q_n)$.

Let us introduce $H(z; q) = \frac{1}{(1 - q_1 z) \dots (1 - q_n z)}$, where z is complex variable. If $|q_k z| < 1$, $k = 1, \dots, n$ then after expansion in a power series we obtain

$$H(z; q) = \left(\sum_{t_1=0}^{\infty} q_1^{t_1} z^{t_1} \right) \dots \left(\sum_{t_n=0}^{\infty} q_n^{t_n} z^{t_n} \right) = \sum_{x=0}^{\infty} s_q(x) z^x.$$

If $\rho < \min_k \{|q_k^{-1}|\}$ then

$$s_q(x) = \frac{1}{2\pi i} \int_{|z|=\rho} H(z; q) \frac{dz}{z^{x+1}}. \tag{8}$$

Function $H(z; q) \frac{1}{z^{x+1}}$ has poles at points $z = 0$, $z_k = q_k^{-1}$, $k = 1, \dots, n$, and residue at an infinitely distant point is equal to 0. Then by theorem of the total sum of residues [11] we have from (8) that

$$s_q(x) = - \sum_{k=1}^n \operatorname{res}_{z=q_k^{-1}} H(z) \frac{1}{z^{x+1}}. \tag{9}$$

Calculating this residue, we obtain from (9) that

$$s_q(x) = \sum_{k=1}^n \frac{q_k^x}{\prod_{j=1, j \neq k}^n (1 - \frac{q_j}{q_k})} = \sum_{k=1}^n \frac{q_k^{x+n-1}}{\prod_{j=1, j \neq k}^n (q_k - q_j)}. \tag{10}$$

Formula (10) can be obtained in another way. For example, we can find it in [12, Sec. 3.3] (without proof). The proof can be found in [13, 14] as particular case of the more general result in which arbitrary integer polytope is used instead of simplex.

If $q = (q_1, \dots, q_n)$ and $W(q)$ is a Vandermonde determinant then

$$W(q[k]) = \begin{vmatrix} q_1^{n-2} & q_1^{n-3} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ q_{k-1}^{n-2} & q_{k-1}^{n-3} & \dots & 1 \\ q_{k+1}^{n-2} & q_{k+1}^{n-3} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ q_n^{n-2} & q_n^{n-3} & \dots & 1 \end{vmatrix} = \prod_{1 \leq \mu < \nu \leq n, \mu, \nu \neq k} (q_\mu - q_\nu).$$

The following equality holds true

$$W(q) = (-1)^{k-1} \prod_{j=1, j \neq k}^n (q_k - q_j) W(q[k]). \tag{11}$$

Let us consider operators π_μ and δ_ν . They are permutable and for any function $f(t)$ the following equality holds true

$$(\delta_\mu - \delta_\nu) \pi_\mu \pi_\nu f(t) = (\pi_\nu \delta_\mu - \pi_\mu \delta_\nu) \pi_\mu \pi_\nu f(t). \tag{12}$$

Proof of the theorem.

Using equality (6) and $f(t) = \delta^t f(0)$, we obtain

$$S(x) = \sum_{\|t\|=x} \varphi(t) = \sum_{\|t\|=x} W(\delta) f(t) = \sum_{\|t\|=x} W(\delta) \delta^t f(0) = \left(\sum_{\|t\|=x} \delta^t \right) W(\delta) f(0).$$

It follows from (10) and (11) that

$$\begin{aligned} S(x) &= \sum_{k=1}^n \frac{\delta_k^{x+n-1}}{\prod_{j=1, k \neq j}^n (\delta_k - \delta_j)} W(\delta) f(0) = \\ &= \sum_{k=1}^n \frac{\delta_k^{x+n-1}}{\prod_{j=1, k \neq j}^n (\delta_k - \delta_j)} (-1)^{k-1} \prod_{j=1, j \neq k}^n (\delta_k - \delta_j) W(\delta[k]) f(0) \\ &= \sum_{k=1}^n (-1)^{k-1} \delta_k^{n-1} W(\delta[k]) \delta_k^x f(0). \end{aligned}$$

Using (12), we find

$$\begin{aligned} W(\delta[k]) \delta_k^x f(0) &= \prod_{1 \leq \mu, \nu \leq n, \mu, \nu \neq k} (\delta_\mu - \delta_\nu) \delta_k^x f(0) = \\ &= \prod_{1 \leq \mu, \nu \leq n, \mu, \nu \neq k} (\pi_\nu \delta_\mu - \pi_\mu \delta_\nu) \pi_1 \dots [k] \dots \pi_n f(x, x, \dots, x) = \\ &= W_{NL}(\delta[k], \pi[k]) f(x, \dots, x). \end{aligned}$$

Thus, expanding determinant $W_{NL}(\delta, \pi)$ by first column, we obtain

$$S(x) = \sum_{k=1}^n (-1)^{k-1} \delta_k^{n-1} W_{NL}(\delta[k], \pi[k]) f(x, \dots, x) = W_{NL}(\delta, \pi) f(x, \dots, x).$$

Formula (7) is proved. □

Example. Let us find the vector partition function with weight $\varphi(t_1, t_2, t_3) = (3 - 2t_1) 2^{t_2} 3^{t_3}$ for $n=3$. Primitive function for φ is $f(t_1, t_2, t_3) = t_1 2^{t_2} 3^{t_3}$. According to formula (7) of Theorem 1, sum (4) is equal to $S(x) = 3^{x+2} - 2^{x+2} - x - 2$.

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Дискретный аналог формулы Ньютона–Лейбница в задаче суммирования по целым точкам симплекса

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В задаче суммирования функции по целым точкам рационального симплекса дано определение дискретной первообразной и найден дискретный аналог формулы Ньютона–Лейбница для суммы.

Ключевые слова: суммирование функций, дискретная первообразная, формула Ньютона–Лейбница.