On Some Approach for Finding the Resultant of Two Entire Functions

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One approach for finding the resultant of two entire functions is discussed in the article. It is based on Newton’s recurrent formulas.

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Let us consider classic resultant $R(f, g)$ for given polynomials $f$ and $g$. It can be defined in various ways:

a) using the Sylvester determinant (see, for example, [1–3]);
b) using the formula for the product $R(f, g) = \prod_{x : f(x) = 0} g(x)$ (see, for example, [1–3]);
c) using the Bezout-Cayley method (see, for example, [4]).

See also monograph [5].

In our approach, we take the formula of the product as the main definition. Let us consider the Sylvester determinant for

\[
\begin{align*}
    f(z) &= a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n, \\
    g(z) &= b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m.
\end{align*}
\]

Let us define

\[ D_{n,m} = \begin{vmatrix}
    a_0 & a_1 & a_2 & \ldots & 0 \\
    0 & a_0 & a_1 & \ldots & 0 \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & \ldots & a_0 & \ldots & a_n \\
    b_0 & b_1 & b_2 & \ldots & 0 \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & \ldots & b_0 & \ldots & b_m
\end{vmatrix}. \tag{2} \]

If $a_n \neq 0$ then

\[ D_{n,m} = a_n^m \prod_{x : f(x) = 0} g(x), \]

that is, it coincides up to a constant the multiplier with the resultant.

Let us consider the following sums of powers of the values of $g$ at the roots of $f$

\[ S_k = \sum_{x : f(x) = 0} g^k(x), \quad k = 1, 2, \ldots. \]
It is known that (see [6])

\[ R_{f,g} = \frac{1}{n!} \begin{vmatrix} S_1 & 1 & 0 & \cdots & 0 \\ S_2 & S_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_n & S_{n-1} & S_{n-2} & \cdots & S_1 \end{vmatrix} . \tag{3} \]

A natural generalization of polynomials are entire functions. A number of results are known that extend the definition of resultant in the case of entire functions with a finite number of zeros and with an infinite number of zeros (see [6, 7]). One of the results is the following [6]. Let \( g(z) \) be an entire function of the form

\[ g(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m + \ldots. \]

then

\[ R(f, g) = \lim_{m \to \infty} a_n^{-m} D_{n,m}. \tag{4} \]

Therefore, to find the resultant \( R(f, g) \) it is necessary to calculate determinants of order \( m+n \), and then to find their limit. It is certainly quite difficult. Here we propose another approach for finding the resultant.

Let us denote the roots of the polynomial \( f(z) \) by \( z_1, z_2, \ldots, z_n \). Multiple roots are taken into account. Then

\[ \prod_{i=1}^{n} g(z_i) = g(z_1) \cdot g(z_2) \cdots g(z_n) = \left( \sum_{j=0}^{m} b_j z_1 \right) \cdots \left( \sum_{j=0}^{m} b_j z_n \right). \]

This expression is the sum of some symmetric polynomials in variables \( z_1, z_2, \ldots, z_n \). These symmetric polynomials are polynomials of elementary symmetric polynomials of polynomial \( f(z) \). Thus, it is possible to find the resultant of polynomials \( f(z) \) and \( g(z) \) without finding the roots themselves.

Let us consider an example of application of this method to the system

\[
\begin{align*}
f(z) &= a_0 + a_1 z + a_2 z^2, \\
g(z) &= b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m.
\end{align*}
\tag{5}
\]

The roots of polynomial \( f(z) \) from (5) are \( z_1 \) and \( z_2 \). Then

\[ \prod_{i=1}^{2} g(z_i) = g(z_1) \cdot g(z_2) = \]

\[ = b_0^2 + b_1^2 z_1 z_2 + b_2^2 z_1^2 z_2^2 + \ldots + b_m^2 z_1^n z_2^n + \]
\[ + b_0 b_1 (z_1 + z_2) + b_0 b_2 (z_1^2 + z_2^2) + \ldots + b_0 b_m (z_1^n + z_2^n) + \]
\[ + b_1 b_2 z_1 z_2 (z_1 + z_2) + b_1 b_3 z_1 z_2 (z_1^2 + z_2^2) + \ldots + b_1 b_m z_1 z_2 (z_1^n + z_2^n) + \]
\[ + b_2 b_3 z_1^2 z_2^2 (z_1 + z_2) + b_2 b_4 z_1^2 z_2^2 (z_1^2 + z_2^2) + \ldots + b_2 b_m z_1^2 z_2^2 (z_1^n + z_2^n) + \]
\[ \cdots \]
\[ + b_n b_{n-1} z_1^{n-1} z_2^{n-1} (z_1^n + z_2^n). \]

Hence we obtain
\[
\prod_{i=1}^{2} g(z_i) = g(z_1) \cdot g(z_2) = \sum_{k=0}^{n} b_k z_1^k z_2^k + \sum_{t=0}^{n} \sum_{s=t+1}^{n} b_t b_s (z_1^t z_2^s + z_1^s z_2^t) = \]
\[
= \sum_{k=0}^{n} b_k^2 z_1^k z_2^k + \sum_{t=0}^{n} \sum_{s=t+1}^{n} b_t b_s z_1^t z_2^s (z_1^{s-t} + z_2^{s-t}).
\]

To simplify the resulting expression we use Vieta’s formulas
\[
\begin{align*}
    e_1 &= z_1 + z_2 = -a_1, \\
    e_2 &= z_1 \cdot z_2 = a_0,
\end{align*}
\]
where \( e_1, e_2 \) are elementary symmetric polynomials of polynomial \( f(z) \).

Now one needs to calculate the sums in brackets in formula (6). To do this we introduce the following notation
\[
\tilde{S}_k = \sum_{i=1}^{2} z_i^k,
\]
i.e.,
\[
\begin{align*}
    \tilde{S}_1 &= z_1 + z_2, \\
    \tilde{S}_2 &= z_1^2 + z_2^2, \\
    \tilde{S}_k &= z_1^k + z_2^k.
\end{align*}
\]
Expressions \( \tilde{S}_k \) are sums of powers of the roots of polynomial \( f(z) \).

Let us consider the famous Newton-Girard formula
\[
\tilde{S}_k = \sum_{r_1 + 2r_2 + \ldots + kr_k = k, r_1, r_2, \ldots, r_k \geq 0} (-1)^k \frac{k! (r_1 + \ldots + r_k - 1)!}{r_1! \ldots r_k!} \prod_{i=1}^{k} (-e_i)^r_i.
\]
In our case there are only two elementary symmetric polynomials. Therefore
\[
\begin{align*}
    \tilde{S}_{2j} &= \sum_{t=0}^{j} (-1)^j t \frac{j \cdot (j + t - 1)!}{(2t)! \cdot (j - t)!} (e_1)^{2t} \cdot (e_2)^{j-t}, \\
    \tilde{S}_{2j+1} &= \sum_{t=0}^{j} (-1)^j t \frac{(2j + 1) \cdot (j + t)!}{(2t + 1)! \cdot (j - t)!} (e_1)^{2t+1} \cdot (e_2)^{j-t},
\end{align*}
\]
if \( j \) is even, and
\[
\begin{align*}
    \tilde{S}_{2j+1} &= \sum_{t=0}^{j} (-1)^j t \frac{(2j + 1) \cdot (j + t)!}{(2t + 1)! \cdot (j - t)!} (e_1)^{2t+1} \cdot (e_2)^{j-t},
\end{align*}
\]
if \( j \) is odd.

For example,
\[
\begin{align*}
    \tilde{S}_2 &= e_1^2 - 2e_2 = a_1^2 - 2a_0, \\
    \tilde{S}_3 &= e_1^3 - 3e_1 e_2 = -a_1^3 + 3a_0 a_1.
\end{align*}
\]
As a result, we obtain
\[
\prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} b_k^2 z_1^k z_2^k + \sum_{t=0}^{n} \sum_{s=t+1}^{n} b_t b_s z_1^t z_2^s (z_1^{s-t} + z_2^{s-t}) = \sum_{k=0}^{n} b_k^2 a_k^k + \sum_{t=0}^{n} \sum_{s=t+1}^{n} b_t b_s a_t a_s \tilde{S}_s. \quad (9)
\]
Now, if \( g(z) \) is an entire function then formula (9) takes the form
\[ \prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{\infty} b_k^2 a_0^k + \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} b_t b_s a_0^t \tilde{S}_s. \]  

(10)

**Example 1.** Let us consider the system of equations

\[ \begin{align*}
    f(z) &= z^2 - a^2, \\
    g(z) &= b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n.
\end{align*} \]  

(11)

The roots of the first equation of the system are \pm a. Using formula (7), we have

\[ \prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} b_k^2 z_1^k z_2^k + \sum_{t=0}^{n} \sum_{s=t+1}^{n} b_t b_s z_1^t z_2^s (z_1^{s-t} + z_2^{s-t}). \]

Then

\[ \begin{align*}
    e_2 &= z_1 \cdot z_2 = -a^2, \\
    e_1 &= z_1 + z_2 = 0,
\end{align*} \]

(12)

and

\[ \tilde{S}_{s-t} = z_1^{s-t} + z_2^{s-t} = \begin{cases} 2a^{s-t}, & s-t \text{ even}, \\
0, & s-t \text{ odd}. \end{cases} \]

Thus

\[ \prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} (-1)^k b_k^2 a^{2k} + \sum_{t=0}^{n} \sum_{s=t+1}^{n} (-1)^t b_t b_s a^{2t} \tilde{S}_{s-t} = \]

\[ = \sum_{k=0}^{n} (-1)^k b_k^2 a^{2k} + 2 \sum_{t=0}^{n} \sum_{s=t+1}^{n} (-1)^t b_t b_s a^{s-t}, \]

provided that \( s-t \) is an even number. Let us introduce the following designation \( s-t = 2j \).

Then

\[ \prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} (-1)^k b_k^2 a^{2k} + 2 \sum_{t=0}^{\lfloor (n-t)/2 \rfloor} \sum_{j=1}^{(n-t)/2} (-1)^t b_{t+2j} a^{2t+2j}. \]

**Example 2.** Let us consider the system of equations

\[ \begin{align*}
    f(z) &= z^2 - a^2, \\
    g(z) &= e^{bz} = \sum_{n=1}^{\infty} \frac{(bz)^n}{n!} = 1 + bz + \frac{(bz)^2}{2!} + \ldots + \frac{(bz)^n}{n!} + \ldots.
\end{align*} \]  

(13)

Using formula (4), we obtain

\[ \prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{\infty} (-1)^k \frac{(ab)^{2k}}{(k!)^2} + 2 \sum_{t=0}^{\infty} \sum_{j=1}^{\infty} (-1)^t (ab)^{2t+2j} \frac{1}{t!(t+2j)!} = 1. \]

The first sum is the Bessel function of the first kind \( J_0(2ab) \) (see, for example, [8]).

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References


О некотором подходе к нахождению результantsа двух целых функций

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В статье обсуждается один подход к нахождению результantsа двух целых функций, основанный на рекуррентных формулax Ньютона.

Ключевые слова: результantsирующие, целые функции, формулы Ньютона.