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Global in Space Regularity Results for the Heat Equation with Robin-Neumann Type Boundary Conditions in Time-varying Domains

Tahir Boudjeriou*
Laboratory of Applied Mathematics, Department of Mathematics
Faculty of Exact Sciences, University of Bejaia, Bejaia, 6000
Algeria

Arezki Kheloufi†
Department of Technology, Faculty of Technology
Lab. of Applied Mathematics, Bejaia University, Bejaia, 6000
Algeria

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This article deals with the heat equation

$$\partial_t u - \partial_x^2 u = f \text{ in } D,\ D = \left\{(t,x) \in \mathbb{R}^2 : a < t < b, \psi(t) < x < +\infty\right\}$$

with the function ψ satisfying some conditions and the problem is supplemented with boundary conditions of Robin-Neumann type. We study the global regularity problem in a suitable parabolic Sobolev space. We prove in particular that for $f \in L^2(D)$ there exists a unique solution u such that $u,\ \partial_t u,\ \partial_x^j u \in L^2(D), j = 1,\ 2$. The proof is based on the domain decomposition method. This work complements the results obtained in [10].

Keywords: heat equation, Unbounded non-cylindrical domains, Robin condition, Neumann condition, anisotropic Sobolev spaces.
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1. Introduction and preliminaries

This work is devoted to the analysis of the following one-dimensional second order parabolic problem

$$\begin{cases} \partial_t u - \partial_x^2 u = f \in L^2(\Omega_\infty), \\ \partial_x u + \beta u|_{\Gamma_1} = 0, \\ \partial_x u|_{\Gamma_2} = 0, \end{cases} \tag{1}$$

where $L^2(\Omega_\infty)$ stands for the space of square-integrable functions on Ω_∞ with the measure $dtdx$. The coefficient β is a real number satisfying the following non-degeneracy assumption

$$\beta < 0. \tag{2}$$

Here, Ω_∞ (see, Fig. 1) is an open set of $\mathbb{R}^2$ defined by

$$\Omega_\infty := \left\{(t,x) \in \mathbb{R}^2 : a < t < b,\ \psi(t) < x < +\infty\right\},$$

*re.tahar@yahoo.com
†arezkinet2000@yahoo.fr, arezki.kheloufi@univ-bejaia.dz

where a, b are real numbers such that $-\infty < a < 0 < b < +\infty$, while ψ is a Lipschitz continuous real-valued function on (a, b), and such that

$$\psi(t) := \begin{cases} \varphi_1(t) & \text{on } (a, 0], \\ \varphi_2(t) & \text{on } [0, b). \end{cases}$$

The function φ_1 (respectively, φ_2) is positive and decreasing (respectively, increasing) on $(a, 0]$ (respectively, on $[0, b)$) and verifies the hypothesis $\varphi_1(0) = \varphi_2(0) = 0$. A natural assumption between coefficient β and the function of parametrization φ_1 of the domain Ω_∞ which guarantees the uniqueness of the solution of Problem (1) is

$$\left(\frac{\varphi_1'(t)}{2} - \beta\right) \geqslant 0 \text{ almost everywhere } t \in]a, 0[. \tag{3}$$

The lateral boundaries Γ_1 and Γ_2 of Ω_∞ are defined by

$$\Gamma_1 = \left\{(t, \varphi_1(t)) \in \mathbb{R}^2 : a < t < 0\right\}, \quad \Gamma_2 = \left\{(t, \varphi_2(t)) \in \mathbb{R}^2 : 0 < t < b\right\}.$$

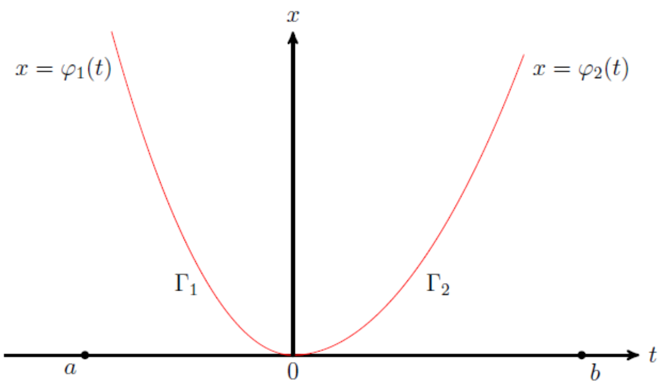

Fig. 1. The unbounded domain Ω_∞

Notice that the section of Ω_∞ in the t direction defined by

$$I_x := [\varphi_1^{-1}(x), \varphi_2^{-1}(x)]$$

for x in $]0, +\infty[$, is such that the sections $I_n, n \in \mathbb{N}^*$ become bounded when n becomes large, i. e.,

$$\forall\, n \in \mathbb{N}^*, \ |\varphi_2^{-1}(n) - \varphi_1^{-1}(n)| \leqslant b - a. \tag{4}$$

The most interesting point of the parabolic problem studied here is the unboundedness of Ω_∞ with respect to the space variable x which prevents one using the methods in [16, 17] and [21]. It's the characteristic (4) of the x-sections of Ω_∞ which helps us to overcome this difficulty. Also, These specific Robin-Neumann type boundary conditions

$$\partial_x u + \beta u|_{\Gamma_1} = \partial_x u|_{\Gamma_2} = 0$$

are important for the originality of this work. Indeed, to our knowledges, results concerning parabolic equations on unbounded (with respect to the space variable x) time-varying domains, subject to such kind of boundary conditions, have not appeared in the literature to date. So, let us consider the anisotropic Sobolev space

$$\mathcal{H}_\gamma^{1,2}(\Omega_\infty) := \left\{u \in \mathcal{H}^{1,2}(\Omega_\infty) : \partial_x u + \beta u|_{\Gamma_1} = \partial_x u|_{\Gamma_2} = 0\right\}$$

with
$$\mathcal{H}^{1,2}(\Omega_\infty) := \left\{u \in L^2(\Omega_\infty) : \partial_t u, \partial_x^j u \in L^2(\Omega_\infty),\ j = 1,\ 2\right\}.$$
The space $\mathcal{H}^{1,2}(\Omega_\infty)$ is equipped with the natural norm, that is

$$\|u\|_{\mathcal{H}^{1,2}(\Omega_\infty)} = \left(\|u\|^2_{L^2(\Omega_\infty)} + \|\partial_t u\|^2_{L^2(\Omega_\infty)} + \sum_{j=1}^{2} \left\|\partial_x^j u\right\|^2_{L^2(\Omega_\infty)}\right)^{\frac{1}{2}}.$$

Then, the main result of this paper is the following theorem:

Theorem 1.1. *Under the conditions (2) and (3), Problem (1) admits a (unique) solution* $u \in \mathcal{H}^{1,2}(\Omega_\infty)$.

It is not difficult to prove the uniqueness of the solution. Indeed, let us consider $u \in \mathcal{H}^{1,2}_\gamma(\Omega_\infty)$ a solution of the problem (1) with a null right-hand side term. So,

$$\partial_t u - \partial_x^2 u = 0 \text{ in } \Omega_\infty.$$

In addition u fulfils the boundary conditions

$$\partial_x u + \beta u|_{\Gamma_1} = \partial_x u|_{\Gamma_2} = 0.$$

Using Green formula, we have

$$\int_{\Omega_\infty} \left(\partial_t u - \partial_x^2 u\right) u\ dt\ dx = \int_{\partial\Omega_\infty} \left(\frac{1}{2}|u|^2 \nu_t - u\ \partial_x u\ \nu_x\right) d\sigma + \int_{\Omega_\infty} (\partial_x u)^2\, dtdx,$$

where ν_t, ν_x are the components of the unit outward normal vector at the boundary of Ω_∞. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Ω_∞ where $x = \varphi_i(t)$, $i = 1,\ 2$, we have

$$\nu_x = \frac{-1}{\sqrt{1 + (\varphi_i')^2(t)}},\ \nu_t = \frac{\varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2(t)}} \text{ and } \partial_x u(t, \varphi_1(t)) + \beta u(t, \varphi_1(t)) = \partial_x u(t, \varphi_2(t)) = 0.$$

Accordingly, the corresponding boundary integral is

$$\int_a^0 \left(\frac{\varphi_1'(t)}{2} - \beta\right) u^2(t, \varphi_1(t))\, dt + \int_0^b \frac{\varphi_2'(t)}{2} u^2(t, \varphi_2(t)) dt.$$

Then, we obtain

$$\begin{aligned} \int_{\Omega_\infty} \left(\partial_t u - \partial_x^2 u\right) u\ dt\ dx &= \int_a^0 \left(\frac{\varphi_1'(t)}{2} - \beta\right) u^2(t, \varphi_1(t))\, dt + \int_0^b \frac{\varphi_2'(t)}{2} u^2(t, \varphi_2(t)) dt\ + \\ &+ \int_{\Omega_\infty} (\partial_x u)^2\, dtdx. \end{aligned}$$

Consequently using the fact that u is the solution yields

$$\int_{\Omega_\infty} (\partial_x u)^2\, dtdx = 0,$$

because
$$\int_a^0 \left(\frac{\varphi_1'(t)}{2} - \beta\right) u^2(t, \varphi_1(t))\, dt + \int_0^b \frac{\varphi_2'(t)}{2} u^2(t, \varphi_2(t)) dt \geqslant 0$$

thanks to the hypothesis (3) and the fact that φ_2 is an increasing function on $[0, b)$. This implies that $\partial_x u = 0$ and consequently $\partial_x^2 u = 0$. Then, the hypothesis $\partial_t u - \partial_x^2 u = 0$ gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions and the fact that $\beta \neq 0$ imply that $u = 0$.

We can find in [10] solvability results for Problem (1) with Dirichlet-Neumann type boundary conditions, corresponding here to the case where $\beta = \infty$. In the case of bounded non-cylindrical domains $\Omega_l, l > 0$, studies related to Problem (1) can be found in [7, 11] and [8] both in one-dimensional and bi-dimensional cases. It is possible to consider similar questions with some other operators (see, for example, [4] for a 2m-th order operator in bounded non-rectangular domains). Whereas second-order parabolic equations in bounded non-cylindrical domains are well studied (see for instance [1, 6, 9, 12, 14, 15, 18, 19, 20, 23] and the references therein), the literature concerning unbounded non-cylindrical domains does not seem to be very rich. The regularity of the heat equation solution in a non-smooth and unbounded domain (in the t direction) is obtained in [21] and [22] by using two different approaches. In [13], uniqueness classes of solutions of non-divergent second order parabolic equations were obtained. The heat equation in unbounded non-cylindrical domains with respect to the space variable x were considered in [5] and [2]. In Guesmia [5], the analysis is done in the framework of evolution function spaces. However, in Aref'ev and Bagirov [2], properties of solutions of the heat equation with Cauchy–Dirichlet boundary conditions were obtained in the more regular anisotropic Sobolev–Slobodetskii spaces (more precisely, those of functions with $t-$and$-x$ derivatives are in weighted L^2-spaces). The class of domains used in [2] corresponds here to

$$\psi(t) := \begin{cases} -\alpha\sqrt{-t} & \text{on } [a, 0], \\ \delta\sqrt{t} & \text{on } [0, b] \end{cases}$$

for any positive constants α and δ.

This paper is organized as follows. The two next sections are devoted to the proof of Theorem 1.1. Indeed, in Section 2, we study an auxiliary problem related to Problem (1) in a bounded domain. Then, in Section 3, prove the energy type estimate

$$\|u_m\|_{\mathcal{H}^{1,2}(\Omega_m)} \leqslant C \|f\|_{L^2(\Omega_\infty)},$$

where C is a constant independent of m and for each $m \in \mathbb{N}^*$, $u_m \in \mathcal{H}^{1,2}(\Omega_m)$ is the solution (obtained in the Section 2) in truncated bounded domain Ω_m approximating Ω_∞. The previous estimate will allow us to pass to the limit and complete the proof of Theorem 1.1.

2. An auxiliary problem in a bounded domain

In this section, we replace the unbounded domain Ω_∞ by the bounded domain Ω_c, $c > 0$ (see, Fig. 2) defined by

$$\Omega_c = \{(t, x) \in \Omega_\infty : 0 < x < c\}$$

and we consider the boundary value problem

$$\begin{cases} \partial_t u_c - \partial_x^2 u_c = f_c \in L^2(\Omega_c), \\ u_c|_{\Gamma_{0,c}} = 0, \\ \partial_x u_c + \beta u_c|_{\Gamma_{1,c}} = 0, \\ \partial_x u_c|_{\Gamma_{2,c}} = 0, \end{cases} \tag{5}$$

where $f_c = f|_{\Omega_c}$, $\Gamma_{0,c} = \{(t, c) : d_1 < t < d_2\}$, $\Gamma_{1,c} = \left\{(t, \varphi_1(t)) \in \mathbb{R}^2 : d_1 < t < 0\right\}$ and $\Gamma_{2,c} = \left\{(t, \varphi_2(t)) \in \mathbb{R}^2 : 0 < t < d_2\right\}$ with $d_1 = \varphi_1^{-1}(c),\ d_2 = \varphi_2^{-1}(c)$.

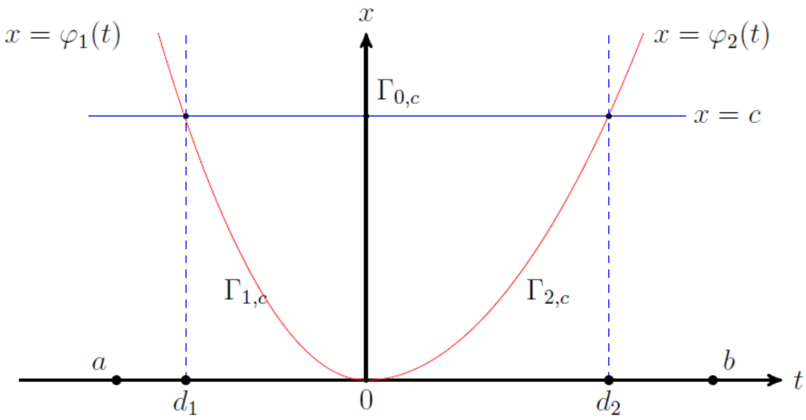

Fig. 2. The bounded domain Ω_c

2.1. Problem (5) in a reference domain

Here, we replace Ω_c by

$$\Omega_c^{(n)} = \left\{ (t,x) \in \Omega_c : d_1 + \frac{1}{n} < t < d_2 - \frac{1}{n} \right\},$$

where n is a large enough positive integer such that $d_1 + \dfrac{1}{n} < 0$ and $d_2 - \dfrac{1}{n} > 0$ (see, Fig. 3). Thus, $\varphi_1 \left(d_1 + \dfrac{1}{n} \right) < c$ and $\varphi_2 \left(d_2 - \dfrac{1}{n} \right) < c$.

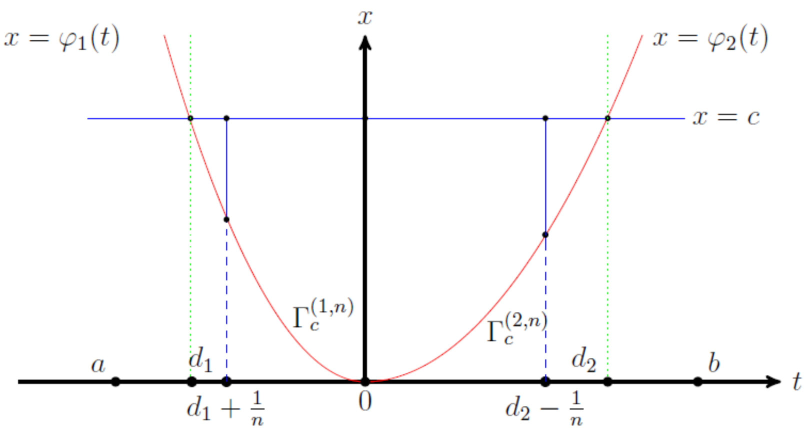

Fig. 3. The domain $\Omega_c^{(n)}$

Theorem 2.1. *For a large enough positive integer n, the problem*

$$\begin{cases} \partial_t u_c^{(n)} - \partial_x^2 u_c^{(n)} = f_c^{(n)} \in L^2\left(\Omega_c^{(n)}\right), \\ u_c^{(n)}\Big|_{t=d_1+\frac{1}{n}} = u_c^{(n)}\Big|_{x=c} = 0, \\ \partial_x u_c^{(n)} + \beta u_c^{(n)}\Big|_{\Gamma_c^{(1,n)}} = 0, \\ \partial_x u_c^{(n)}\Big|_{\Gamma_c^{(2,n)}} = 0, \end{cases} \tag{6}$$

admits a (unique) solution $u_c^{(n)} \in \mathcal{H}^{1,2}\left(\Omega_c^{(n)}\right)$. Here, $f_c^{(n)} = f|_{\Omega_c^{(n)}}$,

$$\Gamma_c^{(1,n)} = \left\{(t, \varphi_1(t)) \in \mathbb{R}^2 : d_1 + \frac{1}{n} < t < 0\right\}, \quad \Gamma_c^{(2,n)} = \left\{(t, \varphi_2(t)) \in \mathbb{R}^2 : 0 < t < d_2 - \frac{1}{n}\right\}.$$

Proof. The uniqueness of the solution is easy to check. Let us prove its existence. The change of variable

$$(t,x) \quad \mapsto \quad (t,y) = \left(t, \frac{x - \psi_c^{(n)}(t)}{c - \psi_c^{(n)}(t)}\right),$$

where

$$\psi_c^{(n)}(t) := \begin{cases} \varphi_1(t) \text{ on } \left[d_1 + \frac{1}{n}, 0\right], \\ \varphi_2(t) \text{ on } \left[0, d_2 - \frac{1}{n}\right], \end{cases}$$

transforms $\Omega_c^{(n)}$ into the rectangle $R^{(n)} = \left]d_1 + \frac{1}{n}, d_2 - \frac{1}{n}\right[\times]0,1[$. Putting $u_c^{(n)}(t,x) = v^{(n)}(t,y)$ and $f_c^{(n)}(t,x) = g^{(n)}(t,y)$, then Problem (6) becomes

$$\begin{cases} \partial_t v^{(n)}(t,y) + a(t,y)\,\partial_y v^{(n)}(t,y) - \dfrac{1}{b^2(t)}\partial_y^2 v^{(n)}(t,y) = g^{(n)}(t,y) \text{ in } R^{(n)}, \\ v^{(n)}\big|_{t=d_1+\frac{1}{n}} = v^{(n)}\big|_{y=1} = 0, \\ \partial_y v^{(n)} + \beta b(t) v^{(n)}\big|_{\Gamma^{(n,d_1)}} = 0, \\ \partial_y v^{(n)}\big|_{\Gamma^{(n,d_2)}} = 0, \end{cases} \tag{7}$$

where $b(t) := c - \psi_c^{(n)}(t)$, $a(t,y) := \dfrac{(y-1)\psi_c^{(n)'}(t)}{c - \psi_c^{(n)}(t)}$, and

$$\Gamma^{(n,d_1)} = \left\{(t,0) \in \mathbb{R}^2 : d_1 + \frac{1}{n} < t < 0\right\}, \quad \Gamma^{(n,d_2)} = \left\{(t,0) \in \mathbb{R}^2 : 0 < t < d_2 - \frac{1}{n}\right\}.$$

The aforementioned change of variable conserves the spaces L^2 and $\mathcal{H}^{1,2}$ because $-\dfrac{1}{b^2(t)}$ and $a(t,y)$ are bounded functions when $t \in]d_1 + \frac{1}{n}, d_2 - \frac{1}{n}[$. In other words

$$f_c^{(n)} \in L^2(\Omega_c^{(n)}) \Leftrightarrow g^{(n)} \in L^2(R^{(n)}), \quad u_c^{(n)} \in \mathcal{H}^{1,2}(\Omega_c^{(n)}) \Leftrightarrow v^{(n)} \in \mathcal{H}^{1,2}(R^{(n)}).$$

We need the following lemma:

Lemma 2.1. *For a large enough positive integer n, the following operator is compact:*

$$B : \mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right) \to L^2\left(R^{(n)}\right), \quad v^{(n)} \mapsto Bv^{(n)} = a(t,y)\,\partial_y v^{(n)}.$$

Here, for a fixed t in $]d_1 + \frac{1}{n}, 0[$

$$\mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right) = \left\{ v^{(n)} \in \mathcal{H}^{1,2}\left(R^{(n)}\right) : \begin{array}{l} v^{(n)}\big|_{t=d_1+\frac{1}{n}} = v^{(n)}\big|_{y=1} = 0, \\ \partial_y v^{(n)} + \beta b(t) v^{(n)}\big|_{\Gamma^{(n,d_1)}} = \partial_y v^{(n)}\big|_{\Gamma^{(n,d_2)}} = 0 \end{array} \right\}.$$

Proof. $R^{(n)}$ has the "horn property" of Besov [3], so

$$\partial_y : \mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right) \to \mathcal{H}^{\frac{1}{2},1}\left(R^{(n)}\right), \quad v^{(n)} \mapsto \partial_y v^{(n)}$$

is continuous. Since $R^{(n)}$ is bounded, the canonical injection is compact from $\mathcal{H}^{\frac{1}{2},1}\left(R^{(n)}\right)$ into $L^2\left(R^{(n)}\right)$, see for instance [3]. Here

$$\mathcal{H}^{\frac{1}{2},1}\left(R^{(n)}\right) = L^2\left(d_1 + \frac{1}{n}, d_2 - \frac{1}{n}; H^1\,]0,1[\right) \cap H^{\frac{1}{2}}\left(d_1 + \frac{1}{n}, d_2 - \frac{1}{n}; L^2\,]0,1[\right),$$

see [17] for the complete definitions of the $\mathcal{H}^{r,s}$ Hilbertian Sobolev spaces. Then, ∂_y is a compact operator from $\mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right)$ into $L^2\left(R^{(n)}\right)$. Since $a\,(.,.)$ is a bounded function for $t \in]d_1+\frac{1}{n}, d_2-\frac{1}{n}[$, the operator $B = a\partial_y$ is also compact from $\mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right)$ into $L^2\left(R^{(n)}\right)$. □

So, thanks to Lemma 2.1, to complete the proof of Theorem 2.1, it is sufficient to show that the operator

$$\partial_t - \frac{1}{(c-\psi_c^{(n)})^2}\partial_y^2 : \quad \mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right) \quad \to \quad L^2\left(R^{(n)}\right)$$

is an isomorphism. A simple change of variable $t = h\,(s)$ with $h'\,(s) = (c-\psi_c^{(n)})^2\,(t)$, transforms the problem

$$\begin{cases} \partial_t v^{(n)}\,(t,y) - \dfrac{1}{(c-\psi_c^{(n)})^2\,(t)}\partial_y^2 v^{(n)}\,(t,y) = g^{(n)}\,(t,y) \in L^2\left(R^{(n)}\right), \\ v^{(n)}\big|_{t=d_1+\frac{1}{n}} = v^{(n)}\big|_{y=1} = 0, \\ \partial_y v^{(n)} + \beta b(t) v^{(n)}\big|_{\Gamma^{(n,d_1)}} = 0, \\ \partial_y v^{(n)}\big|_{\Gamma^{(n,d_2)}} = 0, \end{cases}$$

into the following

$$\begin{cases} \partial_s w^{(n)}\,(s,y) - \partial_y^2 w^{(n)}\,(s,y) = \zeta^{(n)}\,(s,y), \\ w^{(n)}\big|_{s=h^{-1}\left(d_1+\frac{1}{n}\right)} = w^{(n)}\big|_{y=1} = 0, \\ \partial_y w^{(n)} + \beta b(h(s)) w^{(n)}\big|_{\Gamma_h^{(n,d_1)}} = 0, \\ \partial_y w^{(n)}\big|_{\Gamma_h^{(n,d_2)}} = 0, \end{cases} \tag{8}$$

with $\zeta^{(n)}\,(s,y) = \dfrac{g^{(n)}\,(t,y)}{h'\,(s)}$, $w^{(n)}\,(s,y) = v^{(n)}\,(t,y)$ and

$$\Gamma_h^{(n,d_1)} = \left\{(s,0) \in \mathbb{R}^2 : h^{-1}(d_1 + \frac{1}{n}) < s < 0\right\}, \; \Gamma_h^{(n,d_2)} = \left\{(s,0) \in \mathbb{R}^2 : 0 < s < h^{-1}(d_2 - \frac{1}{n})\right\}.$$

Note that this change of variable preserves the spaces L^2 and $\mathcal{H}^{1,2}$. It follows from Lions and Magenes [17], for instance, that there exists a unique $w^{(n)} \in \mathcal{H}^{1,2}$ solution of the problem (8). In other words, the operator

$$\mathcal{L}_1 := \partial_t - \frac{1}{\left(c-\psi_c^{(n)}\right)^2}\partial_y^2$$

is an isomorphism from $\mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right)$ into $L^2\left(R^{(n)}\right)$. On the other hand, the operator $a\partial_y$ is compact (see Lemma 2.1). Consequently, $\mathcal{L}_1 + a\partial_y$ is a Fredholm operator from $\mathcal{H}_\gamma^{1,2}\left(R^{(n)}\right)$ into $L^2\left(R^{(n)}\right)$. Thus the invertibility of $\mathcal{L}_1 + a\partial_y$ follows from its injectivity. This implies that

Problem (6) admits a unique solution $u_c^{(n)} \in \mathcal{H}^{1,2}\left(\Omega_c^{(n)}\right)$. We obtain the function $u_c^{(n)}$ by setting $u_c^{(n)}(t,x) = v^{(n)}(t,y) = w^{(n)}\left(h^{-1}(t), y\right)$. This ends the proof of Theorem 2.1. □

We shall need the following result in order to justify the calculus of the next section.

Lemma 2.2. *The space*

$$\left\{u^{(n)} \in H^2\left(\left]d_1+\frac{1}{n},0\right[\times]0,1[\right);\ u^{(n)}\Big|_{t=d_1+\frac{1}{n}} = u^{(n)}\Big|_{y=1} = 0,\ \partial_y u^{(n)} + \beta b(t)\, u^{(n)}\Big|_{\Gamma^{(n,d_1)}} = 0\right\}$$

is dense in the space

$$\left\{u^{(n)} \in \mathcal{H}^{1,2}\left(\left]d_1+\frac{1}{n},0\right[\times]0,1[\right);\ u^{(n)}\Big|_{t=d_1+\frac{1}{n}} = u^{(n)}\Big|_{y=1} = 0,\ \partial_y u^{(n)} + \beta b(t)\, u^{(n)}\Big|_{\Gamma^{(n,d_1)}} = 0\right\}.$$

Proof. It is a consequence of [17, Vol. 1, Theorem 2.1]. □

Remark 2.1. *We can replace in Lemma 2.2,* $\left]d_1+\frac{1}{n},0\right[\times]0,1[$ *by* $\Omega_c^{(n)}\Big|_{t<0}$ *with the help of the change of variable defined above.*

2.2. Problem (5) in the non-rectangular bounded domain Ω_c

Now, we return to the non-rectangular bounded domain Ω_c. For a large enough positive integer n such that $d_1 + \dfrac{1}{n} < 0 < d_2 - \dfrac{1}{n}$, we set $f_c^{(n)} = f|_{\Omega_c^{(n)}}$ and denote by $u_c^{(n)} \in \mathcal{H}^{1,2}\left(\Omega_c^{(n)}\right)$ the solution of Problem (6) in $\Omega_c^{(n)}$. Such a solution exists by Theorem 2.1.

An energy type estimate

First, let us denote

$$Q_1 = \Omega_c^{(n)}\Big|_{t<0},\quad Q_2 = \Omega_c^{(n)}\Big|_{t>0} \quad \text{and } f_i = f|_{Q_i}, i = 1,2.$$

Then, consider the following problems:

$$\begin{cases} \partial_t u_1 - \partial_x^2 u_1 = f_1 \ \text{ in } Q_1, \\ u_1|_{t=d_1+\frac{1}{n}} = u_1|_{x=c} = 0, \\ \partial_x u_1 + \beta u_1|_{\Gamma_c^{(n,d_1)}} = 0, \end{cases} \tag{9}$$

$$\begin{cases} \partial_t v - \partial_x^2 v = f_2 \ \text{ in } Q_2, \\ v|_{\Gamma} = v|_{x=c} = 0, \\ \partial_x v|_{\Gamma_c^{(n,d_2)}} = 0, \end{cases} \tag{10}$$

where

$$\Gamma_c^{(n,d_1)} = \left\{(t,\varphi_1(t)) \in \mathbb{R}^2 : d_1 + \frac{1}{n} < t < 0\right\},\quad \Gamma_c^{(n,d_2)} = \left\{(t,\varphi_2(t)) \in \mathbb{R}^2 : 0 < t < d_2 - \frac{1}{n}\right\}$$

and

$$\Gamma = \left\{(0,x) \in \mathbb{R}^2 : x \in]0,c[\right\}.$$

By a similar argument like that used in Subsection 2.1, Problems (9) and (10) admit (unique) solutions $u_1 \in \mathcal{H}^{1,2}(Q_1)$ and $v \in \mathcal{H}^{1,2}(Q_2)$.

The following Lemmas will be needed in order to establish the uniform estimate of Proposition 2.1.

Lemma 2.3. *The solutions u_1 and v of Problems (9) and (10) verify the following estimates:*

$$\|f_1\|^2 = \|\partial_t u_1\|^2_{L^2(Q_1)} + \left\|\partial_x^2 u_1\right\|^2_{L^2(Q_1)} + \|\partial_x u_1\|^2_{L^2(\Gamma)} + I_n, \tag{11}$$

$$\|f_2\|^2 = \|\partial_t v\|^2_{L^2(Q_2)} + \left\|\partial_x^2 v\right\|^2_{L^2(Q_2)} + \|\partial_x v\|^2_{L^2(\Gamma')}, \tag{12}$$

where

$$I_n = -\beta(u_1(0,\varphi_1(0)))^2 - \int_{d_1+\frac{1}{n}}^{0} \varphi_1'(t)\,(\partial_x u(t,\varphi_1(t)))^2\, dt,$$

$$\Gamma = \left\{(0,x)\in\mathbb{R}^2 : x \in\,]0,c[\right\},\quad \Gamma' = \left\{\left(d_2-\frac{1}{n},x\right)\in\mathbb{R}^2 : x\in \left]\varphi_2\left(d_2-\frac{1}{n}\right),c\right[\right\}.$$

Proof. Let us denote the inner product in $L^2(Q_1)$ by $\langle .,.\rangle$, then we have

$$\begin{aligned}\|f_1\|^2_{L^2(Q_1)} &= \langle \partial_t u_1 - \partial_x^2 u_1, \partial_t u_1 - \partial_x^2 u_1\rangle = \\ &= \|\partial_t u_1\|^2_{L^2(Q_1)} + \left\|\partial_x^2 u_1\right\|^2_{L^2(Q_1)} - 2\langle \partial_t u_1, \partial_x^2 u_1\rangle.\end{aligned}$$

Calculating the last term of the previous relation, we obtain

$$\begin{aligned}\left\langle \partial_t u_1, \partial_x^2 u_1\right\rangle &= \int_{Q_1} \partial_t u_1 \partial_x^2 u_1 dt dx = \\ &= -\int_{Q_1} \partial_x\partial_t u_1.\partial_x u_1 dt dx + \int_{\partial Q_1} \partial_t u_1.\partial_x u_1 \nu_x d\sigma.\end{aligned}$$

So,

$$\begin{aligned}-2\left\langle \partial_t u_1, \partial_x^2 u_1\right\rangle &= \int_{Q_1} \partial_t (\partial_x u_1)^2\, dt dx - 2\int_{\partial Q_1} \partial_t u_1.\partial_x u_1 \nu_x d\sigma = \\ &= \int_{\partial Q_1} \left[(\partial_x u_1)^2 \nu_t - 2\partial_t u_1.\partial_x u_1 \nu_x\right] d\sigma\end{aligned}$$

where ν_t, ν_x are the components of the unit outward normal vector at ∂Q_1. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_1 where $t = d_1 + \dfrac{1}{n}$ and $x = c$, we have $u_1 = 0$ and consequently $\partial_x u_1 = 0$. The corresponding boundary integral vanishes. On the part of the boundary of Q_1 where $t = 0$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_0^c (\partial_x u_1)^2\, dx$$

is nonnegative. On the part of the boundary where $x = \varphi_1(t)$, we have

$$\nu_x = \frac{-1}{\sqrt{1+(\varphi_1')^2(t)}},\quad \nu_t = \frac{\varphi_1'(t)}{\sqrt{1+(\varphi_1')^2(t)}}\quad \text{and}\quad \partial_x u_1(t,\varphi_1(t)) + \beta u_1(t,\varphi_1(t)) = 0.$$

Consequently, the corresponding boundary integral is the following:

$$I_n = \int_{d_1+\frac{1}{n}}^{0} \varphi_1'(t)\left[\partial_x u_1(t,\varphi_1(t))\right]^2 dt + 2\int_{d_1+\frac{1}{n}}^{0} \partial_t u_1(t,\varphi_1(t))\partial_x u_1(t,\varphi_1(t))\, dt.$$

By putting $h(t) := u_1(t,\varphi_1(t)),\ t\in[d_1+\frac{1}{n},0]$, we obtain

$$\partial_t u(t,\varphi_1(t))\partial_x u(t,\varphi_1(t)) = h'(t)\partial_x u(t,\varphi_1(t)) - \varphi_1'(t)\,(\partial_x u(t,\varphi_1(t)))^2.$$

So, by using the boundary conditions, we get

$$\begin{aligned}2\int_{d_1+\frac{1}{n}}^{0}\partial_t u_1(t,\varphi_1(t))\partial_x u_1(t,\varphi_1(t))\,dt &= \\ &= 2\int_{d_1+\frac{1}{n}}^{0} h'(t)\partial_x u(t,\varphi_1(t))\,dt - 2\int_{d_1+\frac{1}{n}}^{0}\varphi_1^{'}(t)\left(\partial_x u(t,\varphi_1(t))\right)^2\,dt = \\ &= -2\beta\int_{d_1+\frac{1}{n}}^{0} h'(t)h(t)\,dt - 2\int_{d_1+\frac{1}{n}}^{0}\varphi_1^{'}(t)\left(\partial_x u(t,\varphi_1(t))\right)^2\,dt = \\ &= -\beta\int_{d_1+\frac{1}{n}}^{0}(h(t)^2)'\,dt - 2\int_{d_1+\frac{1}{n}}^{0}\varphi_1^{'}(t)\left(\partial_x u(t,\varphi_1(t))\right)^2\,dt = \\ &= -\beta(h(0))^2 - 2\int_{d_1+\frac{1}{n}}^{0}\varphi_1^{'}(t)\left(\partial_x u(t,\varphi_1(t))\right)^2\,dt.\end{aligned}$$

Finally,

$$-2\langle\partial_t u_1,\partial_x^2 u_1\rangle = -\beta(u_1(0,\varphi_1(0)))^2 - \int_{d_1+\frac{1}{n}}^{0}\varphi_1^{'}(t)\left(\partial_x u(t,\varphi_1(t))\right)^2\,dt + \|\partial_x u_1\|^2_{L^2(\Gamma)}$$

and formula (11) follows. By using a similar argument, we can prove formula (12). □

Let us now, consider the following problem

$$\begin{cases}\partial_t w - \partial_x^2 w = 0 \ \text{ in } Q_2,\\ w|_{\Gamma} = u_1|_{\Gamma},\\ w|_{x=c} = \partial_x w|_{\Gamma_c^{(n,d_2)}} = 0,\end{cases} \tag{13}$$

where u_1 is the solution of Problem (9). Thanks to [17, Theorem 4.3, Vol. 2], Problems (13) admits a unique solution $w \in \mathcal{H}^{1,2}(Q_2)$. Note that we can approach $u_1|_{\Gamma}$ (which is in $H^1(\Gamma)$) by regular functions (for example, by functions in $H^2(\Gamma)$), then it is easy to prove that

Lemma 2.4. *The solution w of Problem (13) verifies*

$$\|\partial_x u_1\|^2_{L^2(\Gamma)} = \|\partial_t w\|^2_{L^2(Q_2)} + \left\|\partial_x^2 w\right\|^2_{L^2(Q_2)} + \|\partial_x w\|^2_{L^2(\Gamma')}. \tag{14}$$

Now, we set

$$u_c^{(n)} = \begin{cases} u_1 \text{ in } Q_1,\\ u_2 \text{ in } Q_2,\end{cases}$$

where $u_2 = v + w$. Note that $u_c^{(n)} \in \mathcal{H}^{1,2}\left(\Omega_c^{(n)}\right)$ is then the solution of Problem (6) obtained in Theorem 2.1.

Proposition 2.1. *There exists a constant $C > 0$ independent of n such that*

$$\left\|\partial_t u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} + \left\|\partial_x^2 u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} \leqslant C\,\|f_c\|^2_{L^2(\Omega_c)}.$$

Proof. Summing up the estimates (11), (12) and (14), we then obtain

$$\begin{aligned}\left\|f_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} &= \|f_1\|^2_{L^2(Q_1)} + \|f_2\|^2_{L^2(Q_2)} \geqslant \\ &\geqslant \|\partial_t u_1\|^2_{L^2(Q_1)} + \|\partial_t v\|^2_{L^2(Q_2)} + \|\partial_t w\|^2_{L^2(Q_2)} + \\ &\quad + \left\|\partial_x^2 u_1\right\|^2_{L^2(Q_1)} + \left\|\partial_x^2 v\right\|^2_{L^2(Q_2)} + \left\|\partial_x^2 w\right\|^2_{L^2(Q_2)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left\|f_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} &\geqslant \|\partial_t u_1\|^2_{L^2(Q_1)} + \frac{1}{2}\|\partial_t u_2\|^2_{L^2(Q_2)} + \left\|\partial_x^2 u_1\right\|^2_{L^2(Q_1)} + \frac{1}{2}\left\|\partial_x^2 u_2\right\|^2_{L^2(Q_2)} \geqslant \\ &\geqslant \tfrac{1}{2}\left(\left\|\partial_t u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} + \left\|\partial_x^2 u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)}\right).\end{aligned}$$

But

$$\left\|f_c^{(n)}\right\|_{L^2\left(\Omega_c^{(n)}\right)} \leqslant \|f_c\|_{L^2(\Omega_c)},$$

then,

$$\left\|\partial_t u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} + \left\|\partial_x^2 u_c^{(n)}\right\|^2_{L^2(\Omega_c^{(n)})} \leqslant 2\left\|f_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} \leqslant 2\|f_c\|^2_{L^2(\Omega_c)}.$$

This ends the proof of Proposition 2.1. □

Theorem 2.2. *There exists a constant $K > 0$ independent of n and c such that*

$$\left\|u_c^{(n)}\right\|^2_{\mathcal{H}^{1,2}\left(\Omega_c^{(n)}\right)} \leqslant K\left\|f_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} \leqslant K\|f\|^2_{L^2(\Omega_c)}.$$

Proof. The majoration of $\left\|\partial_t u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)} + \left\|\partial_x^2 u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)}$ is given by Proposition 2.1. The majorations of $\left\|\partial_x u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)}$ and $\left\|u_c^{(n)}\right\|^2_{L^2\left(\Omega_c^{(n)}\right)}$ can be obtained by similar arguments used in Lemma 3.1 and Lemma 3.2. □

Passing to the limit

We are now in position to prove the first main result of this work.

Theorem 2.3. *Problem (5) admits a (unique) solution u_c belonging to*

$$\mathcal{H}^{1,2}_\gamma(\Omega_c) = \left\{u \in \mathcal{H}^{1,2}(\Omega_c);\, u|_{\Gamma_{0,c}} = \partial_x u + \beta u|_{\Gamma_{1,c}} = \partial_x u|_{\Gamma_{2,c}} = 0\right\}.$$

Proof. Choose a sequence $\left(\Omega_c^{(n)}\right)_{n\in\mathbb{N}^*}$ of the domains defined above. Then, we have $\Omega_c^{(n)} \to \Omega_c$, as $n \to +\infty$. Consider the solution $u_c^{(n)} \in \mathcal{H}^{1,2}\left(\Omega_c^{(n)}\right)$ of the mixed Robin-Neumann boundary value problem

$$\begin{cases} \partial_t u_c^{(n)} - \partial_x^2 u_c^{(n)} = f_c^{(n)} \in L^2\left(\Omega_c^{(n)}\right), \\ u_c^{(n)}\Big|_{t=d_1+\frac{1}{n}} = u_c^{(n)}\Big|_{x=c} = 0, \\ \partial_x u_c^{(n)} + \beta u_c^{(n)}\Big|_{\Gamma_c^{(1,n)}} = \partial_x u_n|_{\Gamma_c^{(2,n)}} = 0. \end{cases}$$

Such a solution $u_c^{(n)}$ exists by Theorem 2.1. Let us define

$$\begin{aligned}\pi_1^{(n)} &:= \left\{(t,x) \in \Omega_c :\, d_1 < t < d_1 + \frac{1}{n}\right\}, \\ \pi_2^{(n)} &:= \left\{(t,x) \in \Omega_c :\, d_2 - \frac{1}{n} < t < d_2\right\}, \\ \sigma &:= \left\{(t,x) \in \Omega_c :\, t = d_2 - \frac{1}{n}\right\},\end{aligned}$$

and consider u_c the 0-extension of $u_c^{(n)}$ to $\pi_1^{(n)}$ and the extension by symmetry with respect to the vertical segment σ to $\pi_2^{(n)}$. This extension noted by $\widetilde{u_c^{(n)}}$ is then in $\mathcal{H}^{1,2}(\Omega_c)$ and verifies in particular

$$\left\|\widetilde{u_c^{(n)}}\right\|^2_{\mathcal{H}^{1,2}(\Omega_c)} \leqslant K \left\|f_c\right\|^2_{L^2(\Omega_c)}.$$

The following compactness result is well known: A bounded sequence in a reflexive Banach space (and in particular in a Hilbert space) is weakly convergent. So, for a suitable increasing sequence of integers n_k, $k = 1, 2, \ldots$, there exist functions

$$u_c,\ v_c \text{ and } v_{c,j},\ j = 1,\ 2$$

in $L^2(\Omega_c)$ such that

$$\begin{array}{lll} \widetilde{u_c^{(n_k)}} & \rightharpoonup & u_c \quad \text{weakly in } L^2(\Omega_c),\ k \to \infty, \\ \widetilde{\partial_t u_c^{(n_k)}} & \rightharpoonup & v_c \quad \text{weakly in } L^2(\Omega_c),\ k \to \infty, \\ \widetilde{\partial_x^j u_c^{(n_k)}} & \rightharpoonup & v_{c,j} \quad \text{weakly in } L^2(\Omega_c),\ k \to \infty,\ j = 1,\ 2. \end{array}$$

Then, $v_c = \partial_t u_c$, $v_{c,1} = \partial_x u_c$ and $v_{c,2} = \partial_x^2 u_c$ in the sense of distributions in Ω_c and so in $L^2(\Omega_c)$. So, we have

$$\partial_t u_c - \partial_x^2 u_c = f_c \text{ in } \Omega_c.$$

On the other hand, the solution u_c satisfies the boundary conditions

$$u_c|_{\Gamma_{0,c}} = \partial_x u_c + \beta u_c|_{\Gamma_{1,c}} = \partial_x u_c|_{\Gamma_{2,c}} = 0$$

since

$$\forall n \in \mathbb{N}^*, \qquad u_c|_{\Omega_c^{(n)}} = u_c^{(n)}.$$

This proves the existence of solution to Problem (5). □

3. Back to Problem (1)

For a large enough positive integer m, we define Ω_m by

$$\Omega_m = \{(t, x) \in \Omega_\infty : 0 < x < m\}.$$

Let $u_m \in \mathcal{H}_\gamma^{1,2}(\Omega_m)$ the solution of the following problem:

$$\begin{cases} \partial_t u_m - \partial_x^2 u_m = f_m \in L^2(\Omega_m), \\ u_m|_{\Gamma_{0,m}} = \partial_x u_m + \beta u_m|_{\Gamma_{1,m}} = 0, \\ \partial_x u_m|_{\Gamma_{2,m}} = 0, \end{cases} \tag{15}$$

where

$$f_m = f|_{\Omega_m}, \Gamma_{0,m} = \left\{(t, m) : \varphi_1^{-1}(m) < t < \varphi_2^{-1}(m)\right\},$$

$$\Gamma_{1,m} = \left\{(t, \varphi_1(t)) \in \mathbb{R}^2 : \varphi_1^{-1}(m) < t < 0\right\}, \quad \Gamma_{2,m} = \left\{(t, \varphi_2(t)) \in \mathbb{R}^2 : 0 < t < \varphi_2^{-1}(m)\right\}.$$

Such a solution u_m exists by Theorem 2.3.

Theorem 3.1. *There exists a constant $K > 0$ independent of m such that*

$$\|u_m\|^2_{\mathcal{H}^{1,2}(\Omega_m)} \leqslant K \|f_m\|^2_{L^2(\Omega_m)}.$$

In order to show the desired inequality, we need the following lemmas:

Lemma 3.1. *There exists a constant $K_1 > 0$ independent of m such that*

$$\|u_m\|^2_{L^2(\Omega_m)} \leqslant K_1 \|f_m\|^2_{L^2(\Omega_m)} .$$

Proof. For a real number $\lambda \neq 0$, we have

$$\begin{aligned}
\int_{\Omega_m} f_m u_m e^{-2\lambda^2 t}\, dtdx &= \int_{\Omega_m} \partial_t u_m u_m e^{-2\lambda^2 t}\, dtdx - \int_{\Omega_m} \partial_x^2 u_m u_m e^{-\lambda^2 t}\, dtdx, = \\
&= \int_{\Omega_m} \left[\partial_t \left(\frac{1}{2} u_m^2 e^{-2\lambda^2 t} \right) - \partial_x \left(\partial_x u_m u_m e^{-2\lambda^2 t} \right) \right] dtdx + \\
&\quad + \int_{\Omega_m} (\partial_x u_m)^2 e^{-2\lambda^2 t}\, dtdx + \lambda^2 \int_{\Omega_m} u_m^2 e^{-2\lambda^2 t}\, dtdx = \\
&= \int_{\varphi_1^{-1}(m)}^0 \left(\frac{\varphi_1'(t)}{2} - \beta \right) u_m^2(t, \varphi_1(t)) e^{-2\lambda^2 t}\, dt + \\
&\quad + \lambda^2 \int_{\Omega_m} u_m^2 e^{-2\lambda^2 t}\, dtdx + \int_{\Omega_m} (\partial_x u_m)^2 e^{-2\lambda^2 t}\, dtdx + \\
&\quad + \int_0^{\varphi_2^{-1}(m)} \frac{\varphi_2'(t)}{2} u_m^2(t, \varphi_2(t)) e^{-2\lambda^2 t}\, dt \geqslant \\
&\geqslant \lambda^2 e^{-2\lambda^2 b} \|u_m\|^2_{L^2(\Omega_m)} .
\end{aligned}$$

On the other hand, for all $\epsilon > 0$, we have

$$\int_{\Omega_m} f_m u_m e^{-2\lambda^2 t}\, dtdx \leqslant \left(\frac{1}{\epsilon} \|f_m\|^2_{L^2(\Omega_m)} + \epsilon \|u_m\|^2_{L^2(\Omega_m)} \right) e^{-2\lambda^2 a}.$$

Therefore,

$$\frac{\|f_m\|^2_{L^2(\Omega_m)}}{\epsilon} \geqslant \left(\lambda^2 e^{-2\lambda^2 (b-a)} - \epsilon \right) \|u_m\|^2_{L^2(\Omega_m)} .$$

Hence, by choosing ϵ small enough, we obtain the desired inequality. □

Lemma 3.2. *There exists a constant $K_2 > 0$ independent of m such that*

$$\|\partial_x u_m\|^2_{L^2(\Omega_m)} \leqslant K_2 \|f_m\|^2_{L^2(\Omega_m)} .$$

Proof. We have

$$\int_{\Omega_m} \partial_x (u_m \partial_x u_m) dtdx = \int_{\partial\Omega_m} u_m \partial_x u_m \nu_x d\sigma,$$

where ν_t, ν_x are the components of the unit outward normal vector at $\partial\Omega_m$. On the part of the boundary of Ω_m where $x = m$, we have $u_m = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $x = \varphi_2(t)$, we have $\partial_x u_m = 0$. Consequently, the corresponding boundary integral vanishes. On the part of the boundary where $x = \varphi_1(t)$, we have

$$\nu_x = \frac{-1}{\sqrt{1 + (\varphi_1')^2 (t)}} \quad \text{and} \quad \partial_x u_m (t, \varphi_1(t)) + \beta u_m (t, \varphi_1(t)) = 0.$$

Accordingly, the corresponding boundary integral is

$$-\beta \int_{\varphi_1^{-1}(m)}^0 u_m^2(t, \varphi_1(t))\, dt.$$

Finally,

$$\int_{\Omega_m} \partial_x(u_m \partial_x u_m) dt dx = -\beta \int_{\varphi_1^{-1}(m)}^{0} u_m^2(t, \varphi_1(t))\, dt.$$

On the other hand, we have

$$\int_{\Omega_m} \partial_x(u_m \partial_x u_m) dt dx = \int_{\Omega_m} u_m \partial_x^2 u_m dt dx dy + \int_{\Omega_m} (\partial_x u_m)^2 dt dx.$$

Then,

$$-\beta \int_{\varphi_1^{-1}(m)}^{0} u_m^2(t, \varphi_1(t))\, dt = \int_{\Omega_m} u_m \partial_x^2 u_m dt dx + \|\partial_x u_m\|^2_{L^2(\Omega_m)}.$$

Consequently,

$$\begin{aligned}
\|\partial_x u_m\|^2_{L^2(\Omega_m)} &= -\int_{\Omega_m} u_m \partial_x^2 u_m dt dx + \beta \int_{\varphi_1^{-1}(m)}^{0} u_m^2(t, \varphi_1(t))\, dt \leqslant \\
&\leqslant \int_{\Omega_m} u_m^2 dt dx + \int_{\Omega_m} (\partial_x^2 u_m)^2 dt dx = \\
&= \|u_m\|^2_{L^2(\Omega_m)} + \left\|\partial_x^2 u_m\right\|^2_{L^2(\Omega_m)}.
\end{aligned}$$

Lemma 3.1 and Proposition 2.1 which remains valid in Ω_m give

$$\|\partial_x u_m\|^2_{L^2(\Omega_m)} \leqslant K_1 \|f_m\|^2_{L^2(\Omega_m)} + 2\|f_m\|^2_{L^2(\Omega_m)} \leqslant K_2 \|f_m\|^2_{L^2(\Omega_m)}.$$

□

Theorem 3.1 is a direct consequence of Lemma 3.1, Lemma 3.2 and Proposition 2.1 which remains valid in Ω_m. We obtain the solution u of Problem (1) by letting m go to infinity in Theorem 3.1. This ends the proof of Theorem 1.1.

Remark. Let us consider the following problem:

$$\begin{cases} \partial_t v - \partial_x^2 v = f \in L^2(D), \\ \partial_x v + \alpha v|_{\Gamma_1} = 0, \\ \partial_x v|_{\Gamma_2} = 0, \end{cases} \tag{16}$$

where

$$D := \left\{(t, x) \in \mathbb{R}^2 : \ l_1 < t < l_2; -\infty < x < \Phi(t)\right\},$$

where l_1, l_2 are real numbers such that $-\infty < l_1 < 0 < l_2 < +\infty$, while Φ is a Lipschitz continuous real-valued function on (l_1, l_2), and such that

$$\Phi(t) := \begin{cases} \psi_1(t) & \text{on} \quad (l_1, 0], \\ \psi_2(t) & \text{on} \quad [0, l_2). \end{cases}$$

The function ψ_1 (respectively, ψ_2) is a negative and increasing (respectively, decreasing) on $(l_1, 0]$ (respectively, on $[0, l_2)$) and verifies the hypothesis $\psi_1(0) = \psi_2(0) = 0$. Here, the coefficient α is a positive real number and Γ_i is the part of the boundary of D where $x = \psi_i(t)$, $i = 1, 2$.

By using the same arguments like those used in solving Problem (1), we can show that Problem (16) admits a (unique) solution v belonging to $\mathcal{H}^{1,2}(D)$, under the assumption

$$\left(\frac{\psi_1'(t)}{2} - \alpha\right) \leqslant 0 \text{ almost everywhere } t \in]l_1, 0[\,. \tag{17}$$

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Результаты исследования регулярности в пространстве для уравнения теплопроводности с граничными условиями типа Робина-Неймана в изменяющихся во времени областях

Тахир Буджериу
Лаборатория прикладной математики,
Факультет точных наук, Университет Беджая, Беджая, 6000
Алжир
Арезки Хелуфи
Лаборатория прикладной математики
Технологический факультет, Университет Беджая, Беджая, 6000
Алжир

Эта статья посвящена уравнению теплопроводности

$$\partial_t u - \partial_x^2 u = f \ in \ D, \ D = \left\{(t,x) \in \mathbb{R}^2 : a < t < b, \psi(t) < x < +\infty\right\}$$

с функцией ψ, удовлетворяющей некоторым условиям, и задача дополняется граничными условиями типа Робина-Неймана. Мы изучаем проблему глобальной регулярности в подходящем параболическом пространстве Соболева. Докажем, в частности, что для $f \in L^2(D)$ существует единственное решение и такое, что u, $\partial_t u$, $\partial_x^j u \in L^2(D)$, $j = 1, 2$. Доказательство основано на методе декомпозиции области. Эта работа дополняет результаты, полученные в [10].

Ключевые слова: уравнение теплопроводности, неограниченные нецилиндрические области, условие Робина, условие Неймана, анизотропные пространства Соболева.