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The Highest Dimension of Commutative Subalgebras in Chevalley Algebras

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Let $L_\Phi(K)$ denotes a Chevalley algebra with the root system Φ over a field K. In 1945 A. I. Mal'cev investigated the problem of describing abelian subgroups of highest dimension in complex simple Lie groups. He solved this problem by transition to complex Lie algebras and by reduction to the problem of describing commutative subalgebras of highest dimension in the niltriangular subalgebra. Later these methods were modified and applied for the problem of describing large abelian subgroups in finite Chevalley groups. The main result of this article allows to calculate the highest dimension of commutative subalgebras in a Chevalley algebra $L_\Phi(K)$ over an arbitrary field.

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Introduction

Let $L_\Phi(K)$ denotes a Chevalley algebra with the root system Φ over a field K, and let Π be a fundamental system of roots. The elements $\{e_r,\, h_p \mid r \in \Phi, p \in \Pi\}$ form a basis of $L_\Phi(K)$, called a Chevalley basis [1]. Let $N\Phi(K)$ denotes a niltriangular subalgebra in $L_\Phi(K)$ with the basis $\{e_r \mid r \in \Phi^+\}$.

In 1945 A. I. Mal'cev investigated the problem of describing abelian subgroups of highest dimension in complex simple Lie groups [7]. He solved this problem by transition to complex Lie algebras and by reduction to the problem of describing commutative subalgebras of highest dimension in the niltriangular subalgebra $N\Phi(\mathbb{C})$.

Later these methods were modified and applied for the problem of describing large abelian subgroups in finite Chevalley groups [3–5, 8, 9]. Given a group-theoretic property $\mathcal{P}$, we recall that every $\mathcal{P}$-subgroup of largest order in a finite group is a *large $\mathcal{P}$-subgroup.*

The generalization of Mal'cev problem [7] for Chevalley algebras $L_\Phi(K)$ over an arbitrary field K was pointed in [6].

The main result of this article is Theorem 1, which allows to calculate the highest dimension of commutative subalgebras in a Chevalley algebra $L_\Phi(K)$ over an arbitrary field.

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1. Preliminary remarks and notation

Let $L_\Phi(K)$ denotes a Chevalley algebra with the root system Φ over a field K, and let Π be a fundamental system of roots. The elements $\{e_r,\ h_p \mid r \in \Phi, p \in \Pi\}$ form a basis of $L_\Phi(K)$, called a Chevalley basis. The elements of this basis multiply together as follows:

$$\begin{aligned}
&h_r * h_s = 0, && r, s \in \Pi,\\
&h_r * e_s = A_{rs} e_s, && r \in \Pi, s \in \Phi,\\
&e_r * e_{-r} = h_r, && r \in \Phi,\\
&e_r * e_s = 0, && r, s \in \Phi,\ r + s \notin \Phi,\\
&e_r * e_s = N_{rs} e_{r+s},
\end{aligned}$$

where the elements N_{rs} are called the structure constants of $L_\Phi(K)$, and $A_{rs} = \dfrac{2(r,s)}{(r,r)}$. The elements $\{h_p \mid p \in \Pi\}$ form a basis for a Cartan subalgebra H [1].

A subset Ψ of the root system Φ is called a *commutative*, if $r + s \notin \Phi$ for all $r, s \in \Psi$ [7]. A subset Ψ of the root system Φ is said to be *p-commutative*, if in the algebra $N\Phi(K)$ over a field K of characteristic p we have $e_r * e_s = 0$ for all $r, s \in \Psi$ [8].

Further, we use a regular ordering of roots [1, Lemma 5.3.1]. Let $x \in L_\Phi(K)$ and

$$x = a_1 e_{r_1} + a_2 e_{r_2} + \dots + h + c_1 e_{s_1} + c_2 e_{s_2} + \dots, \tag{1}$$

where $r_i \in \Phi^-$, $s_i \in \Phi^+$, $r_1 < r_2 < \dots$, $s_1 < s_2 < \dots$, $h = b_1 h_{p_1} + b_2 h_{p_2} + \dots$, $p_1, p_2, \dots \in \Pi$. We consider the first non-zero term in (1). If this term has the form te_r, $r \in \Phi$, then we denote $b(x) = e_r$, else we denote $b(x) = h$. For $M \subseteq L$ we set $b(M) = \{b(x) \mid x \in M\}$.

Lemma 1. *If $x * y = 0$ for $x, y \in L_\Phi(K)$, then $b(x) * b(y) = 0$.*

Proof. Let

$$\begin{aligned}
x &= a_1 e_{r_1} + a_2 e_{r_2} + \dots + h + c_1 e_{s_1} + c_2 e_{s_2} + \dots,\\
y &= a'_1 e_{r'_1} + a'_2 e_{r'_2} + \dots + h' + c'_1 e_{s'_1} + c'_2 e_{s'_2} + \dots,
\end{aligned}$$

where $r_1 + r'_1 \in \Phi^-$, $r_1 < r_2 < \dots < s_1 < s_2 < \dots$, $r'_1 < r'_2 < \dots < s'_1 < s'_2 < \dots$. Then in the expression of $x * y$ as a linear combination of e_r, $r \in \Phi$, every e_r ($r \in \Phi^-$) has the form $e_{r_i + r'_i}$, $e_{r_i + s'_i}$, $e_{s_i + r'_i}$, e_{r_i} or $e_{r'_i}$. Since $r_i - r_1 \in V^+$, $r'_i - r'_1 \in V^+$ ($i \neq 1$), where V^+ is a certain positive subspace, then $(r_i + r'_i) - (r_1 + r'_1) = (r_i - r_1) + (r'_i - r'_1) \in V^+$, so $r_1 + r'_1 < r_i + r'_i$. Since $ht(r_1 + r'_1) < ht(r_i)$ and $ht(r_1 + r'_1) < ht(r'_i)$, we have $r_1 + r'_1 < r_i$ and $r_1 + r'_1 < r'_i$. Hence $b(x * y) = e_{r_1 + r'_1}$, so $x * y \neq 0$, that gives a contradiction. Analogously, the remaining cases when $b(x)$, $b(y)$, $b(x) * b(y)$ are not in H give a contradiction.

Let $b(x) = h$, $b(y) = a'_1 e_{r'_1}$ and $h * a'_1 e_{r'_1} \neq 0$. Then $b(x * y) = e_{r'_1}$, so again $x * y \neq 0$.

In the case

$$\begin{aligned}
x &= c_1 e_{s_1} + c_2 e_{s_2} + \dots,\\
y &= a'_1 e_{r'_1} + a'_2 e_{r'_2} + \dots + h' + c'_1 e_{s'_1} + c'_2 e_{s'_2} + \dots,
\end{aligned}$$

where $r'_1 = -s_1$, the expression of $x * y$ contains a term $h_{r'_1}$. If $x * y = 0$, then in the expression of x and y there exists another pair of terms of form e_{s_i}, e_{-s_i}, respectively. Hence $s_1 < s_i$ and $-s_1 < -s_i$, a contradiction. This completes the proof of Lemma 1. □

Note that in the case $b(M) = \{e_r \mid r \in \Phi^+\}$ the set of corresponding roots $r \in \Phi^+$ coincides with the set $\mathcal{L}_1(M)$ ([2,4]).

2. The highest dimension of commutative subalgebras

Theorem 1. *Let $L_\Phi(K)$ be a Chevalley algebra with the root system Φ over an arbitrary field K of characteristic p. Let m be a maximal order of p-commutative subsets of roots in Φ, and let k be a dimension of the center of $L_\Phi(K)$. The highest dimension of commutative subalgebras of $L_\Phi(K)$ equals $m+k$.*

Lemma 2. *Let A be a commutative subalgebra of $L = L_\Phi(K)$. Then there exists an automorphism of L transforming A to a subalgebra B in L such that for all $x \in B$ either $b(x) = e_r$, $r \in \Phi^-$, or $b(x) = h \in H$, where h is an element of the center of L.*

Proof. Let $x_r(t)$ denotes an automorphism of a Chevalley algebra, effecting on the elements of a Chevalley basis as follows:

$$e_r \to e_r, \tag{2}$$

$$h_s \to h_s - tA_{sr}e_r \ (s \in \Pi), \tag{3}$$

$$e_{-r} \to e_{-r} + th_r - t^2 e_r, \tag{4}$$

$$e_s \to \sum_{i=0}^{q} M_{r,s,i} t^i e_{ir+s} \ (s \in \Phi \setminus \{\pm r\}), \quad M_{r,s,0} := 1, \tag{5}$$

$$M_{r,s,i} := \pm \left(p(r,s) + i//i\right).$$

Let

$$x = h + c_1 e_{s_1} + c_2 e_{s_2} + \dots .$$

Suppose that there exists a fundamental root p such that $h * e_p \neq 0$ and $s_1 \neq p$. Since

$$h * e_p = (b_1 h_{p_1} + b_2 h_{p_2} + \dots) * e_p = (b_1 A_{p_1,p} + b_2 A_{p_2,p} + \dots) e_p,$$

then the automorphism $x_{-p}(1)$ transforms h to

$$h + (b_1 A_{p_1,p} + b_2 A_{p_2,p} + \dots) e_p,$$

where $b_1 A_{p_1,p} + b_2 A_{p_2,p} + \dots \neq 0$. Hence $b(x_{-p}(1)(x)) = e_{-p}$.

If $s_1 = p$, then first we obtain $c_1 = 0$, up to a certain automorphism $x_p(t)$, $p \in \Pi$, and again $b(x_{-p}(1)(x)) = e_{-p}$.

Suppose that there exists $y \in A$ such that $b(y) = e_r$, $r \in \Phi^-$, and $b(x_{\pm p}(1)(y))$ equals $h \in H$ or e_r, $r \in \Phi^+$. Taking into account the relations (2)–(5), we deduce that $b(y) = e_{-p}$. Since $h * e_p \neq 0$ hence $h * e_{-p} \neq 0$. By Lemma 1, this is a contradiction.

If

$$x = c_1 e_{s_1} + c_2 e_{s_2} + \dots$$

then $b(x_{-s_1}(1)(x)) = e_{-s_1}$.

Suppose that there exists $y \in A$ such that $b(y) = e_r$, $r \in \Phi^-$, and $b(x_{-s_1}(1)(y)$ equals $h \in H$ or e_r, $r \in \Phi^+$. Taking into account the relations (2)–(5), we deduce that $b(y) = e_{-s_1}$. By Lemma 1, this is a contradiction.

The lemma is proved. □

Now the Theorem 1 follows from Lemma 1 and Lemma 2.

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Наивысшая размерность коммутативных подалгебр алгебр Шевалле

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Пусть $L_\Phi(K)$ — алгебра Шевалле над полем K, ассоциированная с системой корней Φ. В 1945 г. А. И. Мальцев исследовал проблему описания абелевых подгрупп наивысшей размерности в комплексных простых группах Ли. Он решил эту проблему переходом к комплексным алгебрам Ли и редукцией к проблеме описания коммутативных подалгебр наивысшей размерности в нильтреугольной подалгебре. Позже эти методы модифицировались и применялись для решения проблемы описания больших абелевых подгрупп конечных групп Шевалле. Основной результат данной статьи позволяет вычислить наивысшую размерность коммутативных подалгебр алгебры Шевалле $L_\Phi(K)$ над произвольным полем.

Ключевые слова: алгебра Шевалле, коммутативная подалгебра.