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Group Analysis of Equations of Hydrostatic Model of Viscous Fluid

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The group properties of three-dimensional equations of hydrostatic model of viscous fluid are studied. Several exact solutions are presented. The free surface of fluid and pressure on this surface are determined.

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The group analysis of differential equations is a powerful tool for studying non-linear equations and boundary value problems. This method was introduced by Sophus Lie in his works in the 19th century. The interest to the group analysis was revived by L. V. Ovsyannikov, who pointed out a method for describing the properties of differential equations [1, 2].

One of the main problems in the group analysis of differential equations is to find the permissible group of transformations of the system of equations on the set of solutions of these equations.

Lie group properties of differential equations were studied by L. V. Ovsyannikov [2], N. H. Ibragimov [3], V. V. Pukhnachov and by their followers S. V. Habirov, Y. N. Pavlovsky, A. A. Buchnev, O. V. Bytev, V. K. Andreev. At present the group properties of equations of fluid mechanics are studied by V. K. Andreyev, O. V. Kaptsov, V. V. Pukhnachev, A. A. Rodionov [4].

The Navier–Stokes equations are a system of differential equations that describe the motion of a viscous fluid. The aim of this study is to perform a group analysis of the hydrostatic model of three-dimensional Navier–Stokes equations and to find exact solutions of this model.

1. Basic equations and problem statement

The three-dimensional Navier–Stokes equations for the motion of a viscous incompressible fluid are

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$$
\begin{aligned}
u_t + uu_x + vu_y + wu_z + \frac{1}{\rho}p_x &= \nu(u_{xx} + u_{yy} + u_{zz}),\\
v_t + uv_x + vv_y + wv_z + \frac{1}{\rho}p_y &= \nu(v_{xx} + v_{yy} + v_{zz}),\\
w_t + uw_x + vw_y + ww_z + \frac{1}{\rho}p_z &= \nu(w_{xx} + w_{yy} + w_{zz}) - g,\\
u_x + v_y + w_z &= 0.
\end{aligned} \tag{1}
$$

Here u, v, w are the components of the velocity vector along the x, y, z directions, p is pressure, t is time, $g = \text{const} > 0$ is the acceleration of gravity in the z direction, ν is the dynamic viscosity coefficient, $\rho = const$ is the fluid density (it is assumed that $\rho = 1$).

Let us assume that the pressure in the fluid linearly depends on the depth

$$p_z = -g. \tag{2}$$

This assumption is often used in oceanology [5]. Then

$$p(x, y, z, t) = -gz + q(x, y, t), \tag{3}$$

$q(x, y, t)$ is a new function. In this case system (1) is rewritten as

$$
\begin{aligned}
u_t + uu_x + vu_y + wu_z + q_x &= \nu(u_{xx} + u_{yy} + u_{zz}),\\
v_t + uv_x + vv_y + wv_z + q_y &= \nu(v_{xx} + v_{yy} + v_{zz}),\\
w_t + uw_x + vw_y + ww_z &= \nu(w_{xx} + w_{yy} + w_{zz}),\\
u_x + v_y + w_z = 0, &\quad q_z = 0.
\end{aligned} \tag{4}
$$

Let $\Gamma : z = \eta(x, y, t)$ be the equation for the free boundary of a fluid on which the following kinematic and dynamic conditions are satisfied

$$\eta_t + u(x, y, \eta(x, y, t), t)\eta_x + u(x, y, \eta(x, y, t), t)\eta_y = w(x, y, \eta(x, y, t), t), \tag{5}$$

$$(p_a - p)\overrightarrow{n} + 2\nu D \cdot \overrightarrow{n} = 2\sigma H \overrightarrow{n}, \tag{6}$$

where $p_a(x, y, t)$ is the atmospheric pressure, $p = -g\eta(x, y, t) + q(x, y, t)$, $\overrightarrow{n}$ is the normal to the free surface, H is the mean curvature that depends on the position of the point on the surface, $\sigma = const$ is the surface tension coefficient, $D = D(u, v, w)$ is the deformation rate tensor [4].

If the solution of system (4)is known then the equation of free surface can be found from relation (5), and $p_a(x, y, t)$ can be found from (6).

We apply group analysis [2] to equations of system (4). It is required to find the Lie algebra of admissible operators for this system and to construct exact solutions.

A similar study of the hydrostatic model of an ideal fluid was carried out in [6].

2. Group properties of the equations

Let us consider the group properties of equations (4). Let us introduce the following index designations: $u^1 = u$, $u^2 = v$, $u^3 = w$, $u^4 = q$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$. In these new designations equations (4) supplemented by requirement (4) assume the following form:

$$
\begin{aligned}
u^1_4 + u^1u^1_1 + u^2u^1_2 + u^3u^1_3 + u^4_1 - \nu(u^1_{11} + u^1_{22} + u^1_{33}) &= 0,\\
u^2_4 + u^1u^2_1 + u^2u^2_2 + u^3u^2_3 + u^4_2 - \nu(u^2_{11} + u^2_{22} + u^2_{33}) &= 0,\\
u^3_4 + u^1u^3_1 + u^2u^3_2 + u^3u^3_3 - \nu(u^2_{11} + u^2_{22} + u^2_{33}) &= 0,\\
u^1_1 + u^2_2 + u^3_3 = 0, &\quad u^4_3 = 0.
\end{aligned} \tag{7}
$$

The lower index denotes differentiation.

The admissible operator for system (7) has the following form

$$X = \xi^i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x^i} + \eta^k(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^k},$$

Here summation is performed over $i, k = 1, 2, 3, 4$. Let us extend the operator to the first derivatives

$$\underset{1}{X} = X + \zeta_i^k \frac{\partial}{\partial u_i^k}, \quad \zeta_i^k = \frac{\partial \eta^k}{\partial x^i} + u_i^l \frac{\partial \eta^k}{\partial u^l} - u_j^k \left(\frac{\partial \xi^j}{\partial x^i} + u_i^l \frac{\partial \xi^j}{\partial u^l} \right)$$

and to the second derivatives

$$\underset{2}{X} = \underset{1}{X} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} = \underset{1}{X} + \zeta_{11}^1 \frac{\partial}{\partial u_{11}^1} + \zeta_{22}^1 \frac{\partial}{\partial u_{22}^1} + \zeta_{11}^2 \frac{\partial}{\partial u_{11}^2} + \zeta_{22}^2 \frac{\partial}{\partial u_{22}^2} + \ldots,$$

where

$$\zeta_{ij}^\alpha = \frac{\partial \zeta_i^\alpha}{\partial x^j} + u_i^l \frac{\partial \zeta_i^\alpha}{\partial u^l} + u_{jk}^l \frac{\partial \zeta_i^\alpha}{\partial u_k^l} - u_{ik}^\alpha \left(\frac{\partial \xi^k}{\partial x^l} + u_j^l \frac{\partial \xi^k}{\partial x^l} \right),$$

where summation is performed over l, k.

Let us note that values u_{12}^1, u_{13}^1, u_{23}^1, u_{12}^2, u_{13}^2, u_{23}^2, u_{12}^3, u_{13}^3, u_{23}^3 are absent in system (7). We use the invariance criterion [2]. Acting via operator $\underset{2}{X}$ onto equations (7), the governing equations are obtained. Let us consider equation (7) and substitute u_4^1, u_4^2, u_4^3, u_1^1 for remaining variables. Splitting governing equations with respect to independent variables, we obtain coordinates of operator X

$$\begin{gathered}
\xi^1 = C_4 x^1 + f(x^4), \qquad \xi^2 = C_4 x^2 + C_5 x^1 + h(x^4), \\
\xi^3 = C_2 + C_3 x^4 + C_4 x^3, \qquad \xi^3 = C_1 + 2C_4 x^4, \\
\eta^1 = -C_4 u^1 - C_5 u^2 + f'(x^4), \qquad \eta^2 = -C_4 u^2 + C_5 u^1 + h'(x^4), \\
\eta^3 = C_3 + 2C_4 x^4, \qquad \eta^4 = -2C_4 u^4 - x^1 f''(x^4) - x^2 h''(x^4) + \varphi(x^3),
\end{gathered}$$

where $C_1, \ldots, C_5$ are constants, $f(x^4)$, $h(x^4)$, $\varphi(x^4)$ are arbitrary functions.

It is proven that the Lie algebra for the system of equations (4) is formed by the operators

$$\begin{gathered}
X_1 = \partial_t, \qquad X_2 = \partial_z, \qquad X_3 = t\partial_z + \partial_w, \\
X_4 = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - u\partial_u - v\partial_v - w\partial_w - 2q\partial_q, \\
X_5 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \\
X_6 = f(t)\partial_x + f'(t)\partial_u - xf''(t)\partial_q, \\
X_7 = h(t)\partial_y + h'(t)\partial_v - yh''(t)\partial_q, \qquad X_8 = \varphi(t)\partial_q.
\end{gathered} \tag{8}$$

The first operator is responsible for the transfer in time t, the second and third operators are responsible for the transfer and the Galileo transformation along the z axis, the fourth operator is responsible for the tensile transformations, the fifth operator is responsible for the rotation around the $z$axis. The sixth, seventh and eighth operators contain arbitrary functions $f(t), h(t), \varphi(t)$ that depend on time. They define the infinite-dimensional part of the Lie algebra of admissible operators.

For the first time a group analysis of equations of system (1) was carried out by V. O. Bytev [7]. The difference of the obtained result from operators (7) is that two operators responsible for the rotation around the x and y axes are absent in (7), and the infinite-dimensional operator along the z axis similar to X_6, X_7, operators is represented as two finite-dimensional operators X_2, X_3.

If the two-dimensional hydrostatic model is considered then variable y and velocity v in equations (4) should be excluded. Then for equations

$$\begin{gathered} u_t + uu_x + wu_z + q_x = \nu(u_{xx} + u_{zz}), \\ w_t + uw_x + ww_z = \nu(w_{xx} + w_{zz}), \\ u_x + w_z = 0, \quad q_z = 0 \end{gathered} \tag{9}$$

the algebra of admissible operators is determined by the following operators

$$\begin{gathered} X_1 = \partial_t, \quad X_2 = \partial_z, \quad X_3 = t\partial_z + \partial_w, \\ X_4 = x\partial_x + z\partial_z + 2t\partial_t - u\partial_u - w\partial_w - 2q\partial_q, \\ X_5 = f(t)\partial_x + f'(t)\partial_u - xf''(t)\partial_q, \quad X_6 = \varphi(t)\partial_q. \end{gathered} \tag{10}$$

3. Exact solutions

Example 1. Let us find a solution to equations (4) with operators from basis (8)

$$\Big\langle X_3 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial w}; \quad X_7 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \, (h(t) = t)\Big\rangle.$$

The invariants of these operators are

$$J = \Big(x, t;\, u; v - \frac{y}{t};\, w - \frac{z}{t};\, q\Big).$$

Therefore, the invariant solution of equations should have the form

$$(u, v, w, q) = \Big(U(x,t);\, \frac{y}{t} + V(x,t);\, \frac{z}{t} + W(x,t);\, Q(x,t)\Big), \tag{11}$$

where U, V, W, Q are functions of two variables.

Let us substitute functions (11) into the system and obtain the factor-system:

$$\begin{gathered} U_t + UU_x + Q_x = \nu U_{xx}, \quad V_t + UV_x + \frac{1}{t}V = \nu V_{xx}, \\ W_t + UW_x + \frac{1}{t}W = \nu W_{xx}, \quad U_x + \frac{2}{t} = 0. \end{gathered} \tag{12}$$

It follows from the last equation of factor-system (12) that

$$U(x,t) = -\frac{2x}{t} + h_1(t), \tag{13}$$

where $h_1(t)$ is an arbitrary function. It follows from the first equation in (12) that

$$Q(x,t) = \frac{2x}{t}h_1(t) - xh_1'(t) - \frac{3x^2}{t^2} + h_2(t),$$

with an arbitrary function $h_2(t)$. It can be seen from the second equation of system (12) that

$$V_t = \nu V_{xx} + \left(\frac{2x}{t} - h_1(t)\right)V_x - \frac{V}{t}. \tag{14}$$

To solve equation (14) we use the reference book by A. D. Polyanin [8] (Section (1.8.6), p. 129) and consider the following equation

$$w_t = aw_{xx} + [xm(t) + n(t)]w_x + k(t)w.$$

In the case of equation (14) we have

$$a = \nu, \quad m(t) = \frac{2}{t}, \quad n(t) = -h_1(t), \quad k(t) = -\frac{1}{t}.$$

Let us introduce the following designations ($A, B, C = const$)

$$F(t) = B \exp\Big[\int m(t)dt\Big] = Bt^2,$$

$$\tau = \int F^2(t)dt + A = \frac{B^2 t^5}{5} + A,$$

$$\delta = xF(t) + \int g(t)F(t)dt + C = xBt^2 - B\int t^2 h_1(t)dt + C.$$

Using new variables $(x, t) \to (\delta, \tau)$, we obtain

$$V(x,t) = M(\delta,\tau)\exp\Big[\int k(t)dt\Big] = \frac{1}{t}M(\delta,\tau),$$

where $M(\delta, \tau)$ is a new function.

Let us assume that $B = 1$, $A = C = 0$, then $F(t) = t^2$,

$$\tau = \frac{t^5}{5}, \quad \delta = xt^2 - \int t^2 h_1(t)dt,$$

$$V_t = -\frac{1}{t^2}M(\delta,\tau) + \frac{1}{t}\Big[M_\delta \frac{\partial \delta}{\partial t} + M_\tau \frac{\partial \tau}{\partial t}\Big] = -\frac{M(\delta,\tau)}{t^2} + M_\delta(2x - th_1(t)) + M_\tau t^3,$$

$$V_x = \frac{1}{t}\Big[M_\delta \frac{\partial \delta}{\partial x} + M_\tau \frac{\partial \tau}{\partial x}\Big] = M_\delta t; \qquad V_{xx} = t^3 M_{\delta\delta}.$$

Substitution of

$$-\frac{M(\delta,\tau)}{t^2} + M_\delta(2x - th_1(t)) + M_\tau t^3 = \nu t^3 M_{\delta\delta} + \Big(\frac{2x}{t} + h_1(t)\Big)M_\delta t - \frac{M(\delta,\tau)}{t^2},$$

in (14) gives us the following equation

$$M_\tau = \nu M_{\delta\delta}. \tag{15}$$

The heat equation with constant coefficients is obtained.

Let us note that if variable δ is taken in the form $\delta = xt^2 - \int_1^t t^2 h_1(t)dt$ then it can be treated as a Lagrangian variable since $\delta = x$ at $t = 1$.

Let us consider the simplest solution for (15) [5]: $M(\delta) = \alpha\delta + \beta$. Then

$$V(x,t) = \frac{\alpha}{t}\Big(xt^2 - \int t^2 h_1(t)dt\Big) + \frac{\beta}{t},$$

where $\alpha, \beta = const$.

From the third equation of factor-system (12) we obtain

$$W_t = \nu W_{xx} + \left(\frac{2x}{t} - h_1(t)\right) W_x - \frac{W}{t}. \tag{16}$$

The solution of equation (16) is found by analogy with equation (14), that is,

$$W(x,t) = \frac{\varepsilon}{t}\left(xt^2 - \int t^2 h_1(t)dt\right) + \frac{\mu}{t},$$

where $\varepsilon, \mu = const$.

As a result, the exact solution of equations (4) is obtained in the form

$$\begin{aligned} u(x,t) &= h_1(t) - \frac{2x}{t}, \\ v(x,y,t) &= \frac{1}{t}\Big[y + \alpha\big(xt^2 - \int t^2 h_1(t)dt\big) + \beta\Big], \\ w(x,z,t) &= \frac{1}{t}\Big[z + \varepsilon\big(xt^2 - \int t^2 h_1(t)dt\big) + \mu\Big], \\ q(x,t) &= \frac{2xh_1(t)}{t} - xh_1'(t) - \frac{3x^2}{t^2} + h_2(t), \end{aligned} \tag{17}$$

where $h_1(t), h_2(t)$ are arbitrary functions; $\alpha, \beta, \varepsilon, \mu$ are constants.

Kinematic condition (5) on the free boundary $\Gamma : z = \eta(x,y,t)$ for a given solution has the form

$$\frac{\partial \eta}{\partial t} + \left(h_1(t) - \frac{2x}{t}\right)\frac{\partial \eta}{\partial x} + \left(\frac{1}{t}\Big[y + \alpha(xt^2 - \int t^2 h_1(t)dt) + \beta\Big]\right)\frac{\partial \eta}{\partial y} = $$
$$= \frac{1}{t}\left[z + \varepsilon\left(xt^2 - \int t^2 h_1(t)dt\right) + \mu\right].$$

One can see that $\eta(x,y,t) = t\Phi(J_1,J_2) - \varepsilon J_1 - \mu$, where $\Phi(J_1,J_2)$ is an arbitrary function of arguments

$$J_1 = xt^2 - \int t^2 h_1(t)dt, \quad J_2 = \frac{1}{t}(y + \alpha J_1 + \beta).$$

From dynamic condition (6) $(p_a - p\big|_\Gamma)\overrightarrow{n} + 2\nu D\cdot\overrightarrow{n} = 2\sigma H\overrightarrow{n}$ one can determine the atmospheric pressure at the free boundary

$$p_a = p\big|_\Gamma + \frac{\nu}{t}\left[1 + \sqrt{9 + t^4(\alpha^2 + \varepsilon^2)}\right] + 2\sigma H.$$

Taking into account solution (17), the deformation rate tensor and the pressure on the fluid surface are

$$D = \frac{1}{2t}\begin{pmatrix} -4 & \alpha t^2 & \varepsilon t^2 \\ \alpha t^2 & 2 & 0 \\ \varepsilon t^2 & 0 & 2 \end{pmatrix},$$

$$p\big|_\Gamma = -g\eta(x,y,t) + q(x,y,t) = -g\big[t\Phi(J_1,J_2) - \varepsilon J_1 - \mu\big] + \frac{2xh_1(t)}{t} - xh_1'(t) - \frac{3x^2}{t^2} + h_2(t).$$

Example 2. Let us consider equation (14):

$$M_\tau = \nu M_{\delta\delta},$$

and use solution for this equation [8]

$$M(\delta,\tau) = \alpha(\delta^2 + 2\nu\tau) + \beta.$$

Then we obtain

$$V(x,t) = \frac{\alpha(\delta^2 + 2\nu\tau) + \beta}{t} = \frac{\alpha}{t}\left(\left(xt^2 - \int t^2 h_1(t)dt\right)^2 + \frac{2\nu t^5}{5}\right) + \frac{\beta}{t}. \tag{18}$$

In a similar way we can find $W(x,t)$:

$$W(x,t) = \frac{\varepsilon}{t}\left(\left(xt^2 - \int t^2 h_1(t)dt\right)^2 + \frac{2\nu t^5}{5}\right) + \frac{\mu}{t}.$$

As a result, one more exact solution of equations (4) is obtained

$$\begin{aligned} u(x,t) &= h_1(t) - \frac{2x}{t}, \\ v(x,y,t) &= \frac{1}{t}\left[y + \alpha\left(\left(xt^2 - \int t^2 h_1(t)dt\right)^2 + \frac{2\nu t^5}{5}\right) + \beta\right], \\ w(x,z,t) &= \frac{1}{t}\left[z + \varepsilon\left(\left(xt^2 - \int t^2 h_1(t)dt\right)^2 + \frac{2\nu t^5}{5}\right) + \mu\right], \\ q(x,t) &= \frac{2xh_1(t)}{t} - xh_1^{'}(t) - \frac{3x^2}{t^2} + h_2(t), \end{aligned} \tag{19}$$

where $h_1(t), h_2(t)$ are arbitrary functions; $\alpha, \beta, \varepsilon, \mu$ are constants. Thus, the set of solutions of equations (4) can be constructed from the set of solutions of equation (14).

Example 3. Let us consider operators

$$\left\langle X_2 = \frac{\partial}{\partial t}; \quad X_5 = \frac{\partial}{\partial z}; \quad X_6 = \frac{\partial}{\partial x}, a_1(t) = 1\right\rangle.$$

The invariants of these operators are $J = \{y; u; v; w; q\}$. Therefore, the invariant solution of equations should have the following form

$$(u, v, w, q) = \big(U(y); V(y); W(y); Q(y)\big). \tag{20}$$

The factor-system for the stationary solution has the form

$$VU_y = \nu U_{yy}, \quad VV_y + Q_y = \nu V_{yy}, \quad VW_y = \nu W_{yy}, \quad V_y = 0. \tag{21}$$

One can see from (21) that

$$V = C_1, \quad Q = C_2, \quad U(y) = D_1 + D_2 e^{\frac{C_1}{\nu}y}, \quad W(y) = H_1 + H_2 e^{\frac{C_1}{\nu}y}.$$

Finally, we obtain the exact solution of equations (14)

$$u(y) = D_1 + D_2 e^{\frac{C_1}{\nu}y}, \quad v(y) = C_1, \quad w(y) = H_1 + H_2 e^{\frac{C_1}{\nu}y}, \quad q(y) = C_2, \tag{22}$$

where $C_1, C_2, D_1, D_2, H_1, H_2$ are constants.

Kinematic condition (5) on the free boundary $z = \eta(x,y,t)$ for solution (22) has the form

$$\frac{\partial \eta}{\partial t} + \left(D_1 + D_2 e^{\frac{C_1}{\nu}y}\right)\frac{\partial \eta}{\partial x} + C_1 \frac{\partial \eta}{\partial y} = H_1 + H_2 e^{\frac{C_1}{\nu}y}.$$

The solution to this equation is

$$\eta(x,y,t) = \frac{1}{C_1}\left(H_1 y + H_2 \frac{\nu}{C_1} e^{\frac{C_1}{\nu}y}\right) + \Phi(J_1, J_2),$$

where $\Phi(J_1, J_2)$ is an arbitrary function of arguments

$$J_1 = y - C_1 t, \quad J_2 = D_1 y + D_2 \frac{\nu}{C_1} e^{\frac{C_1}{\nu} y} - C_1 x.$$

In the dynamic condition (6) on the free boundary $z = \eta(x, y, t)$ the deformation rate tensor has zero values on the diagonal for solution (22). Therefore, it is easy to determine that the external atmospheric pressure at the free boundary is

$$p_a = -g\eta(x, y, t) + C_2 + \frac{C_1}{2\nu} e^{\frac{C_1}{\nu} y} \sqrt{D_2^2 + H_2^2} + 2\sigma H.$$

Example 4. Let us give an example when the factor-system gives a contradiction. Let us consider operators

$$\Big\langle X_4 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial w};\ X_6 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\ a_1(t) = t\, X_7 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v},\ a_2(t) = t\Big\rangle.$$

The invariants of these operators are $J = \Big\{t;\ u - \dfrac{x}{t};\ v - \dfrac{y}{t};\ w - \dfrac{z}{t};\ q\Big\}$. Therefore, an invariant solution of equations should have the following form

$$(u, v, w, q) = \Big(\frac{x}{t} + U(t);\ \frac{y}{t} + V(t);\ \frac{z}{t} + W(t);\ Q(t)\Big), \tag{23}$$

where U, V, W, Q are functions of one variable.

Then the factor-system has the form

$$\begin{aligned} U_t - \frac{x}{t^2} + \Big(\frac{x}{t} + U3\Big)\frac{1}{t} = 0, \quad V_t - \frac{y}{t^2} + \Big(\frac{y}{t} + V\Big)\frac{1}{t} = 0, \\ W_t - \frac{z}{t^2} + \Big(\frac{z}{t} + W\Big)\frac{1}{t} = 0, \quad \frac{1}{t} + \frac{1}{t} + \frac{1}{t} = 0. \end{aligned} \tag{24}$$

The last relation in (24) is contradictory. Therefore, a solution of form (23) does not exist.

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Групповой анализ уравнений гидростатической модели вязкой жидкости

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Рассматриваются групповые свойства уравнений трехмерной гидростатической модели вязкой жидкости. Представлено несколько примеров точных решений. Определяются свободная поверхность жидкости и давление на ней.

Ключевые слова: групповой анализ, гидростатическая модель, вязкая жидкость, точные решения.