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Carleman Formula for the Maxwell Equation in a Cap Type Domain

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We consider a Cauchy problem for the Maxwell equations in a cap type domain $\mathcal{X}$ in $\mathbb{R}^3$. We find reasonable solvability conditions and a Carleman formula for its solution.

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Introduction

The moral of the Cauchy problem for elliptic equations is that, after J. Hadamard, a Cauchy problem for a Laplace equation is ill-posed. Of course, the instability refers here to the standard setting, for in spaces with the two-norm convergence the Cauchy problem for elliptic equations proves to be stable, see [1].

The classical equations of electrodynamics are usually formulated in terms of the de Rham complex in $\mathbb{R}^3$. In [4], we formulated Maxwell's equations within the framework of arbitrary elliptic complexes on a compact manifold $\mathcal{X}$ with boundary. Using the methods of [9] we studied both the Hilbert problem and the Cauchy problem with data on a part of the boundary $\partial\mathcal{X}$ for solutions of the corresponding stationary equations. The character of instability, solvability criteria and regularisation methods of the Cauchy problem for elliptic equations are studied in [9, 5]. For the complete bibliography we refer the reader to these books. Much of the theory developed in [9] extends immediately to other ill-posed problems of complex analysis or partial differential equations. N. Tarkhanov and A. Shlapunov has published a continuation formula for a larger class of boundary value problems not only for elliptic systems but also for elliptic complex.

Examples of bases with double orthogonality can be found in [6]. When working with domains such as a spherical layer, the main technique is to decompose the elements of a suitable space into a series of homogeneous harmonic functions that form a basis on the sphere. The main difficulty in the implementation of the proposed idea in practice is the choice of a basis with the property of double orthogonality in a given domain.

In this paper we will describe a simpler Carleman formula for the Maxwell equations, i.e. we consider the problem of analytic continuation of a solution of Maxwell equations in a spatial bounded domain from data on part of the boundary of the domain.

An explicit Carleman function for this problem was first described in [10]. Based on the ideas and methods developed by Sh.Yarmukhamedov, we will construct an example of the Carleman function of the Cauchy problem for the Maxwell equations In the limit, it and its derivative disappear outside an arbitrary fixed cone. It decreases rapidly enough at infinity.

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1. Maxwell's equations

The Maxwell equations on an n-dimensional compact manifold $\mathcal{X}$ with boundary have the form

$$\begin{array}{rcl} \varepsilon\, E'_t & = & -\,\sigma E + d^* H, \\ \mu\, H'_t & = & -\,dE, \end{array}$$

with E and H being unknown functions of $t = [0,T]$ with values in differential forms of degree i and $i+1$, respectively, and ε, μ and σ positive constants. Substituting

$$\begin{array}{rcl} E(x,t) & = & (\varepsilon + \imath\,\sigma/\omega)^{-1/2} e^{-\imath\omega t} u(x,t), \\ H(x,t) & = & \mu^{-1/2} e^{-\imath\omega t} f(x,t) \end{array} \tag{1}$$

we get

$$\begin{array}{rcl} (\varepsilon\mu\omega/k)\, u'_t & = & \imath k\, u + d^* f, \\ -\,(k/\omega)\, f'_t & = & du - \imath k\, f \end{array} \tag{2}$$

in $\mathcal{X} \times [0,1]$, where ω is a nonzero real constant and $k^2 = (\varepsilon + \imath\,\sigma/\omega)\mu\omega^2$. The sign of k is chosen from the condition $\Im k \geqslant 0$. Equations (2) still make sense within the framework of arbitrary complexes of differential operators on $\mathcal{X}$, see [4]. The operator on the right-hand side of (2) proves to be elliptic in the sense of Douglis–Nirenberg (see Example below). The Cauchy data on $\partial\mathcal{X}$ of forms u and f of Sobolev class $H^1(\mathcal{X})$ with respect to the stationary Maxwell operator consist of $t(u)$, the tangential part of u, and $n(f)$, the normal part of f. Both $t(u)$ and $n(f)$ are differential forms of degree i on the boundary. Further we discuss the first mixed problem for solutions of Maxwell's equations (2) in the cylinder $\mathcal{X} \times [0,T]$. It stems from scattering of incident electromagnetic waves by a perfectly conducting body. In this case the tangential part $t(u)$ of the "electric" field u must vanish on the body surface $\partial\mathcal{X}$. Hence, we pose the initial conditions on the lower basis of the cylinder and a Dirichlet condition on the lateral surface. Let $\mathcal{X}$ be a compact differentiable manifold of dimension 3 with or without boundary. More explicitly, we use the de Rham complex in $\mathbb{R}^3$

$$0 \longrightarrow \Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3) \longrightarrow 0,$$

where $\Omega^0(\mathbb{R}^3)$ is the space of smooth functions, $\Omega^1(\mathbb{R}^3)$ is the space of 1-forms, and so forth. Forms which are the image of other forms under the exterior derivative, plus the constant 0 function in $\Omega^0(\mathbb{R}^3)$ are called exact and forms whose exterior derivative is 0 are called closed (see closed and exact differential forms); the relationship $d \circ d = 0$ then says that exact forms are closed. Then we can think of E as a differential form u of degree 1, H as a differential form f of degree 2, thus identifying $\operatorname{curl} E$ with du and $\operatorname{curl} H$ with $d^* f$. Here, d^* stands for the formal adjoint operator of d. Let us denote by $\Delta = d^*d + dd^*$ is the Laplacian of the de Rham complex. In this way Maxwell's equations in stationary form can be written as

$$\begin{array}{rcl} \imath k\, u + d^* f & = & 0, \\ -\imath k\, f + du & = & 0, \end{array} \tag{3}$$

which already make sense not only for differential forms u and f of degree 1 and 2 in $\mathbb{R}^3$, respectively, but also for differential forms u and f of degree i and $i+1$ in $\mathbb{R}^n$, where $-1 \leqslant i \leqslant n$.

Definition 1. *Let $-1 \leqslant i \leqslant 3$. By the Maxwell operator for the de Rham complex at step i is meant*

$$M^i = \begin{pmatrix} \imath k & d^{i*} \\ d^i & -\imath k \end{pmatrix}.$$

A characteristic and important example of a system elliptic in the sense of Douglis–Nirenberg and not elliptic in the sense of Petrovskii is the Maxwell equations (1.2) in the domain $\mathcal{X} \subset \mathbb{R}^3$. In the case of the de Rham complex in $\mathbb{R}^3$ at step $i = 0$, Maxwell's equations (1.2) have the form:

$$\begin{array}{c} \imath ku - divf = 0 \\ -\imath kf + gradu = 0 \end{array} \quad \text{or} \quad \begin{array}{c} -\imath ku + divf = 0 \\ -\imath kf + gradu = 0 \end{array}. \tag{4}$$

The system (1.3) with constant coefficients and its operator is:

$$M^0 = \begin{pmatrix} -\imath k & \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & -\imath k & 0 & 0 \\ \dfrac{\partial}{\partial x_2} & 0 & -\imath k & 0 \\ \dfrac{\partial}{\partial x_3} & 0 & 0 & -\imath k \end{pmatrix}.$$

We try to find the vectors s and t with integer components that simultaneously satisfy the conditions of definitions.

For the system (1.3) we choose the weights , for example, the form

$$s = (2, 1, 1, 1) \quad \text{and} \quad t = (0, -1, -1, -1).$$

The corresponding main part will have the form

$$\widetilde{M}^0_{s,t}(D) = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & -\imath k & 0 & 0 \\ \dfrac{\partial}{\partial x_2} & 0 & -\imath k & 0 \\ \dfrac{\partial}{\partial x_3} & 0 & 0 & -\imath k \end{pmatrix}$$

and its determinant

$$\det \widetilde{M}^0_{s,t}(D) = k^2(\xi_1^2 + \xi_2^2 + \xi_3^2) \neq 0, \quad \forall \xi \in \mathbb{R}^3 \quad \text{and} \quad k \neq 0.$$

Using definition, we come to the conclusion that the system (1.3) is elliptic in the sense of Douglis–Nirenberg.

By definition, M^i is a first order differential operator from sections of $F^i \oplus F^{i+1}$ to sections of the same bundle over $\mathcal{X}$. This operator fails to be elliptic of order 1 in the classical sense unless $N = 2$. On the other hand, applying d^* to both sides of $\imath k\, u + d^* f = 0$ we conclude that $d^{i-1*}u = 0$ unless $k = 0$. Analogously, from $-\imath k\, f + du = 0$ it follows that $d^{i+1} f = 0$ unless $k = 0$. Complementing Maxwell's equations by their differential consequences $d^{i-1*}u = 0$ and $d^{i+1} f = 0$ yields a system of first order differential equations for u and f, whose classical symbol is injective. Another way of stating this is to say that there is a differential operator C^i from sections of $F^i \oplus F^{i+1}$ to sections of the same bundle, such that $C^i M^i$ is a second order differential operator on $\mathcal{X}$ elliptic in the classical case. As is usual in homological algebra, we will omit the index i of M^i i.e. $M^i = M$ when it is clear from the context. An easy computation shows that

$$C = \begin{pmatrix} \imath k + (1/\imath k) dd^* & d^* \\ d & -\imath k - (1/\imath k) d^* d \end{pmatrix}. \tag{5}$$

Lemma 1. *As defined above, C satisfies*

$$CM = MC = \begin{pmatrix} \Delta - k^2 & 0 \\ 0 & \Delta - k^2 \end{pmatrix}.$$

Proof.

$$CM = MC = \begin{pmatrix} -k^2 + dd^* + d^*d & \imath kd^* + (1/\imath k)d^*d^* - \imath kd^* \\ \imath kd - \imath kd - (1/\imath k)d^*dd & dd^* - k^2 + d^*d \end{pmatrix} =$$

$$= \begin{pmatrix} \Delta - k^2 & 0 \\ 0 & \Delta - k^2 \end{pmatrix}.$$

□

Lemma 2. *The pseudodifferential operator*

$$\Phi = \begin{pmatrix} G\,(\imath k + (1/\imath k)dd^*) & G\,d^* \\ G\,d & G\,(-\imath k - (1/\imath k)d^*d) \end{pmatrix}$$

is a left fundamental solution of the Maxwell operator M on $\mathcal{X}'$.

Proof. From Lemma 1 it follows immediately that

$$\Phi = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix} \circ C$$

is a left fundamental solution of M. It remains to substitute the explicit expression (5) for C. □

2. The Carleman formula

We now turn to the classical Maxwell equation in a three-dimensional space, which can be a three-dimensional manifold $\mathcal{X}'$ as well. To demonstrate our constructions along more classical lines, we consider the case $\mathcal{X}' = \mathbb{R}^3$. As mentioned in Section 1, the classical Maxwell equations have the form

$$\begin{array}{rcl} \imath k\,E + d^*H & = & 0, \\ -\imath k\,H + dE & = & 0, \end{array}$$

E and H being functions in a closed domain $\mathcal{X} \subset \mathbb{R}^3$ with values in $\mathbb{R}^3$. If E is suitably specified within 1-forms and H within 2-forms, both the exterior derivative d and its formal adjoint d^* can be identified with the operator curl on vector fields in $\mathbb{R}^3$.

Applying Corollary 3.3 [4] to the classical Maxwell equations we obtain the Stratton–Chu formula [7].

Theorem 1. *Suppose (E, H) is an electromagnetic wave in $\mathcal{X}$ whose electric component E and magnetic component H are both continuous up to the boundary. Then*

$$\begin{pmatrix} (1/\imath k)d^*d & -d^* \\ -d & -(1/\imath k)dd^* \end{pmatrix} \int_{\partial\mathcal{X}} \frac{-1}{4\pi} \frac{\exp(\imath k|x-y|)}{|x-y|} \begin{pmatrix} \imath n(H) \\ -\nu \wedge t(E) \end{pmatrix} ds = \begin{pmatrix} E(x) \\ H(x) \end{pmatrix}$$

for all $x \in \mathcal{X} \setminus \partial\mathcal{X}$, and the left-hand side vanishes away from $\mathcal{X}$.

Let σ be a positive number. Consider the entire function $K(w) = \exp(\sigma w^2)$ of complex variable $w \in \mathbb{C}$. The restriction of K to any vertical line $w = u + \imath v$ just amounts to $K(u + \imath v) = K(u)\exp(2\imath\sigma uv - \sigma v^2)$, which is a rapidly decreasing function of v.

Assume that $\mathcal{X}$ is a bounded domain in the upper half-space $\{x_3 > 0\}$ of $\mathbb{R}^3$ whose boundary consists of a smooth surface S lying in the half-space $\{x_3 > 0\}$, and a closed piece of the plane

$\{x_3 = 0\}$. Such domains are usually referred to as cap type domains. Note that the unit outward normal vector on the piece $\partial\mathcal{X} \setminus S$ just amounts to $(0, 0, -1)$. We consider the problem of finding an electric field E and a magnetic field H in $\mathcal{X}$ with given tangential component E_0 of E and normal component H_0 of H on $\mathcal{S}$. Given two different points $x = (x', x_3)$ and $y = (y', y_3)$ in $\mathbb{R}^3$, set $r' = |y' - x'|$ and introduce the integral

$$\Phi_\sigma(x, y) = \frac{-1}{2\pi^2} \frac{1}{K(x_3)} \int_0^\infty \Im\Big(\frac{K(w)}{w - x_3}\Big) \frac{\cos k\vartheta}{\sqrt{r'^2 + \vartheta^2}}\, d\vartheta,$$

where $w = y_3 + \imath\sqrt{r'^2 + \vartheta^2}$. An easy calculation shows that

$$\Phi_\sigma(x, y) = \int_0^\infty p_\sigma(x, y; \vartheta) \cos k\vartheta \, d\vartheta, \tag{6}$$

where $p_\sigma(x, y; \vartheta)$ is given by

$$\frac{-1}{2\pi^2} \frac{e^{\sigma(y_3^2 - x_3^2)} e^{-\sigma(r'^2 + \vartheta^2)}}{\vartheta^2 + r^2} \Big((y_3 - x_3) \frac{\sin 2\sigma y_3 \sqrt{r'^2 + \vartheta^2}}{\sqrt{r'^2 + \vartheta^2}} - \cos 2\sigma y_3 \sqrt{r'^2 + \vartheta^2} \Big).$$

Hence it follows that

$$\begin{aligned} p_\sigma(x, y; \vartheta) &= \frac{1}{2\pi^2} \frac{e^{-\sigma x_3^2} e^{-\sigma(r'^2 + \vartheta^2)}}{\vartheta^2 + r^2}, \\ \partial_{y_3} p_\sigma(x, y; \vartheta) &= \frac{1}{\pi^2} \frac{e^{-\sigma x_3^2} e^{-\sigma(r'^2 + \vartheta^2)}}{(\vartheta^2 + r^2)^2} x_3 \left(1 + \sigma(\vartheta^2 + r^2)\right) \end{aligned}$$

on the plane $y_3 = 0$. If $\sigma = 0$ and $K(0) = 1$ then kernels $\Phi_\sigma(x, y)$ is a classical fundamental solution of the Helmholtz equation. Substituting (2.5) into the formula for the fundamental solution $\Phi(x - y)$ of M we conclude that $\Psi_\sigma(x, y)$ is a Carleman function of the Cauchy problem in the domain $\mathcal{X}$ with data on S, parametrised by σ.

$$\Psi_\sigma(x, y) = \begin{pmatrix} (1/\imath k) d^* d & -d^* \\ -d & -(1/\imath k) d d^* \end{pmatrix} \Phi_\sigma(x, y).$$

Lemma 3. *The $\Psi_\sigma(x, y)$ matrix-valued function on the set $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{y = x\})$, satisfying*

$$\begin{aligned} M(\partial_x)\, \Psi_\sigma(x, y) &= 0, \\ M'(\partial_y)(\Psi_\sigma(x, y))^T &= 0. \end{aligned}$$

Proof. The proof follows from the properties of the fundamental solution $\Phi(x - y)$ of the Maxwell operator M. □

Theorem 2. *Let (E, H) be an electromagnetic wave in $\mathcal{X}$ continuous up to $\overline{\mathcal{S}}$. Then the limit relation*

$$\begin{pmatrix} E(x) \\ H(x) \end{pmatrix} = \lim_{\sigma \to \infty} \int_{\mathcal{S}} \Psi_\sigma(x, y) \begin{pmatrix} \imath n(H) \\ -\nu \wedge t(E) \end{pmatrix} ds$$

holds uniformly on each compact subset of $\mathcal{X}$.

Proof. As is known, the value of the regular solution of Maxwell equations at a point x inside $\mathcal{X}$ in terms of $t(E)$ and $n(H)$ on the boundary $\mathcal{X}$. Using the Stratton–Chu formula of Theorem 2.1

$$\begin{pmatrix} E(x) \\ H(x) \end{pmatrix} = \int_{\mathcal{S}} \Psi_\sigma(x, y) \begin{pmatrix} \imath n(H) \\ -\nu \wedge t(E) \end{pmatrix} ds + \int_{\partial\mathcal{X} \setminus S} \Psi_\sigma(x, y) \begin{pmatrix} \imath n(H) \\ -\nu \wedge t(E) \end{pmatrix} ds\,.$$

The convergence of the improper integral on the right-hand side of $\Phi_\sigma(x, y)$ is thus guaranteed by the factor $e^{-\sigma\vartheta^2}$. If $\sigma \to \infty$ the expression $\Psi_\sigma(x, y)$, $p_\sigma(x, y; \vartheta)$ and its partial derivative $\partial_{y_3} p_\sigma(x, y; \vartheta)$ tends to zero exponentially on the plane $\{x_3 > 0\}$. It follows that the part of the boundary integral over $\partial\mathcal{X} \setminus S$ tends to zero on $\{x_3 = 0\}$. This establishes the desired formula. □

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Формула Карлемана для уравнения Максвелла в области типа шапки

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Рассмотрим задачу Коши для уравнений Максвелла в области типа шапки $\mathcal{X}$ в $\mathbb{R}^3$. Мы указываем разумное условие разрешимости и формулу Карлемана для ее решения.

Ключевые слова: формула Карлемана, рассеяние, эллиптический комплекс, формулы Грина, формулы Стрэттона-Чу, задача Коши.