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Vector Bundle of Prym Differentials over Teichmüller Spaces of Surfaces with Punctures

Alexander V. Chueshev
Victor V. Chueshev*
Institute of Fundamental Sciences
Kemerovo State University
Krasnaya, 6, Kemerovo, 650043
Russia

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In this paper we study multiplicative meromorphic functions and differentials on Riemann surfaces of finite type. We prove an analog of P. Appell's formula on decomposition of multiplicative functions with poles of arbitrary multiplicity into a sum of elementary Prym integrals. We construct explicit bases for some important quotient spaces and prove a theorem on a fiber isomorphism of vector bundles and $n!$-sheeted mappings over Teichmüller spaces. This theorem gives an important relation between spaces of Prym differentials (Abelian differentials) on a compact Riemann surfaces and on a Riemann surfaces of finite type.

Keywords: Teichmüller spaces for Riemann surfaces of finite type, Prym differentials, vector bundles, group of characters, Jacobi manifolds.
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Introduction

In the paper we study multiplicative meromorphic functions and differentials on Riemann surfaces of type (g, n). Recently the interest in this subject has increased in relation with applications in theoretical physics, in particular, in description of vortex-like patterns in ferromagnetics [1].

Theory of functions on compact Riemann surfaces differs significantly from that on Riemann surfaces of finite type even for the class of single-valued meromorphic functions and Abelian differentials, since some of basic spaces of functions and differentials on Riemann surfaces F' of type (g, n), $g \geqslant 1$, $n > 0$ are infinite-dimensional.

In this paper we continue constructing the general theory of functions on Riemann surfaces of type (g, n) for multiplicative meromorphic function and differentials. We prove an analog of P. Appell's formula about the expansion of a multiplicative function with poles of arbitrary multiplicity into a sum of elementary Prym integrals. Also we construct explicit bases for some important quotient spaces and prove a theorem about fiber isomorphism of vector bundles and $n!$-sheeted mappings over Teichmüller spaces. This theorem gives an important relation between spaces of Prym differentials (abelian differentials) on compact Riemann surfaces and Riemann surfaces of finite type.

*vvchueshev@ngs.ru

1. Preliminaries

Let F be a smooth compact oriented surface of genus $g \geqslant 2$, with the marking $\{a_k, b_k\}_{k=1}^g$, i.e. an ordered collection of standard generators of $\pi_1(F)$, and F_0 be a compact Riemann surface with the fixed complex-analytic structure on F. Fix different points $P_1, \ldots, P_n \in F$. We assign type (g, n) to a surface $F' = F \backslash \{P_1, \ldots, P_n\}$. By Γ' we denote the Fuchsian group of genus 1 acting invariantly in the disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and uniformizing the surface F'_0. Thus, $F'_0 = U/\Gamma'$, where Γ' has the representation [2, 3]

$$\Gamma' = \langle A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_n : \prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} C_1 \ldots C_n = I \rangle.$$

Any other complex analytic structure on F' is given by a Beltrami differential μ on F'_0, i. e. by an expression of the form $\mu(z) d\overline{z}/dz$, invariant with respect to the choice of the local parameter on F'_0, where $\mu(z)$ is a complex-valued function on F'_0 and $\|\mu\|_{L_\infty(F'_0)} < 1$. We denote this structure on F' by F'_μ.

Let $M(F')$ be the set of all complex analytic structures on F' with the topology of C^∞ convergence on F'_0, $Diff^+(F')$ be the group of all orientation preserving smooth diffeomorphisms of F' onto itself, which leave all punctures fixed, and $Diff_0(F')$ be the normal subgroup of $Diff^+(F')$ of diffeomorphisms homotopic to the identity diffeomorphism on F'_0. The group $Diff^+(F')$ acts on $M(F')$ by $\mu \to f^*\mu$, where $f \in Diff^+(F'), \mu \in M(F')$. Then the Teichmüller space $\mathbb{T}_{g,n}(F'_0)$ is the quotient space $M(F')/Diff_0(F')$ [2].

Since the mapping $U \to F'_0 = U/\Gamma'$ is a local diffeomorphism, any Beltrami differential μ on F'_0 lifts to a Beltrami Γ'-differential μ on U, i. e. $\mu \in L_\infty(U), \|\mu\|_\infty = esssup_{z \in U} |\mu(z)| < 1$, and $\mu(T(z))\overline{T'(z)}/T'(z) = \mu(z)$, $z \in U$, $T \in \Gamma'$, see [2].

If the Γ'-differential μ on U is continued on $\overline{\mathbb{C}} \backslash U$, setting $\mu = 0$, then there is a single quasiconformal homeomorphism $w^\mu : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with fixed points $+1, -1, i$, which is a solution of the Beltrami equation $w_{\overline{z}} = \mu(z) w_z$. The map $T \to T_\mu = w^\mu T (w^\mu)^{-1}$ defines an isomorphism of the group Γ' onto the quasi-Fuchsian group $\Gamma'_\mu = w^\mu \Gamma' (w^\mu)^{-1}$.

In the work [2, p. 99] there were constructed abelian differentials $\zeta_1[\mu], \ldots, \zeta_g[\mu]$ on $F_{[\mu]}$, that form a canonical base dual to a canonical homotopy base $\{a_k^\mu, b_k^\mu\}_{k=1}^g$ on F_μ, which depends holomorphically on moduli $[\mu]$ for a class of conformal equivalency of a marked Riemann surface F_μ. Further on, for brevity we shall write simply F_μ for the class of equivalence $F_{[\mu]}$. Here we assume that the class $[\mu]$ has Bers coordinates $h_1, h_2, \ldots, h_{3g-3}$ when embedding the Teichmüller space $\mathbb{T}_g(F_0)$ of compact Riemann surfaces into $\mathbb{C}^{3g-3}$. Moreover, the matrix of b-periods $\Omega(\mu) = (\pi_{jk}[\mu])_{j,k=1}^g$ on F_μ consists of complex numbers $\pi_{jk}[\mu] = \int\limits_{\xi}^{B_k^\mu(\xi)} \zeta_j([\mu], w) dw, \xi \in w^\mu(U)$, and depends holomorphically on $[\mu]$.

For any fixed $[\mu] \in \mathbb{T}_g$ and $\xi_0 \in w^\mu(U)$ define the classical Jacobi mapping $\varphi : w^\mu(U) \to \mathbb{C}^g$ by the rule: $\varphi_j(\xi) = \int\limits_{\xi_0}^{\xi} \zeta_j([\mu], w) dw$, $j = 1, \ldots, g$. The quotient space $J(F) = \mathbb{C}^g / L(F)$ is called the marked Jacobi manifold for $F = F_0$, where $L(F)$ is a lattice over $\mathbb{Z}$, generated by the columns $e^{(1)}, \ldots, e^{(g)}, \pi^{(1)}, \ldots, \pi^{(g)}$ of the matrix (I_g, Ω), where I_g is an identity matrix of order g. The universal Jacobi manifold of order g is a fibered space over $\mathbb{T}_g$, with a fiber over $[\mu] \in \mathbb{T}_g$ being a marked Jacobi manifold $J(F_\mu)$ for a marked Riemann surface F_μ [4].

A *character* ρ for F'_μ is any homomorphic $\rho : (\pi_1(F'_\mu), \cdot) \to (\mathbb{C}^*, \cdot), \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Further on we shall assume that $\rho(\gamma_j^\mu) = 1$, where γ_j^μ is a simple loop around only one puncture P_j on F'_μ,

$j = 1, \dots, n$.

Definition 1. *A multiplicative function f on F'_μ for the character ρ is a mermorphic function f on $w^\mu(U)$ such that $f(Tz) = \rho(T)f(z)$, $z \in w^\mu(U)$, $T \in \Gamma'_\mu$.*

Definition 2. *A Prym q-differential with respect to a Fuchsian group Γ' for ρ, or a (ρ, q)-differential, is a differential $\omega(z)dz^q$ such that $\omega(Tz)(T'z)^q = \rho(T)\omega(z)$, $z \in U$, $T \in \Gamma', \rho : \Gamma' \to \mathbb{C}^*$.*

If a multipicative function f_0 on F_μ for ρ does not have zeroes or poles, then the character ρ is called *non-essential* and f_0 is called a *unit*. The characters which are not non-essential are called *essential* on $\pi_1(F_\mu)$. The set L_g of non-essential characters form a subgroup in the group $Hom(\Gamma_\mu, \mathbb{C}^*)$ of all characters on Γ_μ. A *divisor* on F_μ is a formal product $D = P_1^{n_1} \dots P_k^{n_k}$, $P_j \in F_\mu$, $n_j \in \mathbb{Z}$, $j = 1, \dots, k$.

Theorem (Abel's theorem for characters, [3,5]). *Let D be a divisor on a marked variable compact Riemann surface $[F_\mu, \{a_1^\mu, \dots, a_g^\mu, b_1^\mu, \dots, b_g^\mu\}]$ of genus $g \geqslant 1$, and ρ be a character on $\pi_1(F_\mu)$. Then D is a divisor of a multiplicative function f on F_μ for ρ if and only if $degD = 0$ and*

$$\varphi(D) = \frac{1}{2\pi i} \sum_{j=1}^{g} \log \rho(b_j^\mu) e^{(j)} - \frac{1}{2\pi i} \sum_{j=1}^{g} \log \rho(a_j^\mu) \pi^{(j)}[\mu] \big(\equiv \psi(\rho, [\mu]) \big),$$

where $\varphi[\mu] : F_\mu \to J(F_\mu)$ is the Jacobi mapping.

The class $M_1(\rho)$ consists of those Prym differentials for ρ on F'_μ, which have finitely many poles on F'_μ and admit meromorphic continuation to F_μ.

In [6] it was proved that for any essential character ρ, a point $Q_1 \in F_\mu$, and natural $q \geqslant 1$ or a non-essential character ρ, a point $Q_1 \in F_\mu$, and natural $q > 1$ there exists an elementary (ρ, q)-differential $\tau_{\rho,q;Q_1}$ of the third kind with a unique simple pole $Q_1[\mu]$ on F_μ. For any non-essential character ρ, a point $Q_1 \in F_\mu$ if $q = 1$ there is no elementary $(\rho, 1)$-differential $\tau_{\rho;Q_1}$. Also it is proved there that on a variable surface F_μ of genus $g \geqslant 2$ for any natural $q \geqslant 1$ there exists an elementary (ρ, q)-differential $\tau_{\rho,q;Q_1 Q_2}$ of the third kind with simple poles $Q_1, Q_2 \in F_\mu$, and $\tau^{(m)}_{\rho,q;Q_1} = \left(\frac{1}{z^m} + O(1) \right) dz^q$, $z(Q_1) = 0$, of the second kind with the pole $Q_1[\mu]$ of order $m \geqslant 2$. These differentials depend locally holomorphically on $[\mu]$ and ρ.

Let $p : E \to B$ be a locally trivial holomorphic vector bundle of rank m, i.e. E, B are complex analytic manifolds, the base B is covered by a system of open simply-connected sets $\{U_\alpha\}$ such that there exists a system of holomorphic fiber coordinate homeomorphisms $\varphi_\alpha : U_\alpha \times \mathbb{C}^m \to p^{-1}(U_\alpha)$ for all α. On intersections $U_\alpha \cap U_\beta \neq \emptyset$ there are given $\varphi_{\beta\alpha} = \varphi_\beta^{-1} \varphi_\alpha : (U_\alpha \cap U_\beta) \times \mathbb{C}^m \to (U_\alpha \cap U_\beta) \times \mathbb{C}^m$, holomorphic matrix transition functions, which satisfy on $(U_\alpha \cap U_\beta \cap U_\gamma) \times \mathbb{C}^m$ the relations $\varphi_{\alpha\gamma} \varphi_{\gamma\beta} \varphi_{\beta\alpha} = id$, where id is the identity homeomorphism for all indices. The transition functions $\varphi_{\beta\alpha}(x, z) = (x, \widetilde{\varphi}_{\beta\alpha}(x) z)$ define holomorphic mappings $\widetilde{\varphi}_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$, where $x \in B$, $z \in \mathbb{C}^m$ и $\widetilde{\varphi}_{\alpha\gamma} \widetilde{\varphi}_{\gamma\beta} \widetilde{\varphi}_{\beta\alpha} = 1$. These conditions on $B, \mathbb{C}^m, \{U_\alpha\}$ and such $\varphi_{\beta\alpha}, \widetilde{\varphi}_{\beta\alpha}$ are sufficient to define a locally trivial holomorphic vector bundle E of rank m over B [7].

Any holomorphic section $s : B \to E$, i.e. $ps(x) = x, x \in B$, may locally be described as $\varphi_\alpha^{-1} s : U_\alpha \to U_\alpha \times \mathbb{C}^m$, which define holomorphic vector-valued functions $s_\alpha : U_\alpha \to \mathbb{C}^m$ by the formula $(\varphi_\alpha^{-1} s)(x) = (x, s_\alpha(x)), x \in U_\alpha$. On intersections $U_\alpha \cap U_\beta \neq \emptyset$ these functions satisfy the compatibility conditions $s_\beta(x) = \widetilde{\varphi}_{\beta\alpha}(s_\alpha(x))$.

Conversely, given a set of holomorphic vector-valued functions $s_\alpha : U_\alpha \to \mathbb{C}^m$ with the compatibility conditions satisified, then the formula $s(x) = \varphi_\alpha(x, s_\alpha(x))$ uniquely, i.e. independently of the choice of the covering $\{U_\alpha\}$, defines a holomorphic section $s : B \to E$.

If E is a locally trivial holomorphic vector bundle of rank m over B, then there exists a base of locally holomorphic sections for $\{U_\alpha\}$ given by $s_{k\alpha} = \varphi_\alpha(x, e_k)$, $k = 1, \dots, m$, $x \in U_\alpha$, where $e_1, \dots, e_m$ is the standard base in $\mathbb{C}^m$.

Conversely, given a base of locally holomorphic sections $s_{k\alpha}$, $k = 1, \dots, m$, $x \in U_\alpha$ of E, the coordinate homeomorphisms can be defined by $\varphi_\alpha(x, z) = \sum\limits_{j=1}^{m} z_j s_{j\alpha}$, where $z = \sum\limits_{j=1}^{m} z_j e_j$, which are holomorphic in $x \in B$ and $z \in \mathbb{C}^m$. Besides, from $(s_{1\alpha}(x), \dots, s_{m\alpha}(x))^t = \widetilde{\varphi}_{\alpha\beta}(x)(s_{1\beta}(x), \dots$ $\dots, s_{m\beta}(x))^t$ it follows that the transition functions $\widetilde{\varphi}_{\alpha\beta}(x)$ are holomorphic on intersections $U_\alpha \cap U_\beta \neq \emptyset$. In this manner (E, p, B) is endowed with the structure of a holomorphic vector bundle of rank m over B.

2. An analog of Appel's decomposition formula for a multiplicative function on a variable Riemann surface of finite type

Denote by $T^{(1)}_{\rho;Q} = -\int\limits_{Q_0}^{P} \tau^{(2)}_{\rho;Q}$ an elementary Prym integral of second kind on F_μ for an essential character ρ with only simple pole at Q and with residue $+1$ in Q that depends holomorphically on $[\mu]$ and ρ, where $\tau^{(2)}_{\rho;Q}$ has zero residue at Q [5,6,8].

Let f be a function on ${F_\mu}'$ of the class M_1 for an essential character ρ with s simple poles $P_{n+1}, P_{n+2}, \dots, P_{n+s}$ and residues $c_{n+1}, \dots, c_{n+s}$ at these poles respectively for some its branch. Consider an analytic continuation of this function f (denoting it by the same symbol) from ${F_\mu}'$ to F_μ. Consider the expression $f_1 = f - c_{n+1}T^{(1)}_{\rho;P_{n+1}} - \dots - c_{n+s}T^{(1)}_{\rho;P_{n+s}} - \sum\limits_{j=1}^{g-1} \widetilde{c}_j \int\limits_{Q_0}^{P} \tilde{\zeta}_j$, where $\widetilde{c}_j \in \mathbb{C}$, $j = 1, \dots, g-1$, and $\tilde{\zeta}_1, \dots, \tilde{\zeta}_{g-1}$ is the base of Prym differentials of the first kind for an essential character ρ on F_μ depending holomorphically on $[\mu]$ end ρ [2]. Then f_1 is a meromorphic single-valued branch of the Prym integral with an essential character ρ on the fundamental polygon $\triangle_\mu$, where the surface F_μ is uniformized [3], with the divisor $(f_1) \geqslant \dfrac{1}{P_1^{q_1} \dots P_n^{q_n}}$, $q_j \geqslant 0$, $j = 1, \dots, n$, on F_μ. Here we assume $P_1, P_2, \dots, P_{n+s} \in Int\triangle_\mu$. Besides, the Prym integral f_1 for ρ has a branch whose principal parts of Laurent series coincide with principal parts of Laurent series at P_j, $j = 1, \dots, n$, for f and zero a_m-periods, $m = 1, \dots, g-1$, on F_μ or on $\triangle_\mu$ [2]. Therefore $f = \sum\limits_{j=1}^{s} c_{n+j} T^{(1)}_{\rho;P_{n+j}} + \sum\limits_{j=1}^{g-1} \widetilde{c}_j \int\limits_{Q_0}^{P} \tilde{\zeta}_j + f_1$.

If P_l is a pole of order $q_l, q_l \geqslant 2$, then in the formula above one should instead of $c_l T^{(1)}_{\rho;P_l}$, $l = n+1, \dots, n+s$, (for simple poles), and also for poles P_l, $l = 1, \dots, n$, of the branch of f_1 write sums of the form

$$A_{l,1} T^{(1)}_{\rho;P_l} + A_{l,2} \frac{\partial T^{(1)}_{\rho;P_l}}{\partial P_l} + \frac{A_{l,3}}{2} \frac{\partial^2 T^{(1)}_{\rho;P_l}}{\partial P_l^2} + \dots + \frac{A_{l,q_l}}{(q_l - 1)!} \frac{\partial^{q_l - 1} T^{(1)}_{\rho;P_l}}{\partial P_l^{q_l - 1}},$$

where $A_{l,j}$ are coefficients of the principal part of the Laurent series for some branch of f at $P_l, j = 1, \dots, q_l(P_l)$, $l = n+1, \dots, n+s$, and for a branch of f_1 at $P_1, P_2, \dots, P_n$. Indeed, in a neighborhood of P_l we have expansions $T^{(1)}_{\rho;P_l} = \dfrac{1}{z - z(P_l)} + O(1)$; $(T^{(1)}_{\rho;P_l})'_{a_l} = \dfrac{1}{(z - a_l)^2} + O(1)$, $z(P_l) = a_l$; $\dots$; $(T^{(1)}_{\rho;\,P_l})^{(m)}_{a_l} = \dfrac{m!}{(z - a_l)^{m+1}} + O(1)$, $1 \leqslant m \leqslant q_l(P_l) - 1$, where $q_l(P_l)$ is the order of the pole at P_l for branches f and f_1, $l = 1, \dots, s+n$. From that follows the theorem.

Theorem 1. *Let f be a branch of a function of class M_1 for an essential character ρ on a variable Riemann surface $F_\mu^{'}$ of type (g, n), $g \geqslant 2$, $n > 0$, with pairwise distinct poles at $P_{n+1}, \dots, P_{n+s}$ of multiplicities $q_{n+1}, \dots, q_{n+s}$ with given principal parts:*

$$\frac{A_{j,q_j}}{(z - z(P_j))^{q_j}} + \dots + \frac{A_{j1}}{(z - z(P_j))}, \quad j = n+1, \dots, n+s. \tag{1}$$

Then for an analytic continuation of f we have $(f) \geqslant \frac{1}{P_1^{q_1} \dots P_{n+s}^{q_{n+s}}}$, $q_j \geqslant 0, j = 1, \dots, n,$ on F_μ and

$$f = \sum_{j=1}^{n+s} \sum_{m=1}^{q_j} \left[\frac{A_{j,m}}{(m-1)!} \frac{\partial^{m-1} T^{(1)}_{\rho;P_j}}{\partial P_j^{m-1}} \right] + \sum_{j=1}^{g-1} \widetilde{c}_j \int_{Q_0}^{P} \widetilde{\zeta}_j,$$

where $f = \dfrac{A_{j,q_j}}{(z - z(P_j))^{q_j}} + \dots + \dfrac{A_{j,2}}{(z - z(P_j))^2} + \dfrac{A_{j,1}}{z - z(P_j)} + O(1)$ for some branch in a neighborhood of P_j, $j = 1, \dots, n+s$, на F_μ, and all summands depend holomorphically on $[\mu]$ and ρ.

Let now ρ be a non-essential character. The proof of the previous expansion formula for an essential character does not work since in this case there is no Prym integral of the second kind with only simple pole on F_μ. Therefore we need a Prym differential $\tau_{\rho;Q_1^2 Q_2^2}$ of second kind for a non-essential character ρ with two poles of second order at two distinct points Q_1 and Q_2 on $\triangle_\mu$ with zero residues at Q_1 and Q_2 [5,6]. In this case one should use as basic elements of expansion the Prym integrals $T_{\rho;Q_1Q_2} = -\int\limits_{Q_0}^{P} \tau_{\rho;Q_1^2Q_2^2}$ of second kind with two simple poles Q_1 and Q_2.

Consider one more Prym differential $\tau_{\rho;Q_1Q_2} = f_0 \tau_{Q_1Q_2}$ of the third kind on F_μ, where f_0 is a unit for ρ on F_μ and $\tau_{Q_1Q_2}$ is the normalized (i.e all a-periods vanish) abelian differential with simple poles Q_1 and Q_2 on F_μ and residues $+1$ and -1 at these points, respectively, which depend holomorphically on $[\mu]$ and ρ [5,6]. It is known that $\tau_{Q_1Q_2} = d\Pi_{Q_1Q_2}$ and the abelian integral $\Pi_{Q_1Q_2}$ can be expressed implicitly via the Riemann theta-function for the surface F_μ. It equals to a sum of two functions, one of which depends only on Q_1, and another only on Q_2 [5, p. 117]. Therefore the derivative $\dfrac{\partial \Pi_{Q_1Q_2}}{\partial z_1}$ does not depend on Q_2, where $z_1 = z(Q_1)$.

The Prym differential $\tau^{(2)}_{\rho;Q_1}$ admits the expansion $\left(\dfrac{1}{(z-z_1)^2} + \dfrac{c^{(1)}_{-1}}{z - z_1} + O(1) \right) dz$ in a neighborhood of Q_1, $z(Q_1) = z_1$, where $c^{(1)}_{-1} = \sum\limits_{j=1}^{g} \log \rho(a_j) \varphi_j^{'}(Q_1)$ [5,6]. To prove this we consider the abelian differential $\dfrac{\tau^{(2)}_{\rho,Q}}{f_0}$, where f_0 is a multiplicative unit for ρ. Its complete sum of residues is zero. In a neighborhood of Q, $z(Q) = z_0$ we have the Laurent expansions

$$\tau^{(2)}_{\rho,Q} = \left(\frac{1}{(z - z_0)^2} + \frac{c_{-1}}{z - z_0} + c_0 + \dots \right) dz,$$

$$\frac{1}{f_0(z)} = \exp\left(-\sum_{j=1}^{g} \lambda_j \varphi_j(z) \right) = \frac{1}{f_0(z_0)} - \frac{(z - z_0)}{f_0(z_0)} \big(\lambda_1 \varphi_1'(z_0) + \dots + \lambda_g \varphi_g'(z_0) \big) + \dots,$$

where $\lambda_j = \log \rho(a_j)$, $j = 1, \dots, g$. From that we get

$$0 = res_{z_0} \frac{\tau^{(2)}_{\rho,Q}}{f_0} = \frac{c_{-1}}{f_0(z_0)} - \frac{\lambda_1 \varphi_1'(z_0) + \dots + \lambda_g \varphi_g'(z_0)}{f_0(z_0)},$$

since Q is the only pole of the abelian differential. Therefore $c_{-1} = \sum_{j=1}^{g} \lambda_j \varphi_j'(z_0)$ and $c_{-1} = 0$ in a finite number of points Q on $\triangle_\mu$. Indeed, $df_0 = \exp\left(\sum_{j=1}^{g} \lambda_j \varphi_j(P)\right) \sum_{j=1}^{g} \lambda_j \varphi_j' dz(P)$, and the equivalency $\sum_{j=1}^{g} \lambda_j \varphi_j'(Q) = 0 \Leftrightarrow df_0(Q) = 0$ holds. Thus, for a non-essential character ρ there is no Prym differential of the second kind with only pole of the second order at an arbitrary point Q and principal part $\dfrac{1}{(z-z_0)^2}$, since the condition $\lambda_1\varphi_1'(z_0) + \dots + \lambda_g \varphi_g'(z_0) = 0$ holds only for a finite number of points Q on $\triangle_\mu$, i.e. at points Q that are zeroes of the differential df_0.

The Prym differential $\tau^{(2)}_{\rho;Q_2}$ also has an expansion $\left(\dfrac{1}{(z-z_2)^2} + \dfrac{c^{(2)}_{-1}}{z-z_2} + O(1)\right)dz$ in a neighborhood of Q_2 on F_μ, where $c^{(2)}_{-1} = \sum_{j=1}^{g} \log\rho(a_j)\varphi_j'(Q_2)$.

A Prym differentials with two poles of the second order and zero residues at these points may be given in the form

$$\tau_{\rho;Q_1^2Q_2^2} = c^{(2)}_{-1} f_0(Q_1)\tau^{(2)}_{\rho;Q_1} - c^{(1)}_{-1} f_0(Q_2)\tau^{(2)}_{\rho;Q_2} - c^{(1)}_{-1}c^{(2)}_{-1}\tau_{\rho;Q_1Q_2}.$$

Note that the principal part for τ_{ρ,Q_1Q_2} at Q_1 has the form $\dfrac{f_0(Q_1)}{z-z_1}$, and at Q_2 it is $-\dfrac{f_0(Q_2)}{z-z_2}$. It follows that the differential constructed above $\tau_{\rho;Q_1^2Q_2^2}$ has poles of the second order at Q_1 and Q_2, and zero residues at these points. Indeed, in a neighborhood of Q_1 its principal part has the form $c^{(2)}_{-1}f_0(Q_1)\left[\dfrac{1}{(z-z_1)^2} + \dfrac{c^{(1)}_{-1}}{z-z_1}\right] - c^{(1)}_{-1}c^{(2)}_{-1}\dfrac{f_0(Q_1)}{z-z_1} = \dfrac{c^{(2)}_{-1}f_0(Q_1)}{(z-z_1)^2}$; analogously at Q_2: $\left(-c^{(1)}_{-1}f_0(Q_2)\left[\dfrac{1}{(z-z_2)^2} + \dfrac{c^{(2)}_{-1}}{z-z_2}\right]\right) + c^{(1)}_{-1}c^{(2)}_{-1}\dfrac{f_0(Q_2)}{z-z_2} = -\dfrac{c^{(1)}_{-1}f_0(Q_2)}{(z-z_2)^2}$. The constructed differential $\tau_{\rho;Q_1^2Q_2^2}$ depend holomorphically on $[\mu]$ and ρ.

Theorem 2. *Let f be a branch of a function of class M_1 for a non-essential character ρ on a variable Riemann surface $F_\mu^{'}$ of type (g,n), $g \geqslant 2$, $n > 0$, with pairwise distinct poles at $P_{n+1}, \dots, P_{n+s}$ of multiplicities $q_{n+1}, \dots, q_{n+s}$ with given principal parts (1). Assume that for an analytic continuation of f to F_μ the conditions $(f) \geqslant \dfrac{1}{P_1^{q_1} \dots P_{n+s}^{q_{n+s}}}$, $q_j \geqslant 0$, $j = 1, \dots, n$, and $\sum_{j=1}^{g} \log\rho(a_j)\varphi_j^{'}(P_{n+s}) \neq 0$ are fulfilled. Then*

$$f(P) = \sum_{j=1}^{g} c_j \int_{Q_0}^{P} f_0\zeta_j + \sum_{r=1}^{n+s-1} \frac{A_{r1}T_{\rho;P_rP_{n+s}}}{d_{n+s}f_0(P_r)} + \sum_{m=2}^{q_1} \frac{A_{1m}}{(m-1)!}\frac{\partial^{m-1}T_{\rho;P_1P_{n+1}}}{\partial P_1^{m-1}} +$$
$$+ \sum_{j=2}^{n+s}\left[A_{j,2}\frac{\partial T_{\rho;P_jP_1}}{\partial P_j} + \frac{A_{j,3}}{2!}\frac{\partial^2 T_{\rho;P_jP_1}}{\partial P_j^2} + \ldots + \frac{A_{j,q_j}}{(q_j-1)!}\frac{\partial^{q_j-1}T_{\rho;P_jP_1}}{\partial P_j^{q_j-1}}\right] + C,$$

where

$$f = \frac{A_{j,q_j}}{(z-z(P_j))^{q_j}} + \ldots + \frac{A_{j,2}}{(z-z(P_j))^2} + \frac{A_{j,1}}{z-z(P_j)} + O(1)$$

for some branch in a neighborhood of P_j, $j = 1, \dots, n+s$, on F_μ; $C = 0$ for $\rho \neq 1$; $d_k = \sum_{m=1}^{g} \log\rho(a_m)\varphi_m^{'}(P_k)$, $k = 1, \dots, n+s$, on F_μ, and all summands depend holomorphically on $[\mu]$ and ρ.

Proof. It is enough to check that principal parts of both parts of the formula coincide. For a neighborhood of P_r, $r = 1, \dots, n+s-1$, on $\triangle_\mu$ we have the Laurent expansion

$$\sum_{r=1}^{n+s-1}\left(\frac{d_{n+s}f_0(P_r)}{z-P_r} - \frac{d_r f_0(P_{n+s})}{z-P_{n+s}}\right)\frac{A_{r1}}{d_{n+s}f_0(P_r)} = \frac{A_{r1}}{z-P_r} + \dots .$$

For a neighborhood of P_{n+s} on $\triangle_\mu$ we have

$$\sum_{r=1}^{n+s-1}\frac{-d_r f_0(P_{n+s})}{z-P_{n+s}}\frac{A_{r1}}{d_{n+s}f_0(P_r)} = \frac{1}{z-P_{n+s}}\frac{f_0(P_{n+s})}{d_{n+s}}\sum_{r=1}^{n+s-1}\frac{-d_r A_{r1}}{f_0(P_r)} + \dots = \frac{A_{n+s,1}}{z-P_{n+s}} + \dots,$$

since $\sum\limits_{r=1}^{n+s}\frac{-A_{r1}d_r}{f_0(P_r)} = 0$, $\frac{f_0(P_{n+s})}{d_{n+s}}\sum\limits_{r=1}^{n+s-1}\frac{-d_r A_{r1}}{f_0(P_r)} = A_{n+s,1}$, according to the complete sum of residues formula for an abelian differential $\frac{f}{f_0}d\left(\sum\limits_{j=1}^{g}\log\rho(a_j)\varphi_j\right)$ of the third kind on F_μ, which at P_j has the residues $\frac{A_{j1}d_j}{f_0(P_j)}$, $j = 1, \dots, n+s$. Thus, the coefficients at the power –1 in principal parts at $P_1, P_2, \dots, P_{n+s}$, are the same.

The third sum shows that the coefficients coincide in principal parts at P_1 for powers starting from –2. The fourth sum shows that the coefficients coincide in principal parts at $P_2, P_3, \dots, P_{n+s}$ for powers starting from –2.

If $\rho = 1$ then all the summands in the formula become abelian integrals, which differ by a constant C. If $\rho \neq 1$ and ρ is a non-essential character, then $C = 0$, since a constant is neither a multiplicative function, nor a Prym integral for this character on F_μ of genus $g \geqslant 2$. □

Remark. P. Appel [6, see p. 118] proved Theorem 2 for a fixed compact Riemann surface and simple poles with every simple element (summand) depending on additional $g-1$ poles. Our theorem is proved for a variable Riemann surface F' of finite type (g, n), $g \geqslant 2$, $n > 0$, and poles of any order with any summand having either one or two poles. Moreover, if $\rho = 1$, $n = 0$ we recover the classical fact on decomposition of a single-valued meromorphic function into a sum of abelian integrals on a compact Riemann surface.

Corollary. *For any non-essential character ρ on a variable compact Riemann surface F_μ of genus $g \geqslant 2$ at Q_1, which are zeroes of the differential df_0, there exists a differential $\tau^{(2)}_{\rho,Q_1}$ of the second kind with only pole of the second order at Q_1 that depends holomorphically on $[\mu]$ and ρ, and having zero residue at Q_1.*

3. Vector bundles of Prym differentials over a Techmüller space of Riemann surfaces of finite type

Denote by $\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}}; F_\mu\right)$ the vector space of $(\rho, q)-$differentials that are multiples of the divisor $\frac{1}{Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}}$, where $\alpha_j \geqslant 1$, $\alpha_j \in \mathbb{N}$, $j = 1, \dots, s$, $s \geqslant 1$, $q \geqslant 1$, $q \in \mathbb{N}$, and by $\Omega^q_\rho(1; F_\mu)$ the vector subspace of holomorphic (ρ, q)-differentials on F_μ [3]. Here the divisor $Q_1 \dots Q_s$ on F_μ is understood as a constant set of points on a surface F of genus $g \geqslant 2$.

Let $\widetilde{E}$ be the principal $Hom(\Gamma, \mathbb{C}^*)$-bundle over $\mathbb{T}_g(F_0)$ with the fiber $Hom(\Gamma_\mu, \mathbb{C}^*)$ over $F_{[\mu]}$ from $\mathbb{T}_g(F_0)$. Here $F_0 = U/\Gamma$, Γ is a Fuchsian group uniformizing F_0 over the circle U, and $F_{[\mu]} = w^\mu(U)/\Gamma_\mu = \Delta_\mu/\Gamma_\mu$, Γ_μ be a quasi-Fuchsian group uniformizing the compact Riemann surface $F_{[\mu]}$ over $w^\mu(U)$.

Lemma 1 ([5], pp. 105–106)**.** *A holomorphic principal $Hom(\Gamma, \mathbb{C}^*)$-bundle $\widetilde{E}$ is biholomorphic to the trivial bundle $\mathbb{T}_g(F_0) \times Hom(\Gamma, \mathbb{C}^*)$ over $\mathbb{T}_g(F_0)$.*

Proposition 1. *The vector bundle $E = \cup \Omega^q_\rho\Big(\dfrac{1}{Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}}; F_\mu\Big)/\,\Omega^q_\rho(1; F_\mu)$ over $\mathbb{T}_g \times (Hom(\Gamma, \mathbb{C}^*)\backslash 1)$ for $q > 1$ $\big($over $\mathbb{T}_g \times (Hom(\Gamma, \mathbb{C}^*)\backslash L_g)$ when $q = 1\big)$ and $g \geqslant 2$ is a holomorphic vector bundle of rank $\alpha_1 + \cdots + \alpha_s = d$, while the co-sets of (ρ, q)-differentials*

$$\tau^{(1)}_{\rho,q;Q_1}, \ldots, \tau^{(\alpha_1)}_{\rho,q;Q_1}, \ldots, \tau^{(1)}_{\rho,q;Q_s}, \ldots, \tau^{(\alpha_s)}_{\rho,q;Q_s}, \tag{2}$$

form a basis of locally holomorphic sections of this bundle.

Proof. With given conditions on q for the character ρ we have the equality $\dim \Omega^q_\rho(1; F_\mu) = (g-1)(2q-1)$. By the Riemann-Roch theorem for (ρ, q)-differentials we find the dimension

$$i_{\rho,q}(Q_1^{-\alpha_1}\ldots Q_s^{-\alpha_s}; F_\mu) = (g-1)(2q-1) + \alpha_1 + \ldots + \alpha_s + r\big((f[\mu])Z^{q-1}Q_1^{\alpha_1}\ldots Q_s^{\alpha_s}\big),$$

where $f[\mu]$ is a function for ρ, Z is the canonical class for abelian 1-differentials on F_μ. Here $r\big((f[\mu])Z^{q-1}Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}\big) = 0$, since $\deg\big((f[\mu])Z^{q-1}Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}\big) \geqslant \alpha_1 > 0$. Thus,

$$\dim \Omega^q_\rho\Big(\frac{1}{Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}}; F_\mu\Big)/\Omega^q_\rho(1; F_\mu) = \alpha_1 + \ldots + \alpha_s = d.$$

It follows from Theorems 2.1 and 2.2 of [6] that there exist differentials from the set (2) that depend locally holomorphically on $[\mu]$ and ρ.

Let us show that the set (2) of equivalency classes of (ρ, q)-differentials that depend locally holomorphically on $[\mu]$ and ρ is linearly independent over $\mathbb{C}$ for given characters ρ. Consider a linear combination of the form

$$C_1^{(1)}\tau^{(1)}_{\rho,q;Q_1} + \ldots + C_1^{(\alpha_1)}\tau^{(\alpha_1)}_{\rho,q;Q_1} + \ldots + C_s^{(1)}\tau^{(1)}_{\rho,q;Q_s} + \ldots + C_s^{(\alpha_s)}\tau^{(\alpha_s)}_{\rho,q;Q_s} = \omega,$$

where ω is a holomorphic (ρ, q)-differential on F_μ. Since the right hand side does not have singularities, all the coefficients are zeroes. All these differentials depend holomorphically on $[\mu]$, ρ and divisors $Q_1 \ldots Q_s$, which are locally holomorphic (constant) sections of the bundle of integer divisors of degree s over the Teichmüller space $\mathbb{T}_g$ of genus g [4]. Therefore, this set gives the base of locally holomorphic sections of this bundle. □

Lemma 2. *For any divisor $P_1^{q_1}\cdot\ldots\cdot P_n^{q_n}$, $q_j \geqslant 0, j = 1, \ldots, n,$ $q > 1$ and any ρ (or $q = 1$ and an essential character ρ) on F_μ of genus $g \geqslant 2$, there exists a differential $\tilde{\omega} \in \Omega^q_\rho\Big(\dfrac{1}{P_1^{q_1}\cdot\ldots\cdot P_n^{q_n}}, F_\mu\Big)$ with the divisor $(\tilde{\omega}) = \dfrac{R_1, \ldots, R_N}{P_1^{q_1}\cdot\ldots\cdot P_n^{q_n}}$, where $R_j \neq P_l$, $l = 1, \ldots, n$, $j = 1, \ldots, N$, $N = (2g-2)q + q_1 + \ldots + q_n$, and any given principal parts of Laurent series at P_j, $j = 1, \ldots, n$, for its branches. This differential depends locally holomorphically on moduli $[\mu]$ of the surface F_μ and the character ρ.*

Proof. If $q_j = 0$ for all j, there exists a holomorphic (ρ, q)-differential $\tilde{\omega} \neq 0$ on F_μ for every $q \geqslant 1$ and ρ, since $i_{\rho,q}(1) = (2q-1)(g-1) \geqslant 3$ if $q > 1$ and $i_\rho(1) \geqslant g-1 \geqslant 1$ if $q = 1$.

Fix $q_1, \ldots, q_n$ as possible order of poles at punctures $P_1, \ldots, P_n$ on F_μ respectively and assume that for at least one j, $q_j \geqslant 1$.

If $q = 1$ and $q_1 = 1$, $q_2 = 0, \ldots, q_n = 0$ for an essential character ρ there exists a differential $\tilde{\omega} \neq 0$ such that $(\tilde{\omega}) \geqslant \dfrac{1}{P_1}$ [6]. Further on, if $q = 1$ we shall assume that $q_1 + q_2 + \cdots + q_n \geqslant 2$.

For any (ρ, q)-differential $\tilde{\omega}$ the degree of its divisor $\deg(\tilde{\omega}) = (2g-2)q$ on F_μ. It follows that $N = (2g-2)q + q_1 + \ldots + q_n$. By Proposition 1.4.4 [5] and Abel's theorem there exists a differential $\tilde{\omega} \neq 0$ with the divisor $(\tilde{\omega}) = \dfrac{R_1, \ldots, R_N}{P_1^{q_1} \cdot \ldots \cdot P_n^{q_n}}$ if and only if the equality $\varphi(R_1 \ldots R_N) - \varphi(P_1^{q_1} \ldots P_n^{q_n}) = -2Kq + \psi(\rho)$ holds in the Jacobi manifold $J(F_\mu)$, where K is the vector of Riemann constants. From this it follows that $\varphi(R_1 \ldots R_g) = -2Kq + \varphi(P_1^{q_1} \ldots P_n^{q_n}) + \psi(\rho) - \varphi(R_{g+1} \ldots R_N)$. Thus, to determine the zeroes of the differential we have $N-g = (2g-2)q - g + q_1 + \ldots + q_n \geqslant g-1 \geqslant 1$ free parameters that can be chosen so that they depend locally holomorphically on moduli $[\mu]$. Solving the Jacobi inversion problem we find the divisor $R_1 \ldots R_g$, which is the only holomorphic solution to the previous equation if the right hand side does not belong to W_g^1 [3,5]. This can be done since $\dim W_g^1 \leqslant g-2$, but $N-g > g-2$ under our hypothesis. Therefore the divisor of the differential $(\tilde{\omega}) = \dfrac{R_1 \ldots R_N}{P_1^{q_1} \ldots P_n^{q_n}}$ has exactly required singularities, if $R_j \neq P_l$ for all indices. In order to do this we choose points $R_{g+1}, \ldots, R_N \neq P_1, \ldots, P_n$. We shall show that after a specific choice of the divisor $R_{g+1} \ldots R_N$ we can satisfy the condition $R_j \neq P_l$ for any j and l. Assume the converse, if $R_1 = P_1$ on F_μ, then from the previous equality we get $\varphi(R_2 \ldots R_g) = -2Kq + \varphi(P_1^{q_1-1} P_2^{q_2} \ldots P_n^{q_n}) + \psi(\rho) - \varphi(R_{g+1} \ldots R_N)$ or $\varphi(R_2 \ldots R_g R_{g+1} \ldots R_{2g-1}) = -2Kq + \varphi(P_1^{q_1-1} P_2^{q_2} \ldots P_n^{q_n}) + \psi(\rho) - \varphi(R_{2g} \ldots R_N)$. Consider the integer divisor $D = R_2 \ldots R_g R_{g+1} \ldots R_{2g-1}$ of degree $2g-2$. It has $g-1$ free points $R_{g+1}, \ldots, R_{2g-1}$. By the free points theorem [3] we get the inequality $i(D) \geqslant 1$, and therefore $\varphi(D) = -2K$. Then the previous inequality can be rewritten as

$$-2K(q-1) + \varphi(P_1^{q_1-1} \ldots P_n^{q_n}) + \psi(\rho) = \varphi(R_{2g} \ldots R_N). \tag{3}$$

Note that $N - (2g-1) = (2g-2)q + q_1 + \ldots + q_n - 2g + 1 \geqslant 1$ in these conditions. Thus, we see that the sets defined by both sides of this equality in $J(F_\mu)$ has different dimensions. Therefore we can choose $R_{2g}, \ldots, R_N$ on F_μ such that (3) does not hold. This is a contradiction.

It is known that under our conditions on q and character ρ there exist elementary (ρ, q)-differentials of the form $\tau^{(1)}_{\rho,q;Q}$ and $\tau^{(m)}_{\rho,q;Q}, m > 1$ on F_μ [6]. Therefore we can construct any principal parts for Laurent series of the differential $\tilde{\omega}$ at all points P_j, $j = 1, \ldots, n$, on F_μ. $\square$

Further on, we shall assume that the character ρ' on Γ' such that $\rho'(\gamma_j) = 1$, $j = 1, \ldots, n$, i.e. $\rho' = \rho \in Hom(\Gamma, \mathbb{C}^*)$. Consider the diagram

$$\begin{array}{ccc} E' = \cup \dfrac{\Omega^q_\rho(\frac{1}{Q_1^{\alpha_1} \ldots Q_s^{\alpha_s}}, F'_\mu) \cap M_1}{\Omega^q_\rho(1, F'_\mu) \cap M_1} & \rightarrow & \cup \dfrac{\Omega^q_\rho(\frac{1}{Q_1^{\alpha_1} \ldots Q_s^{\alpha_s}}, F_\mu)}{\Omega^q_\rho(1, F_\mu)} = E \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{T}}^n_g \times Hom(\Gamma, \mathbb{C}^*) \backslash X & \rightarrow & \mathbb{T}_g \times Hom(\Gamma, \mathbb{C}^*) \backslash X, \end{array} \tag{4}$$

where $\widetilde{\mathbb{T}}^n_g$ is a part of the Teichmüller space $\mathbb{T}_{g,n}$ [6, p.81, p.88], the vertical arrows are projections in vector bundles, and the lower horizontal arrow is related to the operation of gluing the punctures, which makes the surface $F \backslash \{P_1, \ldots, P_n\}$ into a compact surface F [3]. The upper horizontal arrow will be explained later.

Theorem 3. *The diagram above is a commutative diagram of vertical holomorphic vector bundles with isomorphic corresponding fibers and horizontal holomorphic $n!-$sheeted mappings, where $X = 1$ when $q > 1$, and $X = L_g$ when $q = 1$.*

Proof. By the Riemann-Roch theorem we find the dimension $i_{\rho,q}\left(\dfrac{1}{P_1^{q_1} \ldots P_n^{q_n}}, F\right) = (2q-1)(g-1) + q_1 + \ldots + q_n$. Therefore, $\Omega^q_\rho(1, F')$ is an infinite-dimensional vector space.

Now we prove the isomorphism of fibers for fixed F' and F, where F is obtained from F' by glueing up the punctures. For any fixed $\rho \neq 1$ we define the map θ of a fiber of E' over F' into a fiber of E over F, which puts in correspondence to the class $<\omega>=\omega+\Omega^q_\rho(1,F')\cap M_1$ the class $<\omega-\tilde{\omega}>=\omega-\tilde{\omega}+\Omega^q_\rho(1,F)$ in the following way. If $\omega\in\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}},F'\right)\cap M_1$, i.e. $(\omega)\geqslant\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}}\cdot\frac{1}{P_1^{q_1}\dots P_n^{q_n}}$, then we put into correspondence $\omega-\tilde{\omega}$, since by Lemma 2 we can choose the differential $\tilde{\omega}$ such that $(\tilde{\omega})=\frac{R_1\dots R_N}{P_1^{q_1}\dots P_n^{q_n}}$, $R_j\neq P_l$, $j=1,\dots,N$, $l=1,\dots,n$, and having the same principal parts of Laurent series at all points P_j, $j=1,\dots,n$, as an analytic continuation of the differential ω to F. Then $\omega-\tilde{\omega}\in\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}},F\right)$. Let $\theta\big(\omega+\Omega^q_\rho(1,F')\cap M_1\big)=\omega-\tilde{\omega}+\Omega^q_\rho(1,F)$.

We shall show that this mapping is well-defined. Consider another differential ω' from the same equivalency class $<\omega+\Omega^q_\rho(1,F')\cap M_1>$. It has the same singularities as ω at all points $Q_1,\dots,Q_s$, and its own singularities at the punctures, i.e. $(\omega')\geqslant\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}P_1^{q'_1}\dots P_n^{q'_n}}$. Then we choose $\tilde{\omega'}$ such that $\omega'-\tilde{\omega'}\in\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}},F\right)$. Therefore, on the one hand we have $\theta(<\omega>)=<\omega-\tilde{\omega}>$, on the other hand we have $\theta(<\omega'>)=<\omega'-\tilde{\omega'}>$. Consider the difference of representatives of both classes $(\omega-\omega')+(\tilde{\omega'}-\tilde{\omega})=(\omega-\tilde{\omega})-(\omega'-\tilde{\omega'})=\phi\in\Omega^q_\rho(1,F)$, i.e. ϕ is a holomorphic (ρ,q)-differential on F. Therefore,

$$\theta(<\omega>)=\omega-\tilde{\omega}+\Omega^q_\rho(1,F)=\omega'-\tilde{\omega'}+\phi+\Omega^q_\rho(1,F)=\omega'-\tilde{\omega'}+\Omega^q_\rho(1,F)=\theta(<\omega'>).$$

Thus, the map θ is well defined on the equivalency classes.

Let us establish that θ is surjective. For any equivalency class $\omega_0+\Omega^q_\rho(1,F)$ we define the class $\omega_0+\tilde{\tilde{\omega}}+\Omega^q_\rho(1,F')$, where $(\tilde{\tilde{\omega}})=\frac{R_1\dots R_N}{P_1^{q_1}\dots P_n^{q_n}}$ on F and $(\omega_0+\tilde{\tilde{\omega}})\in\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}},F'\right)\bigcap M_1$ for some $q_j\geqslant 0$, $j=1,\dots,n$. Thus, $\theta(\omega_0+\tilde{\tilde{\omega}}+\Omega^q_\rho(1,F')\bigcap M_1)=\omega_0+\Omega^q_\rho(1,F)$. This can be proved differently. Take a Prym differential with required singularities holomorphic at punctures on F_μ from the bundle in the right hand side of (4) and consider it on the surface with punctures. The map θ takes it back.

Now let us prove that the mapping of a fixed fiber over F' and a fiber over the corresponding surface F is 1–to–1. Assume that different equivalency classes are mapped by θ to one class, i.e. $\theta(<\omega_1>)=<\omega_1-\tilde{\omega_1}>=<\omega_2-\tilde{\omega_2}>=\theta(<\omega_2>)$, where $\tilde{\omega_1}\in\Omega^q_\rho\left(\frac{1}{P_1^{q_1}\dots P_n^{q_n}},F\right)$, a $\tilde{\omega_2}\in\Omega^q_\rho\left(\frac{1}{P_1^{q'_1}\dots P_n^{q'_n}},F\right)$ and $(\omega_2-\omega_1)$ does not belong to $\Omega^q_\rho(1,F')\cap M_1$. Consider the difference $(\omega_2-\omega_1)+(\tilde{\omega_1}-\tilde{\omega_2})=(\omega_2-\tilde{\omega_2})-(\omega_1-\tilde{\omega_1})=\phi\in\Omega^q_\rho(1,F)$. Therefore $\omega_2-\omega_1=$ $=\phi+\tilde{\omega_2}-\tilde{\omega_1}\in\Omega^q_\rho(1,F')\cap M_1$. This is a contradiction.

Now we prove that θ is linear. Indeed, for $c_j\in\mathbb{C}$, $j=1,2$, we have equalities

$$\theta[c_1(\omega_1+\Omega^q_\rho(1,F')\cap M_1)+c_2(\omega_2+\Omega^q_\rho(1,F')\cap M_1)]=$$

$$=\theta[c_1\omega_1+c_2\omega_2+\Omega^q_\rho(1,F')\cap M_1]=c_1\omega_1+c_2\omega_2-(c_1\tilde{\omega_1}+c_2\tilde{\omega_2})+\Omega^q_\rho(1,F)=$$

$$=c_1(\omega_1-\tilde{\omega_1})+c_1\Omega^q_\rho(1,F)+c_2(\omega_2-\tilde{\omega_2})+c_2\Omega^q_\rho(1,F)=c_1\theta(<\omega_1>)+c_2\theta(<\omega_2>).$$

Thus, θ is linear, and we get an isomorphism

$$\theta:\frac{\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}},F'_\mu\right)\cap M_1}{\Omega^q_\rho(1,F'_\mu)\cap M_1}\to\frac{\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1}\dots Q_s^{\alpha_s}},F_\mu\right)}{\Omega^q_\rho(1,F_\mu)},$$

of fibers under these conditions.

Now we lift the set (2) to the set of Prym differentials on F'_μ, which is obtained from (2) by adding some differential from $\Omega^q_\rho(1, F^{'}_\mu) \cap M_1$. All these differentials can be chosen so that they depend holomorphically on $[\mu]$ and ρ on $F^{'}_\mu$. Thus, we obtain a set of equivalency classes of differentials

$$\tau^{(1)}_{\rho,q;Q_1} + \Omega^q_\rho(1, F^{'}_\mu), \tau^{(2)}_{\rho,q;Q_1} + \Omega^q_\rho(1, F^{'}_\mu), \ldots, \tau^{(\alpha_1)}_{\rho,q;Q_1} + \Omega^q_\rho(1, F^{'}_\mu), \tau^{(1)}_{\rho,q;Q_2} + \Omega^q_\rho(1, F^{'}_\mu), \ldots$$

$$\ldots, \tau^{(\alpha_2)}_{\rho,q;Q_2} + \Omega^q_\rho(1, F^{'}), \ldots, \tau^{(1)}_{\rho,q;Q_s} + \Omega^q_\rho(1, F^{'}_\mu), \ldots, \tau^{(\alpha_s)}_{\rho,q;Q_s} + \Omega^q_\rho(1, F^{'}_\mu), \qquad (1')$$

which correspond to Prym differentials from (2), on $F^{'}_\mu$. It is a basis of locally holomorphic sections of the vector bundle E'. Consequently, both these bundles E and E' are holomorphic vector bundles of rank d over mentioned bases.

The operation of gluing up the punctures that makes F' into F defines an $n!$-sheeted holomorphic mapping from $\widetilde{\mathbb{T}}^n_g$ onto $\mathbb{T}_g$. Here, over each surface F with fixed points $P_1, \ldots, P_n$ there are $n!$ surfaces F'.

Thus, we have proved commutativity of the diagram (4) with required properties. □

4. Spaces of univalent differentials

Lemma 3. *For each divisor $P_1^{q_1} \cdot \ldots \cdot P_n^{q_n}, q_j \geqslant 0, j = 1, ..., n,$ and $q > 1$ on F_μ of genus $g \geqslant 2$ there exists a differential $\tilde{\omega} \in \Omega^q\left(\frac{1}{P_1^{q_1} \cdot \ldots \cdot P_n^{q_n}}, F_\mu\right)$ with the divisor $(\tilde{\omega}) = \frac{R_1, \ldots, R_N}{P_1^{q_1} \cdot \ldots \cdot P_n^{q_n}}$, where $R_j \neq P_l$, $l = 1, \ldots, n$, $j = 1, \ldots, N$, $N = (2g-2)q + q_1 + \ldots + q_n$, and any given principal parts of Laurent series at P_j, $j = 1, \ldots, n$. This differential depends locally holomorphically on moduli $[\mu]$ of the surface F_μ.*

The proof is analogous to the proof of Lemma 2.

Denote by $\Omega^q\left(\frac{1}{Q_1^{\alpha_1} \ldots Q_l^{\alpha_l} Q_{l+1} \ldots Q_s}; F_\mu\right)$ for $q > 1$ the space of q-differentials on F_μ that are multiple of the divisor $\frac{1}{Q_1^{\alpha_1} \ldots Q_l^{\alpha_l} Q_{l+1} \ldots Q_s}$, where $\alpha_1, \ldots, \alpha_l \geqslant 2$, $s \geqslant 1$, $0 \leqslant l \leqslant s$ and the points $Q_1, \ldots, Q_s$ are distinct, and by $\Omega^q(1; F_\mu)$ denote the subspace of holomorphic q-differentials on F_μ.

By the Riemann-Roch theorem for q-differentials we find the dimensions of these spaces. It is known that $\dim \Omega^q(1; F_\mu) = (2q-1)(g-1)$ for $q > 1$. Moreover,

$$i_q\left(\frac{1}{Q_1^{\alpha_1} \ldots Q_l^{\alpha_l} Q_{l+1} \ldots Q_s}\right) = (g-1)(2q-1) - \deg\left(\frac{1}{Q_1^{\alpha_1} \ldots Q_l^{\alpha_l} Q_{l+1} \ldots Q_s}\right) +$$

$$+ r\left(Z^{q-1} Q_1^{\alpha_1} \ldots Q_l^{\alpha_l} Q_{l+1} \ldots Q_s\right) = (g-1)(2q-1) + \alpha_1 + \ldots + \alpha_l + s - l \, (\geqslant 4).$$

Therefore, $\dim \Omega^q\left(\frac{1}{Q_1^{\alpha_1} \ldots Q_l^{\alpha_l} Q_{l+1} \ldots Q_s}; F_\mu\right) / \Omega^q(1; F_\mu) = \alpha_1 + \ldots + \alpha_l + s - l \, (\geqslant 1)$.

Consider the sets of q-differentials:

$$\tau^{(1)}_{q;Q_1}, \tau^{(2)}_{q;Q_1}, \ldots, \tau^{(\alpha_1)}_{q;Q_1}, \ldots, \tau^{(1)}_{q;Q_l}, \tau^{(2)}_{q;Q_l}, \ldots, \tau^{(\alpha_l)}_{q;Q_l}, \tau_{q;Q_1 Q_{l+1}}, \ldots, \tau_{q;Q_1 Q_s} \qquad (5)$$

for $l \geqslant 1$, $q > 1$;

$$\tau^{(1)}_{q;Q_1}, \tau_{q;Q_1 Q_2}, \ldots, \tau_{q;Q_1 Q_s} \qquad (6)$$

for $l = 0$, $q > 1$ on F_μ.

Proposition 2 ([6]). *The bundle*

$$\cup\Omega^q\left(\frac{1}{Q_1^{\alpha_1}\dots Q_l^{\alpha_l}Q_{l+1}\dots Q_s};F_\mu\right)/\Omega^q(1;F_\mu)$$

is a holomorphic vector bundle of rank $\alpha_1+\ldots+\alpha_l+s-l$ *over* $\mathbb{T}_g$, *where* $g\geqslant 2,\ \alpha_1,\ldots,\alpha_l\geqslant 2,$ $s\geqslant 1,\ 0\leqslant l\leqslant s,\ q>1$ *and the points* $Q_1,\ldots,Q_s$ *are distinct. The equivalency classes of* q*-differentials from* (5), (6) *form a base of locally holomorphic sections of this bundle over* $\mathbb{T}_g$.

Consider the diagram

$$\begin{array}{ccccc} E'=\cup\dfrac{\Omega^q\left(\frac{1}{Q_1^{\alpha_1}\dots Q_l^{\alpha_l}Q_{l+1}\dots Q_s},F'_\mu\right)\cap M_1}{\Omega^q(1,F'_\mu)\cap M_1} & \to & \cup\dfrac{\Omega^q\left(\frac{1}{Q_1^{\alpha_1}\dots Q_l^{\alpha_l}Q_{l+1}\dots Q_s},F_\mu\right)}{\Omega^q(1,F_\mu)} & = E \\ \downarrow & & \downarrow & \\ \widetilde{\mathbb{T}}_g^n & \to & \mathbb{T}_g. & \end{array} \tag{7}$$

Теорема 4. *The diagram (7) is commutative; vertical arrows stand for holomorphic vector bundles with isomorphic corresponding fibers, horisontal arrows are for holomorphic* $n!$*-sheeted mappings over bases from* $\widetilde{\mathbb{T}}_g^n$ *(a part of the Teichmüller spaces* $\mathbb{T}_{g,n}$*) and a Teichmüller space* $\mathbb{T}_g$.

The proof follows the proof of Theorem 3 together with Lemma 3 and Proposition 2.

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Векторное расслоение дифференциалов Прима над пространствами Тейхмюллера поверхностей с проколами

Александр В. Чуешев
Виктор В. Чуешев
Институт фундаментальных наук
Кемеровский государственный университет
Красная, 6, Кемерово, 650043
Россия

В работе исследуются мультипликативные мероморфные функции и дифференциалы на римановых поверхностях конечного типа. Доказан аналог формулы П. Аппеля о разложении мультипликативной функции с полюсами любых кратностей в сумму элементарных интегралов Прима. Построены явные базисы для ряда важных фактор-пространств. Доказана теорема о послойном изоморфизме векторных расслоений и n*!-листных отображений над пространствами Тейхмюллера. Эта теорема дает важную связь между пространствами дифференциалов Прима (абелевых дифференциалов) на компактной римановой поверхности и на римановой поверхности конечного типа.*

Ключевые слова: пространства Тейхмюллера римановых поверхностей конечного типа, дифференциалы Прима, векторные расслоения, группа характеров, многообразия Якоби.