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A Class of Quintic Kolmogorov Systems with Explicit Non-algebraic Limit Cycle

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Various physical, ecological, economic, etc phenomena are governed by planar differential systems. Subsequently, several research studies are interested in the study of limit cycles because of their interest in the understanding of these systems. The aim of this paper is to investigate a class of quintic Kolmogorov systems, namely systems of the form

$$\dot{x} = x\ P_4(x,y),$$
$$\dot{y} = y\ Q_4(x,y),$$

where P_4 and Q_4 are quartic polynomials. Within this class, our attention is restricted to study the limit cycle in the realistic quadrant $\{(x,y)\in\mathbb{R}^2;\ x>0,\ y>0\}$. According to the hypothesises, the existence of algebraic or non-algebraic limit cycle is proved. Furthermore, this limit cycle is explicitly given in polar coordinates. Some examples are presented in order to illustrate the applicability of our result.

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1. Introduction and preliminaries

The so-called Kolmogorov systems on the plane are differential equations of the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = xP(x,y), \\ \dot{y} = \dfrac{dy}{dt} = yQ(x,y), \end{cases} \tag{1}$$

where P and Q are two coprime polynomials of $\mathbb{R}[x,y]$ and the derivatives are performed with respect to the time variable. By definition, the degree of the system (1) is the maximum of the degrees of the polynomials P and Q. These systems arise in great variety of applications, for example, ecology and population dynamics [20,22,25], chemical reaction and plasma physics [19], hydrodynamics [10], economics, etc ...

System (1) is said to be integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant continuously differentiable function $H:\Omega\longmapsto\mathbb{R}$ called a first integral of this system on Ω which

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is constant on the trajectories of the polynomial system (1) contained in Ω i.e., if

$$\frac{dH}{dt}(x,y)=\frac{\partial H}{\partial x}(x,y)\ x\ P(x,y)+\frac{\partial H}{\partial y}(x,y)\ y\ Q(x,y)\equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H=h$ is the general solution of the above equation, where h is an arbitrary constant. It is well know that for planar differential system, the existence of a first integral determines its phase portrait, see [11].

In the qualitative theory of planar polynomial differential systems [12], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem concerned essentially by the number $H(n)$ of limit cycles of (1) and their positions in the phase space. There is an extensive literature on that subject, most of it deals essentially with detection, number and stability of limit cycles.

We recall that in the phase plane, a *limit cycle* of system (1) is an isolated periodic solution in the set of all its periodic solutions. If limit cycle contained in the zero set of invariant algebraic curve of the plane, then we say that it is *algebraic*, otherwise it is called *non-algebraic.*

In the literature, we can find also another interesting but even more difficult problem is to give an explicit expression of a limit cycle. The limit cycles previously known in an explicit way were algebraic see [3, 4, 15].

After the Odani's work [23], where it has been proved that the limit cycle appearing in the Van der Pol equation is not algebraic without giving an explicit expression, several articles have been published presenting differential systems polynomials for which non-algebraic limit cycles exist and are explicitly determined see [1, 6, 9, 14, 16].

Concerning the Kolmogorov systems, most of the studies were limited to study the existence of limit cycles for classes of these systems see [17, 20, 21, 25–27]. To our knowledge, the exact analytic expressions of the limit cycles for a given Kolmogorov system is still unknown except in algebraic case. For instance, Bendjeddou, Cheurfa and Berbache in [2] showed that the quartic system admits the circle as an invariant curve which corresponds of course to the limit cycle. In the same context, Benyoucef and Bendjeddou studied in [7, 8] two polynomial systems of any degrees. They showed in the first paper that the considered system can admit up to four algebraic limit cycles in the plane and in the second one the system can admit a unique algebraic limit cycle in the first quadrant.

In this paper, we are interested on the quintic Kolmogorov systems of the form

$$\begin{cases} \dot{x} &= x\ P_4(x,y), \\ \dot{y} &= y\ Q_4(x,y), \end{cases} \tag{2}$$

where

$$\begin{aligned} P_4(x,y) = {} & 4\lambda-4(\beta+2\lambda)x+2(2\beta-3\lambda)y+2(3\beta+4\lambda)x^2+2(4\lambda-\beta)xy+2(\lambda-2\beta)y^2- \\ & -4(\beta+\lambda)x^3-(\beta+5\lambda)x^2y+(6\beta-2\lambda-1)xy^2+(\lambda-\beta)y^3+ \\ & +(\beta+\lambda)x^4+(\beta+\lambda)x^3y+(1-2\beta)x^2y^2+(\lambda-\beta)xy^3+(\beta-\lambda)y^4, \\ Q_4(x,y) = {} & -4\lambda+2(2\beta+\lambda)x+4(2\lambda-\beta)y+2\lambda x^2-6\beta xy+2(3\beta-4\lambda)y^2- \\ & -3(\beta+\lambda)x^3+(2\beta-2\lambda-1)x^2y+(5\beta-\lambda)xy^2+4(\lambda-\beta)y^3+ \\ & +(\beta+\lambda)x^4+(\beta+\lambda)x^3y+(1-2\beta)x^2y^2+(\lambda-\beta)xy^3+(\beta-\lambda)y^4, \end{aligned}$$

and λ, β are reals constants. Within this class, we study the existence of a limit cycle in the realistic quadrant $\left\{(x,y)\in\mathbb{R}^2;\ x>0,\ y>0\right\}$ and show under appropriate conditions that this cycle is non-algebraic giving its explicit form.

For presenting our main result, the coordinates are translated by a vector $(1,1)$, which trasforms our system (2) to the following,

$$\begin{cases} \dot{x} = (x+1)\left(\left(x - 2y + x^2 + xy - y^2\right)\left((\beta+\lambda)x^2 + (\lambda-\beta)y^2\right) + x(y+1)^2(x+1)\right), \\ \dot{y} = (y+1)\left(\left(2x + y + x^2 + xy - y^2\right)\left((\beta+\lambda)x^2 + (\lambda-\beta)y^2\right) + y(x+1)^2(y+1)\right). \end{cases} \tag{3}$$

We can write the system (3) in polar coordinates (r,θ) through $x = r\cos\theta$, $y = r\sin\theta$, as

$$\begin{cases} \dot{r} &= \frac{1}{8} r\left(f_1(\theta) r^4 + f_2(\theta) r^3 + f_3(\theta) r^2 + f_4(\theta) r + 8\right), \\ \dot{\theta} &= r^2(\lambda + \beta\cos 2\theta)(2 + r(\cos\theta + \sin\theta)), \end{cases} \tag{4}$$

where

$$\begin{aligned} f_1(\theta) &= 1 + 4\beta + 4\lambda(2\cos 2\theta + \sin 2\theta) + (4\beta - 1)\cos 4\theta + 2\beta\sin 4\theta, \\ f_2(\theta) &= 4\left((2\beta + 4\lambda + 1)\cos\theta + \sin\theta + (2\beta - 1)\cos 3\theta + \sin 3\theta\right), \\ f_3(\theta) &= 8(1 + \lambda + \beta\cos 2\theta + 2\sin 2\theta), \\ f_4(\theta) &= 16(\cos\theta + \sin\theta). \end{aligned}$$

Since we are dealing with solutions of system (2) in the first quadrant, we have $r\cos\theta > -1$ and $r\sin\theta > -1$ hence $(2 + r(\cos\theta + \sin\theta)) > 0$. If $\lambda + |\beta| < \frac{-1}{2}$, then $(\lambda + \beta\cos 2\theta)$ is strictly negative and as a result $\dot{\theta} = \frac{d\theta}{dt}$ is negative for all t. This signifies that $(1,1)$ is the unique equilibrium point of system (2) in the first quadrant and the orbits $(r(t), \theta(t))$ of system (4) have opposite orientation with respect to $(x(t), y(t))$ of system (2).

2. The main result

Our main result on the limit cycles of the quintic Kolmogorov system defined by (2) is as follows

Theorem 2.1. *Consider the polynomial system* (2), *then the following statements hold*
1) If $\beta \neq 0$ *and* $\lambda + |\beta| < \frac{-1}{2}$, *the system* (2) *has non-algebraic, stable and hyperbolic limit cycle explicilty given in polar coordinates* (r,θ) *by*

$$r(\theta, r_0) = \frac{A(\theta)(\cos\theta + \sin\theta) + \sqrt{A^2(\theta) + 4A(\theta) - A^2(\theta)\sin 2\theta}}{2 - A(\theta)\sin 2\theta},$$

where $A(\theta) = \exp(\theta)\left(\frac{r_0^2}{r_0+1} + \int_0^\theta \frac{\exp(-s)}{\lambda + \beta\cos 2s} ds\right)$ *and*

$$r_0 = \frac{\sqrt{e^{2\pi}\int\limits_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds}}{2\left(e^{2\pi} - 1\right)}\left(\sqrt{e^{2\pi}\int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds} + \sqrt{e^{2\pi}\int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds + 4\left(e^{2\pi} - 1\right)}\right).$$

2) If $\beta = 0$ *and* $\lambda < \frac{-1}{2}$, *the system* (2) *has algebraic, stable and hyperbolic limit cycle explicilty given in polar coordinates* (r,θ) *by* $r(\theta, r_0) = \frac{(\cos\theta + \sin\theta) + \sqrt{1 - 4\lambda - \sin 2\theta}}{-(2\lambda + \sin 2\theta)}$, *and in Cartesian coordinates by* $\lambda(x-1)^2 + \lambda(y-1)^2 + xy = 0$.

For the demonstration of Theorem (2.1), we need the following lemmas

Lemma 2.2. *The system of the form*

$$\begin{cases} \dot{r} &= F(\theta) H(r,\theta) - \dfrac{\partial H}{\partial \theta}(r,\theta) + G(\theta), \\ \dot{\theta} &= \dfrac{\partial H}{\partial r}(r,\theta) \end{cases} \tag{5}$$

possess a first integral expressed as

$$L(r,\theta) = H(r,\theta)\exp(-\int_0^\theta F(s)\,ds) - \int_0^\theta G(s)\exp(-\int_0^s F(w)\,dw)ds. \tag{6}$$

Proof. Let set $A(r,\theta) = H(r,\theta)\exp\Big(-\int\limits_0^\theta F(s)\,ds\Big)$ and $B(\theta) = \int\limits_0^\theta G(s)\exp\Big(-\int\limits_0^s F(w)\,dw\Big)ds$, then the derivatives of A and B with respect to θ are

$$\frac{dA}{d\theta}(r,\theta) = \Big(\frac{\partial H}{\partial \theta}(r,\theta) - F(\theta)H(r,\theta)\Big)\exp\Big(-\int_0^\theta F(s)\,ds\Big),$$
$$\frac{dB}{d\theta}(\theta) = G(\theta)\exp\Big(-\int_0^\theta F(s)\,ds\Big)$$

By replacing the expression of derivatives of A and B with respect to θ in the expression of L, it follows that $\frac{dL}{d\theta}(r,\theta) = \Big(\frac{\partial H}{\partial \theta}(r,\theta) - F(\theta)H(r,\theta) - G(\theta)\Big)\exp\Big(-\int\limits_0^\theta F(s)\,ds\Big)$.
By the chain rule, the derivative of L with respect to t is given by following expression,

$$\begin{aligned} \frac{dL}{dt}\big(r(t),\theta(t)\big) &= \frac{\partial L}{\partial r}(r,\theta)\frac{dr}{dt} + \frac{\partial L}{\partial \theta}(r,\theta)\frac{d\theta}{dt} = \\ &= \Big(\frac{\partial H}{\partial r}(r,\theta)\exp\Big(-\int_0^\theta F(s)\,ds\Big)\Big)\Big(F(\theta)H(r,\theta) - \frac{\partial H}{\partial \theta}(r,\theta) + G(\theta)\Big) + \\ &+ \Big(\Big(\frac{\partial H}{\partial \theta}(r,\theta) - F(\theta)H(r,\theta) - G(\theta)\Big)\exp\Big(-\int_0^\theta F(s)\,ds\Big)\Big)\frac{\partial H}{\partial r}(r,\theta) = 0. \end{aligned}$$

So $L(r,\theta)$ is a first integral of system ☐

Lemma 2.3. *Let $\lambda, \beta \in \mathbb{R}$ such that $\lambda + |\beta| < \frac{-1}{2}$, then the following statements hold*

1) $0 < \frac{e^{2\pi}}{e^{2\pi}-1}\int\limits_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s}\,ds < 2.$

2) The function g defined on $[0,2\pi]$ by $g(\theta) = 2\exp(-\theta) + \int\limits_0^\theta \frac{-e^{-s}}{\lambda+\beta\cos 2s}ds$ *is strictly decreasing. Furthermore* $g(\theta) > \frac{e^{2\pi}}{e^{2\pi}-1}\int\limits_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds.$

3) $0 < A(\theta) = \exp(\theta)\left(\frac{e^{2\pi}}{e^{2\pi}-1}\int\limits_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds + \int\limits_0^\theta \frac{e^{-s}}{\lambda+\beta\cos 2s}ds\right) < 2.$

Proof of statement 1) of Lemma 2.3 We have $\lambda + \beta\cos 2s \leqslant \lambda + |\beta| < \frac{-1}{2}$ which implies $0 < \frac{-e^{-s}}{\lambda+\beta\cos 2s} < 2e^{-s}$, consequently $0 < \frac{e^{2\pi}}{e^{2\pi}-1}\int\limits_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds < \frac{2e^{2\pi}}{e^{2\pi}-1}\int\limits_0^{2\pi} e^{-s}\,ds$, whence

$$0 < \frac{e^{2\pi}}{e^{2\pi}-1}\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds < 2.$$

☐

Proof of statement 2) of Lemma 2.3. Over the interval $[0, 2\pi]$, the function g is differentiable and

$$g'(\theta) = -2\exp(-\theta) + \frac{-\exp(-\theta)}{\lambda + \beta\cos 2\theta} = -\exp(-\theta)\left(2 + \frac{1}{\lambda + \beta\cos 2\theta}\right).$$

Since $\lambda + \beta\cos 2\theta \leqslant \lambda + |\beta|$, then $g'(\theta) \leqslant -\exp(-\theta)\left(2 + \frac{1}{\lambda + |\beta|}\right) < 0$. Therefore g is strictly decreasing function. On the other hand, from the statement 1) of Lemma 2.3, we have $\frac{1}{e^{2\pi} - 1}\int\limits_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds < 2e^{-2\pi}$, which implies

$$\frac{e^{2\pi}}{e^{2\pi} - 1}\int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds - \int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds < 2e^{-2\pi}$$

because $\frac{1}{e^{2\pi} - 1} = \left(\frac{e^{2\pi}}{e^{2\pi} - 1} - 1\right)$, consequently

$$\frac{e^{2\pi}}{e^{2\pi} - 1}\int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds < 2e^{-2\pi} + \int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds = g(2\pi),$$

as g is strictly decreasing function, then $\frac{e^{2\pi}}{e^{2\pi} - 1}\int\limits_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds < g(\theta)$. □

Proof of statement 3) of Lemma 2.3. Let us first show that A is strictly positive. From the relationship of Chasles

$$A(\theta) = \exp(\theta)\left(\frac{e^{2\pi}}{e^{2\pi} - 1}\left(\int_0^{\theta} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds + \int_{\theta}^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds\right) - \int_0^{\theta} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds\right),$$

which implies

$$A(\theta) = \exp(\theta)\left(\frac{e^{2\pi}}{e^{2\pi} - 1}\int_{\theta}^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds + \left(\frac{e^{2\pi}}{e^{2\pi} - 1} - 1\right)\int_0^{\theta} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds\right).$$

Since $\frac{e^{2\pi}}{e^{2\pi} - 1} - 1 > 0$ and $\frac{-e^{-s}}{\lambda + \beta\cos 2s} > 0$, then $A(\theta) > 0$.

Let us now show that $A(\theta) < 2$. From the statement 2) of Lemma 2.3, we have

$$\frac{e^{2\pi}}{e^{2\pi} - 1}\int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds < 2\exp(-\theta) + \int_0^{\theta} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds,$$

which implies $\left(\frac{e^{2\pi}}{e^{2\pi} - 1}\int\limits_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds + \int\limits_0^{\theta} \frac{e^{-s}}{\lambda + \beta\cos 2s} ds\right) < 2\exp(-\theta)$, therefore

$$A(\theta) = \exp(\theta)\left(\frac{e^{2\pi}}{e^{2\pi} - 1}\int_0^{2\pi} \frac{-e^{-s}}{\lambda + \beta\cos 2s} ds + \int_0^{\theta} \frac{e^{-s}}{\lambda + \beta\cos 2s} ds\right) < 2,$$

whence $0 < A(\theta) < 2$. □

Proof of Theorem 2.1. We assume that $\lambda + |\beta| < \frac{-1}{2}$. In the new independent variable θ, the differential system (4) becomes

$$\frac{dr}{d\theta} = \frac{1}{8r}\frac{f_1(\theta) r^4 + f_2(\theta) r^3 + f_3(\theta) r^2 + f_4(\theta) r + 8}{(\lambda + \beta\cos 2\theta)(2 + r(\cos\theta + \sin\theta))}, \tag{7}$$

which can be expressed as

$$\frac{dr}{d\theta} = \frac{F(\theta) H(r,\theta) - \dfrac{\partial H}{\partial \theta}(r,\theta) + G(\theta)}{\dfrac{\partial H}{\partial r}(r,\theta)}, \tag{8}$$

where $H(r,\theta) = \dfrac{r^2}{(r\cos\theta + 1)(r\sin\theta + 1)}$, $F(\theta) = 1$ and $G(\theta) = \dfrac{1}{\lambda + \beta\cos 2\theta}$.

By Lemma 2.2, the solutions of the equation (7) are expressed as

$$\frac{r^2}{(r\cos\theta + 1)(r\sin\theta + 1)} - \exp(\theta)\left(k + \int_0^\theta \frac{e^{-s}}{\lambda + \beta\cos 2s} ds\right) = 0, \quad \text{where} \quad k \in \mathbb{R}. \tag{9}$$

In the region $2 - A(\theta)\sin 2\theta \neq 0$ and $A^2(\theta) + 4A(\theta) - A^2(\theta)\sin 2\theta \geqslant 0$ the equation (9) has two solutions

$$r_{1,2}(\theta) = \frac{A(\theta)(\cos\theta + \sin\theta) \pm \sqrt{A^2(\theta) + 4A(\theta) - A^2(\theta)\sin 2\theta}}{2 - A(\theta)\sin 2\theta}, \tag{10}$$

with $A(\theta) = \exp(\theta)\left(k + \int\limits_0^\theta \dfrac{e^{-s}}{\lambda + \beta\cos 2s} ds\right)$.

Notice that system (3) has a periodic solution if and only if equation (7) has a strictly positive 2π-periodic solution. For $\theta = 0$, we have

$$r_1(0) = \frac{1}{2}\left(k + \sqrt{k(k+4)}\right) \quad \text{and} \quad r_2(0) = \frac{1}{2}\left(k - \sqrt{k(k+4)}\right),$$

$r_{1,2}(0)$ are defined if only if $k \in]-\infty, -4[\cup]0, +\infty[$. Over the interval $]-\infty, -4[$, $r_1(0)$ and $r_2(0)$ are negative, and in $]0, +\infty[$, $r_1(0)$ is positive but $r_2(0)$ is negative. Consequently, the admitted solution of equation (9) is

$$r(\theta) = r_1(\theta) = \frac{A(\theta)(\cos\theta + \sin\theta) + \sqrt{A^2(\theta) + 4A(\theta) - A^2(\theta)\sin 2\theta}}{2 - A(\theta)\sin 2\theta}. \tag{11}$$

where $A(\theta) = \exp(\theta)\left(k + \int\limits_0^\theta \dfrac{e^{-s}}{\lambda + \beta\cos 2s} ds\right)$ and $k = \dfrac{r^2(0)}{r(0) + 1} > 0$.

The solution of the equation (9) starting at $r(0, r_0) = r_0 > 0$ is given by

$$r(\theta, r_0) = \frac{A(\theta)(\cos\theta + \sin\theta) + \sqrt{A^2(\theta) + 4A(\theta) - A^2(\theta)\sin 2\theta}}{2 - A(\theta)\sin 2\theta},$$

where $A(\theta) = \exp(\theta)\left(\dfrac{r_0^2}{r_0 + 1} + \int\limits_0^\theta \dfrac{e^{-s}}{\lambda + \beta\cos 2s} ds\right)$ and $r_0 = r(0)$.

The condition of the periodic solution with 2π-periodic starting at $r(0, r_0) = r_0 > 0$ is $r(0, r_0) = r(2\pi, r_0)$. For $\theta = 2\pi$, we obtain

$$r(2\pi, r_0) = \frac{1}{2}\left(A(2\pi) + \sqrt{A(2\pi)(A(2\pi) + 4)}\right)$$

where $A(2\pi) = e^{2\pi}\left(\dfrac{r_0^2}{r_0 + 1} + \int\limits_0^{2\pi} \dfrac{e^{-s}}{\lambda + \beta\cos 2s} ds\right)$.

The resolution of equation $r(0,r_0)=r(2\pi,r_0)$ gives

$$r_0=\frac{\sqrt{e^{2\pi}\int\limits_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds}}{2\left(e^{2\pi}-1\right)}\left(\sqrt{e^{2\pi}\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds}+\sqrt{e^{2\pi}\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds+4\left(e^{2\pi}-1\right)}\right).$$

By some simplifications, we obtain $\frac{r_0^2}{r_0+1}=\frac{e^{2\pi}}{e^{2\pi}-1}\int\limits_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds$. Finally, the general solution of (4) is given explicitly by

$$r(\theta,r_0)=\frac{A(\theta)(\cos\theta+\sin\theta)+\sqrt{A^2(\theta)+4A(\theta)-A^2(\theta)\sin 2\theta}}{2-A(\theta)\sin 2\theta}, \tag{12}$$

with $A(\theta)=\exp(\theta)\left(\frac{e^{2\pi}}{e^{2\pi}-1}\int\limits_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds+\int\limits_0^{\theta}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds\right)$ and $r_0=r(0)$.

Periodicity of $r(\theta,r_0)$. Let us now show that $A(\theta)$ is 2π-periodic function. We have

$$A(\theta+2\pi)=\exp(\theta+2\pi)\left(\frac{e^{2\pi}}{e^{2\pi}-1}\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds+\int_0^{\theta+2\pi}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds\right),$$

it follows

$$A(\theta+2\pi)=e^{\theta+2\pi}\left(\frac{e^{2\pi}}{e^{2\pi}-1}\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds+\int_0^{2\pi}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds+\int_{2\pi}^{\theta+2\pi}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds\right),$$

i.e.

$$A(\theta+2\pi)=e^{\theta}e^{2\pi}\left(\frac{1}{e^{2\pi}-1}\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds+\int_{2\pi}^{\theta+2\pi}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds\right),$$

by the change of variable $u=s-2\pi$, we obtain $\int\limits_{2\pi}^{\theta+2\pi}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds=e^{-2\pi}\int\limits_0^{\theta}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds$, then

$$A(\theta+2\pi)=e^{\theta}\left(\frac{e^{2\pi}}{e^{2\pi}-1}\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}\,ds+e^{2\pi}e^{-2\pi}\int_0^{\theta}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds\right),$$

therefore $A(\theta+2\pi)=A(\theta)$. Furthermore, as $\theta\longmapsto\sin\theta$, $\theta\longmapsto\cos\theta$ and $\theta\longmapsto A(\theta)$ are 2π-periodic functions, then $r(\theta,r_0)$ is also.

Strict positivity of $r(\theta,r_0)$. By the statement 3) of Lemma 2.3, we have $0<A(\theta)<2$, then the denominator of $r(\theta,r_0)$ is strictly positive. Two cases are distinguished

i) If $(\cos\theta+\sin\theta)>0$, the numerator of $r(\theta,r_0)$ is strictly positive, consequently $r(\theta,r_0)$ is also.

ii) If $(\cos\theta+\sin\theta)<0$, we have $4A(\theta)-2A^2(\theta)\sin 2\theta>0$, which implies

$$A^2(\theta)+4A(\theta)-A^2(\theta)\sin 2\theta>A(\theta)+A^2(\theta)\sin 2\theta,$$

i.e.

$$A^2(\theta)+4A(\theta)-A^2(\theta)\sin 2\theta>\left(-A(\theta)(\cos\theta+\sin\theta)\right)^2,$$

then

$$\sqrt{A^2(\theta)+4A(\theta)-A^2(\theta)\sin 2\theta}>-A(\theta)(\cos\theta+\sin\theta),$$

hence

$$A(\theta)(\cos\theta+\sin\theta)+\sqrt{A^2(\theta)+4A(\theta)-A^2(\theta)\sin 2\theta}>0.$$

Therefore $r(\theta, r_0)$ is stictly positive. Finally $r(\theta, r_0)$ defines through (4) is a periodic solution. Let us show that this peiodic solution is a limit cycle. For this aim, we introduce the Poincaré return map

$$\gamma \mapsto \Pi(\gamma)=r(2\pi,\gamma)=\frac{1}{2}\left(A(2\pi)+\sqrt{A^2(2\pi)+4A(2\pi)}\right),$$

where $A(2\pi)=\exp(2\pi)\left(\dfrac{\gamma^2}{\gamma+1}+\int\limits_0^{2\pi}\dfrac{e^{-s}}{\lambda+\beta\cos 2s}ds\right)$ and show that the function of Poincare first return verify $\left.\dfrac{d\Pi(\gamma)}{d\gamma}\right|_{\gamma=r_0}\neq 1$ see [12]. We remark that

$$\sqrt{A^2(2\pi)+4A(2\pi)}=\frac{e^{\pi}}{(\gamma+1)}\sqrt{(\gamma^2+D\gamma+D)\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+4(\gamma+1)\right)},$$

with

$$D=\int_0^{2\pi}\frac{e^{-s}}{\lambda+\beta\cos 2s}\,ds.$$

We have

$$\frac{\partial}{\partial\gamma}(A(2\pi))=\frac{\gamma(\gamma+2)}{(\gamma+1)^2}e^{2\pi},$$

and

$$\frac{\partial}{\partial\gamma}\left(\frac{e^{\pi}}{(\gamma+1)}\sqrt{(\gamma^2+D\gamma+D)\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+4(\gamma+1)\right)}\right)=$$
$$=\frac{\gamma(\gamma+2)e^{\pi}}{(\gamma+1)^2}\frac{\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+2(\gamma+1)\right)}{\sqrt{(\gamma^2+D\gamma+D)\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+4(\gamma+1)\right)}},$$

consequently

$$\frac{\partial}{\partial\gamma}\left(\frac{1}{2}\left(A(2\pi)+\sqrt{A^2(2\pi)+4A(2\pi)}\right)\right)=$$
$$=\frac{1}{2}\frac{\gamma(\gamma+2)e^{\pi}}{(\gamma+1)^2}\left(\frac{\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+2(\gamma+1)\right)}{\sqrt{(\gamma^2+D\gamma+D)\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+4(\gamma+1)\right)}}+e^{\pi}\right),$$

then

$$\left.\frac{d\Pi(\gamma)}{\gamma}\right|_{\gamma=r_0}=\left.\frac{1}{2}\frac{\gamma(\gamma+2)e^{\pi}}{(\gamma+1)^2}\left(\frac{\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+2(\gamma+1)\right)}{\sqrt{(\gamma^2+D\gamma+D)\left(e^{2\pi}\left(\gamma^2+D\gamma+D\right)+4(\gamma+1)\right)}}+e^{\pi}\right)\right|_{\gamma=r_0}=$$
$$=\frac{1}{2}\frac{r_0(r_0+2)e^{\pi}}{(r_0+1)^2}\left(\frac{\left(e^{2\pi}\left(r_0^2+Dr_0+D\right)+2(r_0+1)\right)}{\sqrt{(r_0^2+Dr_0+D)\left(e^{2\pi}\left(r_0^2+Dr_0+D\right)+4(r_0+1)\right)}}+e^{\pi}\right).$$

Since $\lambda+|\beta|<\dfrac{-1}{2}$, then $0<\left(r_0^2+Dr_0+D\right)<r_0^2$ because $D<0$ and $A>0$. Therefore

$$\left(e^{2\pi}\left(r_0^2+Dr_0+D\right)+4(r_0+1)\right)<\left(e^{2\pi}r_0^2+4(r_0+1)\right)<e^{2\pi}\left(r_0^2+2r_0+1\right)=e^{2\pi}(r_0+1)^2$$

because $e^{2\pi}>4$. It follows that

$$\sqrt{(r_0^2+Dr_0+D)\left(e^{2\pi}\left(r_0^2+Dr_0+D\right)+4(r_0+1)\right)}<\sqrt{r_0^2e^{2\pi}(r_0+1)^2}=e^{\pi}r_0(r_0+1),$$

i.e. $\dfrac{1}{\sqrt{(r_0^2+Dr_0+D)\,(e^{2\pi}\,(r_0^2+Dr_0+D)+4\,(r_0+1))}} > \dfrac{1}{e^{\pi}r_0\,(r_0+1)}$, which implies

$$\frac{\left(e^{2\pi}\left(r_0^2+Dr_0+D\right)+2\,(r_0+1)\right)}{\sqrt{(r_0^2+Dr_0+D)\,(e^{2\pi}\,(r_0^2+Dr_0+D)+4\,(r_0+1))}} > \frac{2\,(r_0+1)}{e^{\pi}r_0\,(r_0+1)} = \frac{2}{e^{\pi}r_0},$$

because $\left(e^{2\pi}\left(r_0^2+Dr_0+D\right)+2\,(r_0+1)\right) > 2\,(r_0+1)$.
Consequently

$$\begin{aligned}
\left.\frac{d\Pi(\gamma)}{d\gamma}\right|_{\gamma=r_0} &> \frac{1}{2}\frac{r_0\,(r_0+2)\,e^{\pi}}{(r_0+1)^2}\left(\frac{2}{e^{\pi}r_0}+e^{\pi}\right) = \\
&= \frac{(r_0+2)}{(r_0+1)^2}+\frac{1}{2}\frac{r_0\,(r_0+2)}{(r_0+1)^2}e^{2\pi} = \\
&= \frac{(r_0+2)}{(r_0+1)^2}\left(1+\frac{e^{2\pi}}{2}r_0\right) > \left(\text{because } \frac{e^{2\pi}}{2}>1\right) \\
&> \frac{(r_0+2)}{(r_0+1)^2}\,(1+r_0) = \\
&= \frac{(r_0+2)}{(r_0+1)} > 1.
\end{aligned}$$

Hence

$$\left.\frac{d\Pi(\gamma)}{d\gamma}\right|_{\gamma=r_0} > 1.$$

Therefore the solution of differential equation (4) is unstable and hyperbolic limit cycle see [12], consequently, it is a stable and hyperbolic limit cycle for the system (2).

1) If $\beta \neq 0$, this limit cycle is non-algebraic, due to the expression of $A(\theta)$.
More precisely, in Cartesian coordinates $\left(r^2=(x-1)^2+(y-1)^2,\ \theta=\arctan\left(\frac{y-1}{x-1}\right)\right)$, the curve defined by this limit cycle is $f(x,y)=\dfrac{(x-1)^2+(y-1)^2}{xy}-B(x,y)=0$, with

$$B(x,y)=\exp\left(\arctan\left(\frac{y-1}{x-1}\right)\right)\left(\frac{e^{2\pi}}{e^{2\pi}-1}\left(\int_0^{2\pi}\frac{-e^{-s}}{\lambda+\beta\cos 2s}ds\right)+\int_0^{\arctan\frac{y-1}{x-1}}\frac{e^{-s}}{\lambda+\beta\cos 2s}ds\right).$$

There is no integer n for which both $\dfrac{\partial^n f}{\partial x^n}$ and $\dfrac{\partial^n f}{\partial y^n}$ vanish identically. To be convinced by this fact, one has compute for example $\dfrac{\partial f}{\partial y}$, that is

$$\frac{\partial f}{\partial y}(x,y)=\frac{-x^2+2x+y^2-2}{xy^2}-\frac{x-1}{(x-1)^2+(y-1)^2}\left[B(x,y)+\frac{1}{\lambda+\beta\cos\left(2\arctan\left(\frac{y-1}{x-1}\right)\right)}\right].$$

Since $B(x,y)$ appears again, it will remains in any order of derivation, therefore the curve $f(x,y)=0$ is non-algebraic and the limit cycle of the system (2) will also be non-algebraic. This complete the proof of statement 1) of Theorem 2.1.

2) If $\beta=0$, we have $\int\limits_0^{\theta}\dfrac{\exp(-s)}{\lambda}ds=\dfrac{1}{\lambda}\left(1-e^{-\theta}\right)$ and $e^{2\pi}\int\limits_0^{2\pi}\dfrac{-e^{-s}}{\lambda}ds=\dfrac{1}{\lambda}\left(1-e^{2\pi}\right)$, by simplification we obtain $r_0=\dfrac{-1}{2\lambda}\left(\sqrt{1-4\lambda}+1\right)$ and $\dfrac{r_0^2}{r_0+1}=\dfrac{-1}{\lambda}=A(\theta)$. By substituting the values of r_0 and $A(\theta)$ in (12), the solution of (4) becomes

$$r(\theta, r_0) = \frac{\frac{-1}{\lambda}(\cos\theta + \sin\theta) + \sqrt{\frac{1}{\lambda^2} - \frac{4}{\lambda} - \frac{1}{\lambda^2}\sin 2\theta}}{2 + \frac{1}{\lambda}\sin 2\theta} = \frac{(\cos\theta + \sin\theta) + \sqrt{1 - 4\lambda - \sin 2\theta}}{-(2\lambda + \sin 2\theta)}.$$

In Cartesian coordinates, the curve defined by this limit cycle is $\lambda(x-1)^2 + \lambda(y-1)^2 + xy = 0$ which is algebraic. This complete the proof of statement 2) of Theorem 2.1. □

3. Applications

In this section, we present some examples to illustrate the applicability of the our main result. In addition, a plot of phase portraits in the Poincaré disc for each example were performed showing a limit cycle in the first quadrant.

Eexample 3.1. In the system (2), we take $\lambda = -2$ and $\beta = 1$ $\left(\lambda + |\beta| = -1 < \frac{-1}{2}\right)$, we obtain

$$\begin{cases} \dot{x} = x\begin{pmatrix} -8 + 12x + 16y - 10x^2 - 18xy - 8y^2 + 4x^3 + 9x^2y \\ +9xy^2 - 3y^3 - x^4 - x^3y - x^2y^2 - 3xy^3 + 3y^4 \end{pmatrix}, \\ \dot{y} = y\begin{pmatrix} 8 - 20y - 4x^2 - 6xy + 22y^2 + 3x^3 + 5x^2y \\ +7xy^2 - 12y^3 - x^4 - x^3y - x^2y^2 - 3xy^3 + 3y^4 \end{pmatrix}, \end{cases} \quad (13)$$

which has a non-algebraic, stable and hyperbolic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_0) = \frac{A(\theta)(\cos\theta + \sin\theta) + \sqrt{A^2(\theta) + 4A(\theta) - A^2(\theta)\sin 2\theta}}{2 - A(\theta)\sin 2\theta},$$

where $A(\theta) = \exp(\theta)\left(\frac{r_0^2}{r_0+1} + \int_0^\theta \frac{\exp(-s)}{-2+\cos 2s}ds\right)$ and $r_0 \simeq 1.1877$ (Fig. 1).

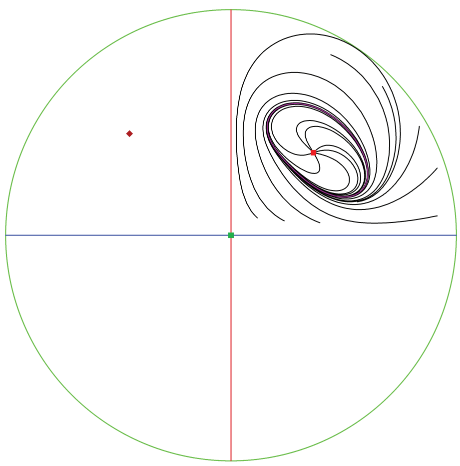

Fig. 1. The phase portrait on the Poincaré disc of the system (13), showing a limit cycles in the first quadrant

Eexample 3.2. In the system (2) , we take $\lambda = -10$ and $\beta = 0$ $\left(\lambda < \frac{-1}{2}\right)$, we obtain

$$\begin{cases} \dot{x} = x \begin{pmatrix} -40 + 80x + 60y - 80x^2 - 80xy - 20y^2 + 40x^3 + 50x^2y \\ +19xy^2 - 10y^3 - 10x^4 - 10x^3y + x^2y^2 - 10xy^3 + 10y^4 \end{pmatrix}, \\ \dot{y} = y \begin{pmatrix} 40 - 20x - 80y - 20x^2 + 80y^2 + 30x^3 + 19x^2y + 10xy^2 \\ -40y^3 - 10x^4 - 10x^3y + x^2y^2 - 10xy^3 + 10y^4 \end{pmatrix}, \end{cases} \quad (14)$$

which has an algebraic, stable and hyperbolic limit cycle given by the expression (Fig. 2).

$$-10\,(x-1)^2 - 10\,(y-1)^2 + xy = 0.$$

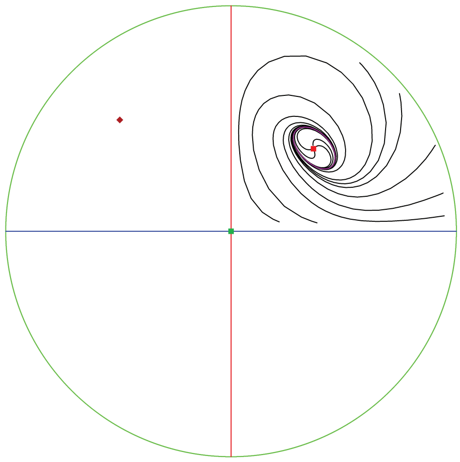

Fig. 2. The phase portrait in the Poincaré disc of the system (14), showing a limit cycle in the first quadrant

Conclusion

In this paper, a quintic Kolmogorov system with two parameters λ and β having $(1, 1)$ as positive equilibrium point was investigated. By translation the coordinates of vector $(1, 1)$ and rewritten the system in polar coordinates, we mainly shown that there is a sufficient condition for the existence of a limit cycle. Moreover, this limit cycle is non-algebraic in the case $\beta \neq 0$.

Finally, It is of interest to extend this study by answering to the following question: Is there a quartic or quintic Kolmogorov system that exhibit more than one non-algebraic limit cycle? This is left as a topic for future research.

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Класс квинтических колмогоровских систем с явным неалгебраическим предельным циклом

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Различные физические, экологические, экономические и т.д. явления перекрываются планарными дифференциальными системами. Впоследствии некоторые исследования привлекут внимание к изучению предельных циклов из-за их интереса к пониманию этих систем. Целью данной работы является исследование одного класса квинтических колмогоровских систем, а именно систем вида

$$\begin{aligned} \dot{x} &= x\ P_4\left(x,y\right), \\ \dot{y} &= y\ Q_4\left(x,y\right), \end{aligned}$$

где P_4 и Q_4 — квартичные полиномы. В этом классе наше внимание ограничено изучением предельного цикла в реалистическом квадранте $\left\{(x,y)\in\mathbb{R}^2;\ x>0,\ y>0\right\}$. Согласно гипотезам доказано существование алгебраического или неалгебраического предельного цикла. Кроме того, этот предельный цикл явно задан в полярных координатах. Некоторые примеры представлены для того, чтобы проиллюстрировать возможности применения нашего результата.

Ключевые слова: колмогоровские системы, первый интеграл, периодические орбиты, алгебраический и неалгебраический предельные циклы.