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A Sufficient Condition for Absolute Continuity of Conjugations between Interval Exchange Maps

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A class of topological equivalent generalized interval exchange maps of genus one and of the same bounded combinatorics is considered in the paper. A sufficient condition for absolute continuity of the conjugation between two maps from this class is provided.

Keywords: conjugate map, interval exchange map, Rauzy-Veech induction, renormalization, dynamical partition, martingale.

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1. Introduction and preliminaries

The study of ergodic properties of interval exchange maps (i.e.m.) is a classical problem of dynamical systems. In recent years interest in this topic has been renewed (see [1–6]).

A standard i.e.m. f on an finite length interval I is a one-to-one map which is locally translation except at a finite number of discontinuities. The generalized i.e.m. obtained by replacing linear mappings of subintervals with a locally orientation-preserving homeomorphism. Let d be the number of intervals of continuity of f . When $d = 2$ standard i.e.m. correspond to linear rotations of the circle, and generalized i.e.m. correspond to homeomorphisms of the circle with two break points. Generalized interval exchange maps were introduced [5]. It was showed that sufficiently smooth generalized i.e.m. of a certain combinatorial type (deformations of standard interval exchange transformations and tangent to them at the points of discontinuities) are smoothly linearizable. Considering piecewise $C^{2+\varepsilon}$ -smooth circle homeomorphisms as generalized interval exchange maps of genus one, Cunha and Smania [3] showed that two generalized i.e.m. with the same bounded-type combinatorics and zero mean nonlinearities C^1 -smoothly conjugate to each other. In the case of circle maps it was shown that for almost all rotation numbers every two $C^{2+\varepsilon}$ -smooth circle homeomorphisms with a break point, with the same irrational rotation number and the same size of the break are C^1 -smoothly conjugate to each other [7–9]. Let us note that statement on the regularity of conjugating map can be obtained by using the convergence of renormalizations of given maps.

The lower bound of the scale of smoothness for a homeomorphism f , that is, Df is absolutely continuous and $D \log Df \in \mathbb{L}_p$ for some $p > 1$ was considered [10,11]. The latter conditions on smoothness of f are called the Katznelson and Ornstein (KO, for short) smoothness condition [12]. For this low smoothness case in it was shown that the Rauzy-Veech renormalizations of two piecewise KO-smooth maps satisfying certain combinatorial assumptions approach to each other in C^{1+L_1} -norm [11]. Let us note that the KO smoothness condition is smaller than $C^{2+\nu}$ smoothness but the obtained convergence rate is slower than exponential [3, 7]. However, in

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this case one can expect the absolute continuity (not C^1 rigidity) of conjugate map between two generalized i.e.m.. Our objective is to obtain a sufficient condition for conjugate map to be absolutely continuous.

Let us define a class of generalized interval exchange maps. Let \mathbb{B}^{KO} be the set of g.i.e.m. $f : I \rightarrow I$ such that

- (i) the map f has cyclic permutation;
- (ii) the map f has no connection and has k - bounded combinatorics.
- (iii) on each intervals of continuity of f the map f satisfies the Katznelson and Ornstein smoothness condition: Df is absolutely continuous and $D \ln Df \in L_p$ for some $p > 1$.

Conditions (i) and (ii) are explained in Section 2. Let us note that the class \mathbb{B}^{KO} consists of circle homeomorphisms with several break points and with irrational rotation number of bounded type.

Two g.i.e.m. $f_1, f_2 \in \mathbb{B}^{KO}$ with the same combinatorics are called break-equivalent if the following conditions hold true:

- (a) the break points of one map u_i are mapped into the break points of the other map v_i by a topological conjugacy h satisfying $f_2 = h^{-1} \circ f_1 \circ h$, i.e., $u_i = h(v_i)$;
- (b) the corresponding sizes of breaks $c_i = \sqrt{f_1'(u_i - 0)/f_1'(u_i + 0)}$, $\tilde{c}_i = \sqrt{f_2'(v_i - 0)/f_2'(v_i + 0)}$ are the same for each $i = 1, \dots, k$.

Let f_1 and f_2 be break-equivalent g.i.e.m. of class \mathbb{B}^{KO} . Let us consider dynamical partitions $\xi_n(f_1)$ and $\xi_n(f_2)$ generated by maps f_1 and f_2 . Let h be a conjugation homeomorphism between f_1 and f_2 , that is, $h \circ f_1 = f_2 \circ h$. Let us assume that $\Delta^{(n)}$ are elements of the partition of $\xi_n(f_2)$. Since h is a conjugate map between f_1 and f_2 for any $L^{(n)} \in \xi_n(f_1)$, we have $h(L^{(n)}) = \Delta^{(n)}$ and $\Delta^{(n)} \in \xi_n(f_2)$. The main result is the following

Theorem 1.1. *Let f_1 and f_2 be break-equivalent g.i.e.m. of class \mathbb{B}^{KO} . Suppose that there exist a sequence δ_n with $\sum_{n=1}^{\infty} \delta_n^2 < \infty$ such that*

$$\left| \frac{|h(L^{(n)})|}{|h(R^{(n)})|} - \frac{|L^{(n)}|}{|R^{(n)}|} \right| \leq \delta_n, \tag{1}$$

for each pair of adjacent intervals $L^{(n)}, R^{(n)} \in \xi_n(f_1)$. Then the conjugate map h is absolutely continuous function.

The structure of the paper is as follows. In Section 2 we present necessary definitions on interval exchange maps and define a renormalization map related to Rauzy-Veech induction. In Section 3 we define a sequence of dynamical partition associated with renormalization map and formulate statements on asymptotic lengths of the elements of dynamical partition. Finally, in Section 4 we define a martingale, and using its properties, prove our main theorem.

2. Background on the interval exchange maps

In this section we describe combinatorial assumptions on the class \mathbb{B}^{KO} . Let I be an open bounded interval and \mathcal{A} be an alphabet with $d \geq 2$ symbols. Let us consider the partition of I into d subintervals indexed by \mathcal{A} , that is, $\mathcal{P} = \{I_\alpha, \alpha \in \mathcal{A}\}$. Let $f : I \rightarrow I$ be a bijection. We say that the triple $(f, \mathcal{A}, \mathcal{P})$ is a *generalized interval exchange map* with d intervals (for short g.i.e.m.) if $f|_{I_\alpha}$ is an orientation-preserving homeomorphism for all $\alpha \in \mathcal{A}$. If $f|_{I_\alpha}$ is a translation then f is called a *standard interval exchange map* (for short s.i.e.m.).

Let $f : I \rightarrow I$ be a g.i.e.m. with alphabet \mathcal{A} and $\pi_0, \pi_1 : \mathcal{A} \rightarrow \{1, \dots, d\}$, be bijections such that $\pi_0(\alpha) < \pi_0(\beta)$ iff $I_\alpha < I_\beta$ and $\pi_1(\alpha) < \pi_1(\beta)$ if $f(I_\alpha) < f(I_\beta)$.

Pair $\pi = (\pi_0, \pi_1)$ is called the *combinatorial data* associated with g.i.e.m. f . The notation $\pi = (\pi(1), \pi(2), \dots, \pi(d))$ is also used for the combinatorial data of f . It is always assumed that pair $\pi = (\pi_0, \pi_1)$ is *irreducible*, that is, for all $j \in \{1, \dots, d-1\}$ we have $\pi_0^{-1}(1, \dots, j) \neq \pi_1^{-1}(1, \dots, j)$. It is assumed that g.i.e.m. f has *cyclic permutation* if $\pi_0(\{1, 2, \dots, d\}) = \{j+1, \dots, d, 1, \dots, j\}$ for some $1 \leq j \leq d-1$.

Let $\pi = (\pi_0, \pi_1)$ be the combinatorial data associated with g.i.e.m f . For each $\varepsilon \in \{0, 1\}$ the last symbol in the expression of π_ε is denoted by $\alpha(\varepsilon) = \pi_\varepsilon^{-1}(d)$.

Let us assume that intervals $I_{\alpha(0)}$ and $f(I_{\alpha(1)})$ have different lengths. Then g.i.e.m. f is called *Rauzy-Veech renormalizable*(renormalizable, for short). If $|I_{\alpha(0)}| > |f(I_{\alpha(1)})|$ then f is renormalizable of *type 0*. When $|I_{\alpha(0)}| < |f(I_{\alpha(1)})|$ then f is renormalizable of *type 1*. In either case, the letter corresponding to the largest of these intervals is called *winner* and the letter corresponding to the shortest of these intervals is called *loser* of π . Let $I^{(1)}$ be the subinterval of I obtained by removing the loser, that is, the shortest of these two intervals:

$$I^{(1)} = \begin{cases} I \setminus f(I_{\alpha(1)}), & \text{if type 0,} \\ I \setminus I_{\alpha(0)}, & \text{if type 1.} \end{cases}$$

Since the loser is the last subinterval on the right of I , the intervals I and $I^{(1)}$ have the same left endpoint.

The *Rauzy-Veech induction* of f is the first return map $R(f)$ to the subinterval $I^{(1)}$. Let us assume that $R(f)$ is again g.i.e.m. with the same alphabet \mathcal{A} . For this one need to associate with this map an \mathcal{A} -indexed partition of its domain. Subintervals of $I^{(1)}$ are denoted by $I_\alpha^{(1)}$. Let f be renormalizable of type 0. Then interval $I^{(1)} = I \setminus f(I_{\alpha(1)})$ is the domain of $R(f)$ and we have

$$I_\alpha^{(1)} = \begin{cases} I_\alpha, & \text{for } \alpha \neq \alpha(0), \\ I_{\alpha(0)} \setminus f(I_{\alpha(1)}), & \text{for } \alpha = \alpha(0). \end{cases} \tag{2}$$

These intervals form a partition of the interval $I^{(1)}$ and they are denoted by $\mathcal{P}^{(1)} = \{I_\alpha^{(1)}, \alpha \in \mathcal{A}\}$. Since $f(I_{\alpha(1)})$ is the last interval on the right of $f(\mathcal{P})$ we have $f(I_\alpha^{(1)}) \subset I^{(1)}$ for every $\alpha \neq \alpha(1)$. This means that $R(f) := f$ is restricted to these $I_\alpha^{(1)}$. On the other hand, because $I_{\alpha(1)}^{(1)} = I_{\alpha(1)}$ we have

$$f\left(I_{\alpha(1)}^{(1)}\right) = f\left(I_{\alpha(1)}\right) \subset I_{\alpha(0)}, \text{ and so } f^2\left(I_{\alpha(1)}^{(1)}\right) \subset f\left(I_{\alpha(0)}\right) \subset I^{(1)}.$$

Then $R(f) := f^2$ is restricted to $I_{\alpha(1)}^{(1)}$. Thus

$$R(f)(x) = \begin{cases} f(x), & \text{if } x \in I_\alpha^{(1)} \text{ and } \alpha \neq \alpha(1), \\ f^2(x), & \text{if } x \in I_{\alpha(1)}^{(1)}. \end{cases} \tag{3}$$

If f is renormalizable of type 1 then interval $I^{(1)} = I \setminus I_{\alpha(0)}$ is the domain of $R(f)$, we have

$$I_\alpha^{(1)} = \begin{cases} I_\alpha, & \text{for } \alpha \neq \alpha(0), \alpha(1), \\ f^{-1}(I_{\alpha(0)}), & \text{for } \alpha = \alpha(0), \\ I_{\alpha(1)} \setminus f^{-1}(I_{\alpha(0)}), & \text{for } \alpha = \alpha(1). \end{cases} \tag{4}$$

Then $f\left(I_\alpha^{(1)}\right) \subset I^{(1)}$ for every $\alpha \neq \alpha(0)$, and so $R(f) = f$ is restricted to these $I_\alpha^{(1)}$. On the other hand,

$$f^2\left(I_{\alpha(0)}^{(1)}\right) = f\left(I_{\alpha(0)}\right) \subset I^{(1)},$$

and $R(f) = f^2$ is restricted to $I_{\alpha(0)}^{(1)}$. Thus,

$$R(f)(x) = \begin{cases} f(x), & \text{if } x \in I_{\alpha}^{(1)} \text{ and } \alpha \neq \alpha(0), \\ f^2(x), & \text{if } x \in I_{\alpha(0)}^{(1)}. \end{cases} \quad (5)$$

It is easy to see that $R(f)$ is a bijection on $I^{(1)}$ and an orientation-preserving homeomorphisms on each $I_{\alpha}^{(1)}$. Moreover, the alphabet \mathcal{A} for f and $R(f)$ is the same.

The triple $(R(f), \mathcal{A}, \mathcal{P}^1)$ is called the *Rauzy-Veech renormalization* of f . If f is renormalizable of type $\varepsilon \in \{0, 1\}$ then combinatorial data $\pi^1 = (\pi_0^1, \pi_1^1)$ of $R(f)$ are given by

$$\pi_{\varepsilon}^1 := \pi_{\varepsilon}, \quad \text{and} \quad \pi_{1-\varepsilon}^1(\alpha) = \begin{cases} \pi_{1-\varepsilon}(\alpha), & \text{if } \pi_{1-\varepsilon}(\alpha) \leq \pi_{1-\varepsilon}(\alpha(\varepsilon)), \\ \pi_{1-\varepsilon}(\alpha) + 1, & \text{if } \pi_{1-\varepsilon}(\alpha(\varepsilon)) < \pi_{1-\varepsilon}(\alpha) < d, \\ \pi_{1-\varepsilon}(\alpha(\varepsilon)) + 1, & \text{if } \pi_{1-\varepsilon}(\alpha) = d. \end{cases}$$

We say that g.i.e.m. f is *infinitely renormalizable* if $R^n(f)$ is well defined for every $n \in \mathbb{N}$. Let $I^{(n)}$ be the domain of $R^n(f)$. It is clear that $R^n(f)$ is the first return map for f to the interval $I^{(n)}$. Similarly, $R^n(f)^{-1} = R^n(f^{-1})$ is the first return map for f to the interval $I^{(n)}$.

For every interval of the form $J = [a, b)$ we put $\partial J := \{a\}$.

Definition 1. We say that g.i.e.m. f has no connection if

$$f^m(\partial I_{\alpha}) \neq \partial I_{\beta}, \quad \text{for all } m \geq 1 \text{ and } \alpha, \beta \in \mathcal{A} \quad \text{with } \pi_0(\beta) \neq 1. \quad (6)$$

It is clear that $f(\partial I_{\alpha}) = \partial I_{\beta}$ for $\alpha = \pi_1^{-1}(1)$ in the case $\pi_0(\beta) = 1$. Condition (6) is called the *Keane condition*. Keane [13] showed that no connection condition is a necessary and sufficient condition for f to be infinitely renormalizable. Condition (6) means that the orbits of the left end point of subintervals I_{α} , $\alpha \in \mathcal{A}$ are disjoint whenever they can be.

Let ε_n be the type of the n -th renormalization and let $\alpha_n(\varepsilon_n)$ be the winner and $\alpha_n(1 - \varepsilon_n)$ be the loser of the n -th renormalization.

Definition 2. We say that g.i.e.m. f has k -bounded combinatorics if for each $n \in \mathbb{N}$ and $\beta, \gamma \in \mathcal{A}$ there exist $n_1, p \geq 0$ with $|n - n_1| < k$ and $|n - n_1 - p| < k$ such that

$$\alpha_{n_1}(\varepsilon_{n_1}) = \beta, \quad \alpha_{n_1+p}(1 - \varepsilon_{n_1+p}) = \gamma, \quad \text{and} \\ \alpha_{n_1+i}(1 - \varepsilon_{n_1+p}) = \alpha_{n_1+i+1}(\varepsilon_{n_1+i}), \quad \text{for every } 0 \leq i < p.$$

We say that g.i.e.m. $f : I \rightarrow I$ has *genus one* (or belongs to the *rotation class*) if f has at most two discontinuities. Let us note that every g.i.e.m. with either two or three intervals has genus one, and the genus of g.i.e.m. is invariant under renormalization.

3. Dynamical partitions associated with interval exchange map

Let $(f, \mathcal{A}, \mathcal{P})$ be a g.i.e.m. with d intervals, and $\mathcal{P} = \{I_{\alpha} : \alpha \in \mathcal{A}\}$ be the initial \mathcal{A} -indexed partition of I . Interval $I = [0, 1)$ is taken for definiteness. Suppose that f is infinitely renormalizable. Let $I^{(n)}$ be the domain of $R^n(f)$. Let us note that $I^{(n)}$ is the nested sequence of subintervals, and it has the same left endpoint of I . We want to construct the dynamical partition of I associated with the domain of $R^n(f)$.

As mentioned above, $R(f)$ is g.i.e.m. with d intervals and intervals $I_{\alpha}^{(1)}$ generate an \mathcal{A} -indexed partition of $I^{(1)}$ denoted by \mathcal{P}^1 . One can verify by induction that $R^n(f)$ is g.i.e.m. with

d intervals. Let $\mathcal{P}^n = \{I_\alpha^{(n)} : \alpha \in \mathcal{A}\}$ be the \mathcal{A} -indexed partition of $I^{(n)}$ generated by $R^n(f)$. \mathcal{P}^n is called the *fundamental partition* and $I_\alpha^{(n)}$ is called the *fundamental segments* of rank n .

Since $R^n(f)$ is the first return map for f to the interval $I^{(n)}$, each fundamental segment $I_\alpha^{(n)} \in \mathcal{P}^n$ returns to $I^{(n)}$ under certain iterates of the map f . Prior to returning, these intervals are in the interval $I \setminus I^{(n)}$ for some time. Consequently the system of intervals (their interiors are mutually disjoint)

$$\xi_n = \left\{ f^i(I_\alpha^{(n)}), 0 \leq i \leq q_\alpha^n - 1, \alpha \in \mathcal{A} \right\}$$

covers the whole interval and forms a partition of I .

The system of intervals ξ_n is called the n -th *dynamical partition* of I . The dynamical partitions ξ_n are refined with increasing n , where $\xi_{n+1} \supset \xi_n$ means that any element of the preceding partition is a union of a number of elements of the next partition or belongs to the next partition. Let us denote by ξ_{n+1}^{pr} the system of preserved intervals of ξ_n . More precisely, if $R^n f$ has type 0

$$\xi_{n+1}^{pr} = \left\{ f^i(I_\alpha^{(n)}), 0 \leq i \leq q_\alpha^n - 1, \text{ for } \alpha \neq \alpha(0) \right\},$$

and if $R^n f$ has type 1

$$\xi_{n+1}^{pr} = \left\{ f^i(I_\alpha^{(n)}), 0 \leq i \leq q_\alpha^n - 1, \text{ for } \alpha \neq \alpha(1) \right\}.$$

Let $\xi_{n+1}^{tn} := \xi_{n+1} \setminus \xi_{n+1}^{pr}$ be the set of elements of ξ_{n+1} which are properly contained in some element of ξ_n . Therefore, if $R^n f$ has type 0

$$\begin{aligned} \xi_{n+1}^{tn} &= \left\{ f^i(I_{\alpha(0)}^{(n+1)}), 0 \leq i < q_{\alpha(0)}^n \right\} \cup \left\{ f^i(I_{\alpha(1)}^{(n+1)}), 0 \leq i < q_{\alpha(0)}^n \right\} = \\ &= \bigcup_{i=0}^{q_{\alpha(0)}^n - 1} \left\{ f^i \left(I_{\alpha(0)}^{(n)} \setminus f^{q_{\alpha(1)}^n} I_{\alpha(1)}^{(n)} \right) \right\} \cup \bigcup_{i=q_{\alpha(1)}^n}^{q_{\alpha(1)}^n + q_{\alpha(0)}^n - 1} \left\{ f^i(I_{\alpha(1)}^{(n)}) \right\}, \end{aligned}$$

and if $R^n f$ has type 1

$$\begin{aligned} \xi_{n+1}^{tn} &= \left\{ f^i(I_{\alpha(0)}^{(n+1)}), 0 \leq i < q_{\alpha(1)}^n \right\} \cup \left\{ f^i(I_{\alpha(1)}^{(n+1)}), 0 \leq i < q_{\alpha(1)}^n \right\} = \\ &= \bigcup_{i=0}^{q_{\alpha(1)}^n - 1} \left\{ f^i \left(f^{-q_{\alpha(1)}^n} (I_{\alpha(0)}^{(n)}) \right) \right\} \cup \bigcup_{i=0}^{q_{\alpha(1)}^n - 1} \left\{ f^i \left(I_{\alpha(1)}^{(n)} \setminus f^{-q_{\alpha(1)}^n} (I_{\alpha(0)}^{(n)}) \right) \right\}. \end{aligned}$$

So, the partition ξ_{n+1} consists of *preserving* elements of ξ_n and *images of two (new) intervals* for defining $R^{n+1}(f)$, that is, $\xi_{n+1} = \xi_{n+1}^{pr} \cup \xi_{n+1}^{tn}$. Let us also note that for the first return time q_α^n we have:

- (1) if $\alpha = \alpha^n(\varepsilon)$, then $q_{\alpha^n(\varepsilon)}^{n+1} = q_{\alpha^n(\varepsilon)}^n$;
- (2) if $\alpha = \alpha^n(1 - \varepsilon)$, then $q_{\alpha^n(1-\varepsilon)}^{n+1} = q_{\alpha^n(1-\varepsilon)}^n + q_{\alpha^n(\varepsilon)}^n$.

Bounded geometry. Let us denote the set of g.i.e.m $f : I \rightarrow I$ by \mathbb{B}^{1+bv} . It satisfies conditions (i) – (ii) which are piecewise C^1 -smooth and have bounded variation of the first derivative.

The norm of the dynamical partition ξ_n is $\|\xi_n\| = \max\{|f^i(I_\alpha^n)|\}$, where the maximum is taken over all $\alpha \in \mathcal{A}$ and $0 \leq i \leq q_\alpha^n - 1$.

Lemma 1. (see [2]) Let $f \in \mathbb{B}^{1+bv}$. Then for sufficiently large n there is $\lambda \in (0, 1)$ such that $\|\xi_{n+k}\| \leq \lambda \|\xi_n\|$.

The following corollary follows from Lemma 1 and states that intervals of the dynamical partition ξ_n have exponentially small length.

Corollary 1. *Let $f \in \mathbb{B}^{1+bv}$. Then for sufficiently large n and m with $m - n > k$ there is $\lambda \in (0, 1)$ such that*

$$\|\xi_n\| \leq \lambda^{\frac{n}{k}-1} \quad \text{and} \quad \|\xi_m\| \leq \lambda^{\frac{m-n}{k}-1} \|\xi_n\|. \tag{7}$$

We need the following lemma which can be easily verified.

Lemma 2. *For given $a, b, c, d > 0$ the following inequalities hold*

$$\min \left\{ \frac{a}{b}, \frac{c}{d} \right\} \leq \frac{a+c}{b+d} \leq \max \left\{ \frac{a}{b}, \frac{c}{d} \right\}.$$

Now the well-known Doob's theorem on martingales is formulated.

Doob's Theorem ([14]). *Let $\{\Phi_n\}$ be a sequence of random variables on a probability space (X, F, P) . If $\sup E|\Phi_n|^p < \infty$ for some $p > 1$ and $\{\Phi_n\}$ is a martingale then there exists an integrable $\Phi \in L_1(X, F)$ such that*

$$\lim_{n \rightarrow \infty} \Phi_n = \Phi \text{ (a.e. P), } \quad \Phi_n \rightarrow \Phi \text{ in } L_1 \text{-norm.}$$

4. Proof of the main result

Let us consider the dynamical partition $\xi_n(f_1)$. For simplicity $L^{(n)}$ is used to denote an interval $f^i(I_\alpha^{(n)})$ of the dynamical partition $\xi_n(f_1)$. Let us define a sequence of random variables $\{\Phi_n\}$ on the interval I as

$$\Phi_n(x) = \frac{|h(L^{(n)})|}{|L^{(n)}|}, \quad \text{if } x \in L^{(n)}. \tag{8}$$

Lemma 3. *The sequence of piecewise functions $\{\Phi_n(x), n \geq 1\}$ generates a finite martingale with respect to the dynamical partition ξ_n .*

Proof. Since conjugate map h is a homeomorphism $\Phi_n(x)$ is a step function which takes constant values on each element $L^{(n)}$ of the partition ξ_n . Then $\Phi_n(x)$ is ξ_n -measurable. Therefore, suffice it to show that

$$E(\Phi_{n+1}/\xi_n) = \Phi_n, \quad \text{for all } n \geq 1,$$

where $E(\Phi_{n+1}/\xi_n)$ is a conditional expectation of the random variable Φ_{n+1} with respect to the partition ξ_n . Let us denote the indicator function of the interval $f^i(I_\alpha^{(n)})$ by $X_{\alpha,i}^{(n)}(x)$. By definition of conditional expectation with respect to the partition we have

$$E(\Phi_{n+1}/\xi_n) = \sum_{\alpha \in \mathcal{A}} \sum_{i=0}^{q_n^\alpha-1} E\left(\Phi_{n+1}/f^i(I_\alpha^{(n)})\right) X_{\alpha,i}^{(n)}(x). \tag{9}$$

Let us recall that the partition ξ_{n+1} consists of the *preserving* elements of ξ_n and *images of two (new) intervals* for defining $R^{n+1}(f)$, that is, $\xi_{n+1} = \xi_{n+1}^{pr} \cup \xi_{n+1}^{tn}$. Splitting sum (9) in two sums corresponding to ξ_{n+1}^{pr} and ξ_{n+1}^{tn} , we obtain

$$E(\Phi_{n+1}/\xi_n) = \sum_{J_i \in \xi_{n+1}^{pr}} E(\Phi_{n+1}/J_i) X_{\alpha,i}^{(n)}(x) + \sum_{J_i \in \xi_{n+1}^{tn}} E(\Phi_{n+1}/J_i) X_{\alpha,i}^{(n)}(x). \tag{10}$$

where $J_i = f^i(I_\alpha^{(n)})$. Consider the first sum that corresponds to ξ_{n+1}^{pr} in (10). Then

$$E(\Phi_{n+1}/J_i) = \frac{1}{|f^i(I_\alpha^{(n)})|} \int_{f^i(I_\alpha^{(n)})} \Phi_{n+1}(x) dx = \frac{1}{|f^i(I_\alpha^{(n)})|} \int_{f^i(I_\alpha^{(n)})} \Phi_n(x) dx. \tag{11}$$

Next we consider the sum that corresponds to ξ_{n+1}^{tn} in (10). Let $J_i := \bigcup I_\alpha^{(n+1)}$, where $I_\alpha^{(n+1)} \in \xi_{n+1}^{tn}$. Then we obtain

$$\begin{aligned} E(\Phi_{n+1}/J_i) &= \frac{1}{|f^i(I_\alpha^{(n)})|} \int_{f^i(I_\alpha^{(n)})} \Phi_{n+1}(x) dx = \\ &= \frac{1}{|f^i(I_\alpha^{(n)})|} \sum_{I_\alpha^{(n+1)} \in J_i} \int_{I_\alpha^{(n+1)}} \Phi_{n+1}(x) dx = \frac{1}{|f^i(I_\alpha^{(n)})|} \int_{f^i(I_\alpha^{(n)})} \Phi_n(x) dx. \end{aligned}$$

This and equations in (10), (11) imply the result. \square

Let us introduce $\Theta_n(x) = \Phi_n(x) - \Phi_{n-1}(x)$, $n \geq 1$, and put $\Phi_0(x) := 0$, $x \in I$.

Lemma 4. *For all $n \geq 1$ the following inequality holds*

$$|\Theta_n(x)| \leq \delta_n |\Phi_n(x)|, \quad x \in I, \quad \text{with } \delta_n \in l_2. \quad (12)$$

Proof. It is clear that

$$|\Theta_n(x)| = |\Phi_n(x)| \cdot \left| \frac{|\Phi_n(x)|}{|\Phi_{n-1}(x)|} - 1 \right|.$$

Let $\Phi_n(L^{(n)}) := \Phi_n(x)$, $x \in L^{(n)}$. Then we have

$$|L^{(n-1)}| \cdot \Phi_{n-1}(L^{(n-1)}) = \sum_{s=1}^{k_n} |L^{(n)}(s)| \cdot \Phi_n(L^{(n)}(s)), \quad (13)$$

where $L^{(n)}(s) \subset L^{(n-1)}$. Using Lemma 2, we obtain

$$\min \Phi_n(L^{(n)}(s)) \leq \Phi_{n-1}(L^{(n-1)}) \leq \max \Phi_n(L^{(n)}(s)). \quad (14)$$

It is clear that for any $0 \leq s \leq k_n$ the following is true

$$\frac{\min \Phi_n(L^{(n)}(s))}{\max \Phi_n(L^{(n)}(s))} \leq \frac{\Phi_{n-1}(L^{(n-1)})}{\Phi_n(L^{(n)}(s))} \leq \frac{\max \Phi_n(L^{(n)}(s))}{\min \Phi_n(L^{(n)}(s))}. \quad (15)$$

Since each pair of adjacent intervals of $\xi_n(f_1)$ are comparable, by assumption of Theorem 1.1 we obtain

$$\left| \frac{\Phi_n(L^{(n)}(s+1))}{\Phi_n(L^{(n)}(s))} - 1 \right| \leq \delta_n. \quad (16)$$

Hence, our map has bounded combinatorics we have

$$\frac{\max \Phi_n(L^{(n)}(s))}{\min \Phi_n(L^{(n)}(s))} \leq (1 + C_2 \delta_n)^K \leq 1 + C_4 \delta_n.$$

One can show that lower bound holds true for the ratio $\min \Phi_n(L^{(n)}(s)) : \max \Phi_n(L^{(n)}(s))$. Then, for all $0 \leq s \leq k_n$, we have

$$1 - C_4 \delta_n \leq \frac{\Phi_{n-1}(L^{(n-1)})}{\Phi_n(L^{(n)}(s))} \leq 1 - C_4 \delta_n. \quad (17)$$

This completes the proof of Lemma 4.

Proof of Theorem 1.1. Note that sequence $\{\Phi_n\}$ of random variables is a martingale with respect to $\xi_n(f_1)$ by lemma 3. Let us show that Φ_n converges to Dh in \mathbb{L}_1 -norm as $n \rightarrow \infty$. One

can show that $\Theta_n(x)$ and $\Phi_{n-1}(x)$ are orthogonal, that is, $\int_I \Theta_n(x)\Phi_{n-1}(x)dx = 0$. Then using Lemma 4, we have

$$\|\Phi_n\|_{L_2}^2 \leq \|\Phi_{n-1}\|_{L_2}^2 + \|\Theta_n\|_{L_2}^2 \leq (1 + C_4\delta_n^2)\|\Phi_{n-1}\|_{L_2}^2. \quad (18)$$

Iterating the the last relation, we have $\|\Phi_n\|_{L_2}^2 \leq \prod_{j=1}^n (1 + C_4\delta_j^2)$. Since the series $\sum_{j=1}^{\infty} \delta_j^2$ converges then the sequence of random variables $\{\Phi_n\}$ is bounded in L_2 norm. Doob's theorem implies that the sequence $\{\Phi_n\}$ converges to some function Φ in L_1 norm.

Next we prove that the sequence of random variables $\{\Phi_n\}$ converges to Dh . Let us denote the left end point of the interval $L^{(n)}$ by γ_n . By definition of Φ_n we have

$$\left| h(x) - \int_0^x \Phi_n(x)dx \right| \leq |h(x) - h(\gamma_n)| + \frac{|h(L^{(n)})|}{|L^{(n)}|} |x - \gamma_n| \leq 2|h(L^{(n)})|.$$

Using the last inequality, we obtain

$$\begin{aligned} \left| h(x) - \int_0^x \Phi(x)dx \right| &\leq \left| h(x) - \int_0^x \Phi_n(x)dx \right| + \\ &+ \int_0^x |\Phi(x) - \Phi_n(x)|dx \leq 2|h(L^{(n)})| + \|\Phi_n - \Phi\|_{L_1}. \end{aligned} \quad (19)$$

In the limit $n \rightarrow \infty$, we obtain $h(x) = \int_0^x \Phi(x)dx$. Since, $\Phi \in L_1(I)$ then h is an absolutely continuous function and $Dh(x) = \Phi(x)$ almost everywhere on I . Theorem 1.1 is completely proved. \square

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Достаточное условие абсолютной непрерывности сопряжений между перекладываниями интервала

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Рассмотрен класс топологически эквивалентных обобщенных перекладываний интервала первого рода с одинаковой ограниченной комбинаторикой. В статье приведено достаточное условие абсолютной непрерывности сопряжения между двумя отображениями из этого класса.

Ключевые слова: сопрягающий гомеоморфизм, перекладывание интервала, индукция Раузи-Вича, ренормализация, динамическое разбиение, мартингал.