Journal of Siberian Federal University. Mathematics & Physics 2019, 12(2), 185–190

УДК 519.21

Symmetries of Differential Ideals and Differential Equations

Oleg V. Kaptsov*
Institute of Computational Modelling SD RAS
Academgorodok, 50/44, Krasnoyarsk, 660036
Russia

Received 28.12.2018, received in revised form 11.01.2019, accepted 06.02.2019

The paper deals with differential rings and partial differential equations with coefficients in some algebra. We introduce symmetries and the conservation laws to the differential ideal of an arbitrary differential ring. We prove that a set of symmetries of an ideal forms a Lie ring and give a precise criterion when a differentiation is a symmetry of an ideal. These concepts are applied to partial differential equations.

Keywords: differential rings, symmetry, partial differential equations.
DOI: 10.17516/1997-1397-2018-12-2-185-190.

Introduction

Symmetries of various mathematical structures have been actively studied since the 19th century. A powerful impetus to this was given by the works of Évariste Galois, Felix Klein and Sophus Lie. The Lie's researches [1], devoted to the symmetries of differential equations, was developed in the works of Ovsyannikov [2] and his followers [3, 4]. More than 40 years ago, there was great interest in higher symmetries or Lie-Bäcklund symmetries [3, 5, 6]. Higher symmetries, apparently, first appeared in the famous work of Noether about the conservation laws [7], and then reappeared among different authors. These symmetries are used to classify nonlinear evolutionary and wave equations [3]. However, the notion of higher symmetries is introduced either not quite strictly or too difficult and long. At the same time, using simple language of differential algebra, this can be done very briefly.

In this paper, we introduce the concepts of symmetries and conservation laws for the differential ideals of commutative rings with several differentiations. The definition of a symmetry includes conditions of contact and invariance. A criterion is obtained when a differentiation of a ring is a symmetry of an ideal. It is shown that the set of symmetries of an ideal forms a Lie ring. We consider polynomial systems of partial differential equations with coefficients in a certain ring, define symmetries of these systems using the previous constructions and derive condition that a differentiation is contact.

1. Symmetries of Differential Ideals

We assume throughout that all rings are commutative unless otherwise specified. Recall that an operator d on a ring A is called a derivation operator (or differentiation) if $d(a+b) = d(a)+d(b)$ and $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$.

Definition. *A ring A with a finite set $\Delta = \{\partial_1, \dots, \partial_n\}$ of mutually commuting derivation operators on A is called a differential ring, and denoted by $\langle A, \Delta\rangle$.*

*kaptsov@icm.krasn.ru

Obviously, any linear combination of derivation operators on A

$$a_1\partial_1 + \cdots + a_n\partial_n, \qquad \partial_i \in \Delta,\ a_i \in A$$

is a differentiation of A. These linear combinations form a left A-module, which will be denoted by $\mathcal{M}_\Delta$.

Example. Let $A = C^\infty(\mathbb{R}^n)$ be a ring of infinitely differentiable functions of n real variables $x_l, ..., x_n$. Then A can be considered as a differential ring with the set of derivation operators $\Delta = \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$.

In the ring $\langle A, \Delta\rangle$ there can exist differentiations not belonging to $\mathcal{M}_\Delta$. The set of all derivation operators on A is denoted by $Der(A)$. Recall that the bracket operation $D_1D_2 - D_2D_1$ of differentiations $D_1, D_2 \in Der(A)$ is denoted by $[D_1, D_2]$.

It is known [8] that $Der(A)$ is a Lie ring, that is, the following conditions hold

$$[D_1, D_2] = -[D_2, D_1], \qquad [[D_1, D_2], D_3] + [[D_2, D_3], D_1] + [[D_3, D_1], D_2] = 0 \tag{1}$$

for all $D_1, D_2, D_3 \in Der(A)$. The second formula in (1) is the famous Jacobi identity. Any differentiation $\mathcal{D} \in Der(A)$ defines the adjoint action $ad\mathcal{D} : Der(A) \longrightarrow Der(A)$ by the formula $ad\mathcal{D}(D) = [\mathcal{D}, D]$. It follows from the Jacobi identity that the adjoint action $ad\mathcal{D}$ is a differentiation of the Lie ring $Der(A)$. Recall that an ideal I of $\langle A, \Delta\rangle$ is differential if $\partial(I) \subset I$ for all $\partial \in \Delta$.

Definition. *A differentiation $D \in Der(A)$ of a ring $\langle A, \Delta\rangle$ is called a symmetry of differential ideal I if*

$$adD(\mathcal{M}_\Delta) \subset \mathcal{M}_\Delta, \qquad D(I) \subset I. \tag{2}$$

The set of symmetries of the ideal I is denoted by $Sym(I)$.

The first condition of our definition generalizes the well known notion of the contact transformation [3,6], and so we call it the contact condition. The second condition is called the invariance condition, it means that the ideal I is invariant under action of the operator D.

Denote by ∂^α the multiplication $\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$. Say that a set $F \subset I$ generates a differential ideal I if any element of I is a sum of terms $g\partial^\alpha(f)$ with $g \in A$, $f \in F$. The elements of F are called the generators of I.

Lemma 1. *Let F generate a differential ideal I of $\langle A, \Delta\rangle$. A differentiation $D \in Der(A)$ is a symmetry of the ideal I if and only if*

$$adD(\partial_i) \in \mathcal{M}_\Delta \ \ \forall \partial_i \in \Delta, \qquad \textit{and} \qquad D(f) \in I \ \ \forall f \in F. \tag{3}$$

Proof. Suppose the first condition in (3) is satisfied. For any $b \in A$ and for any $\partial \in \Delta$ evidently

$$adD(b\partial) = b[D, \partial] + D(b)\partial.$$

If we take any element $\partial = \sum_{i=1}^n a_i\partial_i$ in $\mathcal{M}_\Delta$, then it follows from the definition of derivation operator that

$$adD\left(\sum_{i=1}^n a_i\partial_i\right) = \sum_{i=1}^n a_i adD(\partial_i) + \sum_{i=1}^n D(a_i)\partial_i.$$

Each term on the right-hand side of the last equality lies in $\mathcal{M}_\Delta$ by the condition of our Lemma.

Now we have to verify that the second condition in (3) leads to the second property in (2). Assume that h is in the differential ideal I generated by the set F. Then h is a sum of terms $g\partial^\alpha f$, where $g \in A$, $f \in F$. Since the equality holds

$$D(g\partial^\alpha(f)) = D(g)\partial^\alpha(f) + gD\partial^\alpha(f),$$

it suffices to prove that $D\partial^\alpha(f) \in I$. Because of $D \in Sym(I)$, there exist elements $a_i^j \in A$ such that

$$D\partial_i = \partial_i D + \sum_{j=1}^{n} a_i^j \partial_j, \quad \forall \partial_i \in \Delta.$$

Multiplying both sides by ∂_k yields

$$D\partial_i\partial_k = \partial_i D\partial_k + \sum_{j=1}^{n} a_i^j \partial_j\partial_k.$$

It follows that

$$D\partial_i\partial_k = \partial_i\partial_k D + \partial_i \left(\sum_{j=1}^{n} a_k^j \partial_j\right) + \sum_{j=1}^{n} a_i^j \partial_j\partial_k.$$

Then it is easy to see that for any $\alpha \in \mathbb{N}^n$ there exist elements $b_\alpha \in A$ such that

$$D\partial^\alpha = \partial^\alpha D + \sum_{|\beta|\leqslant|\alpha|} b_\alpha \partial^\beta.$$

Obviously each term on the right-hand side of the last formula belongs to the ideal I. The converse is trivial. □

Lemma 2. *Let I be a differential ideal of a differential ring A. Then $Sym(I)$ is a Lie ring and $\mathcal{M}_\Delta$ is its ideal.*

Proof. As noted above, $Der(A)$ is a Lie ring. It is necessary to show that if D_1, D_2 is in $Sym(I)$, then $[D_1, D_2]$ is also in $Sym(I)$, that is, $[[D_1, D_2], \partial_i] \in \mathcal{M}_\Delta$. From (1) it follows that

$$[[D_1, D_2], \partial_i] = [[\partial_i, D_2], D_1] - [[\partial_i, D_1], D_2]. \tag{4}$$

We have $[\partial_i, D_2], [\partial_i, D_1] \in \mathcal{M}_\Delta$ by the condition of this Lemma. Therefore

$$[\partial_i, D_2] = \sum_{j=1}^{n} b_j \partial_j, \qquad [\partial_i, D_1] = \sum_{j=1}^{n} c_j \partial_j, \qquad b_j, c_j \in A.$$

For any $j = 1, \ldots, n$ and $a \in A$, the following formula holds

$$[a\partial_j, D_i] = a[\partial_j, D_i] - D_i(a)\partial_j \in \mathcal{M}_\Delta,\ i = 1, 2.$$

Hence each term on the right-hand side of (4) is in $\mathcal{M}_\Delta$.

Since D_1, D_2 are symmetries, $D_1(D_2(I))$ and $D_2(D_1(I))$ are subsets of I. Thus, $[D_1, D_1](I) \subset I$. It follows from the contact condition that

$$[D, \partial] \in \mathcal{M}(\Delta) \qquad \forall D \in Sym(I)\ \forall \partial \in \mathcal{M}_\Delta,$$

and $\mathcal{M}_\Delta$ is a ideal of $Sym(I)$. □

In applications to differential equations, the ring $\langle A, \Delta\rangle$ is a commutative algebra over some field $\mathcal{F}$ with $\delta(a) = 0\ \forall\delta \in \Delta$ and $\forall c \in \mathcal{F}$. In this case, $\mathcal{F}$ is called the field of constants.

Corollary. *Let I be a differential ideal of a commutative algebra over a field of constants $\mathcal{F}$. Then $Sym(I)$ is a Lie algebra over $\mathcal{F}$.*

Remark. The ring $Sym(I)$ in the general case is not a left A-module. Assume that $a \in A, D \in Sym(I)$, $\partial_i \in \Delta$. Then we have

$$[aD, \partial_i] = a[D, \partial_i] - \partial_i(a)D.$$

Strictly speaking, $\partial_i(a) \neq 0$ и $\partial_i(a)D \notin \mathcal{M}_\Delta$. If $\partial_i(a) = 0$, then $[aD, \partial_i] \in \mathcal{M}_\Delta$.

Any derivative operator $\partial \in \mathcal{M}_\Delta$ is a symmetry of every differential ideal of a ring $\langle A, \Delta\rangle$. We will call such symmetry trivial. The factor ring $Sym(I)/\mathcal{M}_\Delta$ are called the canonical symmetry ring.

Similarly we define a conservation law of an ideal.

Definition. *Let I be a differential ideal of a ring $\langle A, \Delta\rangle$. We say that a sum*

$$\sigma = \partial_1(a_1) + \cdots + \partial_n(a_n), \qquad a_i \in A,\ \partial_i \in \Delta,\ i = 1, \ldots, n$$

is a conservation law of an ideal I if $\sigma \in I$.

2. Partial differential equations

We introduce a ring which is important for partial differential equations. Consider a ring $\langle A, \Delta\rangle$ and a denumerable set U of indeterminates u^i_α, where $\alpha \in \mathbb{N}^n$, $1 \leqslant i \leqslant m$. Let $A[U]$ be the commutative ring of polynomials over A generated by U.

We extend differentiations $\partial_i \in \Delta$ to the ring $A[U]$ by the following formulas

$$D_i(u^j_\alpha) = u^j_{\alpha+1_i}, \quad D_i(a) = \partial_i(a), \quad a \in A,$$

$$D_i(u^k_\alpha u^j_\beta) = u^k_\alpha D_i(u^j_\beta) + u^j_\beta D_i(u^k_\alpha), \quad D_i(au^j_\alpha) = D_i(a)u^j_\alpha + au^j_{\alpha+1_i},$$

where 1_i is a n-tuple of zeros and unit in the i place; $\alpha, \beta \in \mathbb{N}^n$, $1 \leqslant k \leqslant m$, $1 \leqslant j \leqslant m$.

Following Kolchin [9], we call elements of $A[u]$ differential polynomials over A, and $A[u]$ itself the differential polynomial algebra over A. An expression of the form

$$f_1 = 0,\ f_2 = 0, \ldots, f_r = 0, \quad f_i \in A[U],\ i = 1, \ldots, r. \tag{5}$$

is a polynomial system of partial differential equations. For example, we can rewrite the 3-D Laplace's equation

$$w_{xx} + w_{yy} + w_{zz} = 0$$

as $u_{(2,0,0)} + u_{(0,2,0)} + u_{(0,0,2)} = 0$.

Definition. *A differential ring $\langle B, \Delta_B\rangle$ is called an extension of $\langle A, \Delta_A\rangle$ if A is a subring of B and any $\partial \in \Delta_A$ is a restriction of some differentiation in Δ_B.*

For example, the ring of smooth functions on $\mathbb{R}^n$ is an extension of the ring $\mathcal{A}$ of analytic functions on $\mathbb{R}^n$ with $\Delta_\mathcal{A} = \left\{\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}\right\}$.

Definition. *Let B be an extension of the ring A and $f_j \in A[U]$ with $j = 1, \ldots, r$. We call a m-tuple $(v^1, \ldots, v^m) \in B^m$ a solution of the system (5) if the polynomials f_j vanish when we replace elements $D^\alpha(v^i)$ by u^i_α ($i = 1, \ldots, m$ and $\alpha \in \mathbb{N}^n$) into these polynomials.*

The differentiations $D_i \in \Delta$ of the ring $A[U]$ can be written in the form of series

$$D_i = \partial_i + \sum_{\substack{1 \leqslant j \leqslant m \\ \alpha \in \mathbb{N}^n}} u^j_{\alpha+1_i} \frac{\partial}{\partial u^j_\alpha} \quad i = 1, \ldots, n.$$

The left-hand side of a system of partial differential equations (5) generates a differential ideal, denoted by $\langle\langle f_1,\dots,f_r\rangle\rangle$. The symmetries of this system will be called the symmetries of this ideal. The question then arises: how to find symmetries of this ideal?

First of all, we have to use the contact condition. We will look for symmetry operators of the form

$$X=\sum_{i=1}^{n}\xi_i\partial_i+\sum_{\substack{1\leqslant j\leqslant m\\ \alpha\in\mathbb{N}^n}}\eta_j^{\alpha}\frac{\partial}{\partial u_\alpha^j},\quad \xi_i,\eta_j^{\alpha}\in A.$$

One can rewrite the contact condition

$$[X,D_i]=\sum_{k=1}^{n}a_i^kD_k$$

as follows:

$$\sum_{\substack{1\leqslant j\leqslant m\\ \alpha\in\mathbb{N}^n}}X(u_\alpha^j)\frac{\partial}{\partial u_\alpha^j}-\sum_{k=1}^{n}D_i(\xi_k)\partial_k-\sum_{\substack{1\leqslant j\leqslant m\\ \alpha\in\mathbb{N}^n}}D_i(\eta_j^{\alpha})\frac{\partial}{\partial u_\alpha^j}=\sum_{k=1}^{n}a_i^k\partial_k+\sum_{\substack{1\leqslant k\leqslant n\\ 1\leqslant j\leqslant m\\ \alpha\in\mathbb{N}^n}}a_i^ku_{\alpha+1_k}^j\frac{\partial}{\partial u_\alpha^j}.$$

Collecting coefficients of $\partial_1,\dots,\partial_n$, we find

$$a_1^k=-D_i(\xi_1),\dots,\quad a_n^k=-D_i(\xi_n).$$

Then we can collect coefficients of $\dfrac{\partial}{\partial u_\alpha^j}$ and get the standard formulas for η_j^{α} [2,3]. If we take the canonical symmetry ring $Sym(I)/\mathcal{M}_\Delta$, we then obtain simple formulas

$$\eta_j^{\alpha}=D^{\alpha}(\eta_j^0),\quad j=1,\dots,m,\quad \alpha\in\mathbb{N}^n,\quad 0\in\mathbb{Z}^n$$

with $\xi_1=\dots=\xi_n=0$. When we wish to find the functions η_j^0, we must use the invariance condition of the ideal $\langle\langle f_1,\dots,f_r\rangle\rangle$.

To apply these constructions to the equations of mathematical physics, you need to specify the ring A. In applications, one usually considers an algebra of smooth or analytic functions on an open set $V\subset\mathbb{R}^n$. The numerous examples of symmetries Lie algebras of partial differential equations are in [2,10].

The work was supported by the Russian Foundation for Basic Research (grant 17-01-00332-a).

References

[1] S.Lie, Theory of Transformation Groups I: General Properties of Continuous Transformation Groups. A Contemporary Approach and Translation, Springer-Verlag, Berlin, Heidelberg, 2015.

[2] L.V.Ovsyannikov, Group Analysis of Differential Equations, English transl., W.F. Ames (Ed.), Academic Press, New York, 1982.

[3] N.H.Ibragimov, Transformation Groups Applied to Mathematical Physics, Reidel, Boston, 1985.

[4] V.K Andreev, O.V.Kaptsov V.V.Pukhnachev, A.A.Rodionov, Applications of Group-Theoretical Methods in Hydrodynamics, Springer, Netherlands, 2010.

[5] P.Olver, Applications of Lie Groups to Differential Equations, NY, Springer-Verlag, 1986.

[6] G.Bluman, S.Kumei, Symmetries and Differential Equations, NY, Springer, 1989.

[7] E.Noether, Invariante Variationsprobleme, Nachr Konig, Gesell Wissen, Gottingen, Math.-Phys. Kl., 1918, 235–257.

[8] I.Kaplansky, Lie Algebras and Locally Compact Groups, University Chicago Press, 1971.

[9] E.R.Kolchin, Differential Algebra and Algebraic Groups, NY, Academic Press, 1973.

[10] CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws,. N.H. Ibragimov (Ed.), CRC Press Published, 1993.

Симметрии дифференциальных идеалов и дифференциальных уравнений

Олег В. Капцов
Институт вычислительного моделирования СО РАН
Академгородок, 50/44, Красноярск, 660036
Россия

В работе рассматриваются дифференциальные кольца и уравнения с частными производными с коэффициентами в некотором кольце. Вводятся симметрии и законы сохранения дифференциального идеала произвольного дифференциального кольца. Доказано, что множество симметрий идеала образуют кольцо Ли. Получен критерий того, что дифференцирование является симметрией идеала. Эти построения применяются к уравнениям в частных производных.

Ключевые слова: дифференциальные кольца и идеалы, инвариантность, уравнения с частными производными.