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On a Transversality Condition for One Variation Problem with Moving Boundary

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We give an algorithm to obtain a transversality condition for one variation problem with a moving boundary when a functional contains derivatives of order n of functions of one variable. A mathematical justification of the this approach is given.

Keywords: variation, curvature, moving boundary, functional, transversality condition.
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Many practical applications of analytic methods (see monographs [1–5] where the output equations describing basic physical laws are given by using variational methods, however, only for cases with fixed boundary) often involve moving boundaries, as shown by, for example, the works [6, 7]. It should be noted that study of such problems is not limited to the above sources and attracts a great interest even from the general scientific point of view.

Solution of the Problem

We shall restrict ourselves to a specific problem of calculus of variations with moving boundary, which until now has not been studied. Namely, we consider the functional

$$J(y) = \int_{x_0}^{x_1} F(x, y, y', y'', y''', \dots)dx, \tag{1}$$

where the point $M_1 = M(x_1, y_1)$ moves and the point $M_0 = M(x_0, y_0)$ is fixed. Note that such a functional has been treated in [6], however, the order of derivatives in the function under the integral sign was bounded by two there. In this note we drop this restriction.

In the classical setting for a variational problem with moving boundary [8, 9] one considers the Euler–Lagrange functional having the simplest form

$$J(y) = \int_{x_0}^{x_1} F(x, y, y')dx$$

and finds a transversality condition in the form

$$F - (y' - \varphi')F_{y'}\Big|_{x=x_1} = 0, \tag{2}$$

where the function $\varphi(x)$ is the given trajectory of the moving point M_1, and $F'_y = \dfrac{\partial F}{\partial y'}$.

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The question now is: How the condition (2) changes, if the the Euler-Lagrange functional is generalized to have the form (1) and the upper limit of integration, i.e. the point M_1 moves. We could not find an answer to this question in scientific literature and, therefore, give an answer in this note.

According to the geometry depicted on Fig. 1, let us compute the variation of the functional (1) in simplest case when

$$J(y) = \int_{x_0}^{x_1} F(x, y, y', y'')dx. \tag{3}$$

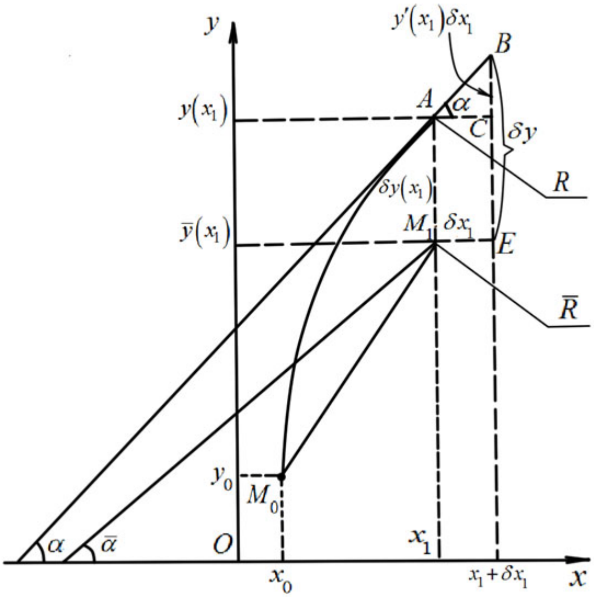

Fig. 1. The schematic representation of independent displacements δx_1 and δy_1. $CE = \delta y(x_1)$, $BC = y'(x_1)\delta x_1$ and, as can be seen here, $\delta y(x_1) = \delta y(x_1) + y'(x_1)\delta x_1$. $\bar{R}$ is the radius of curvature at the point M_1, and R at the point A. The points A and M_1 are the points of contact of tangents to the extremals $y(x)$ and $\bar{y}(x)$, respectively. The point M_0 is the fixed point.

As a result we have

$$\delta J(y) \approx F\Big|_{x=x_1} \delta x_1 + \int_{x_0}^{x_1} (F_y \delta y + F_{y'} \delta y' + F_{y''} \delta y'')\, dx, \tag{4}$$

where

$$\delta y(x) = y(x) - \bar{y}(x), \delta y'(x) = y'(x) - \bar{y}'(x), \delta y''(x) = y''(x) - \bar{y}''(x)$$

are variations and

$$F_y = \frac{\partial F}{\partial y},\ F_{y''} = \frac{\partial F}{\partial y''}.$$

The second integral in (4) we integrate by parts to get

$$\delta J_2 = \int_{x_0}^{x_1} F_{y'} \delta y' dx = F_{y'}\Big|_{x=x_1} \delta y(x_1) - \int_{x_0}^{x^1} \delta y \frac{dF_{y'}}{dx} dx. \tag{5}$$

Analogously, for the third integral we have

$$\delta J_3 = \int_{x_0}^{x_1} F_{y''}\delta y'' dx = F_{y''}\Big|_{x=x_1}\delta y'(x_1) - \frac{d}{dx}F_{y''}\Big|_{x=x_1}\delta y(x_1) + \int_{x_0}^{x_1}\delta y\frac{d^2F_{y''}}{dx^2}dx. \qquad (6)$$

Here we have taken into account that since $M_0 = M(x_0, y_0)$ does not move, the variation of the function and all its derivatives at this point are equal to zero

$$\delta y(x_0) = \delta y'(x_0) = \delta y''(x_0) = \cdots = 0.$$

Substituting (5) and (6) into (4) we find

$$\delta J(y) \approx F\big|_{x=x_1}\delta x_1 + F_{y'}\Big|_{x=x_1}\delta y(x_1) - \frac{d}{dx}F_{y''}\Big|_{x=x_1}\delta y(x_1) + F_{y''}\Big|_{x=x_1}\delta y'(x_1) + \delta\Phi, \qquad (7)$$

where

$$\delta\Phi = \int_{x_0}^{x_1}\left(F_y - \frac{dF_{y'}}{dx} + \frac{d^2F_{y''}}{dx^2}\right)\delta y dx. \qquad (8)$$

According to the necessary condition for extremum, the expression (7) must be zero. Besides, two more equalities must be identically satisfied: the Euler-Poisson equation (which is the necessary condition for an extremum of a functional of the type (1), see, for example, [8])

$$F_y - \frac{dF_{y'}}{dx} + \frac{d^2F_{y''}}{dx^2} - \cdots = 0, \qquad (9)$$

from which the extremals are found and the condition

$$F\Big|_{x=x_1}\delta x_1 + F_{y'}\Big|_{x=x_1}\delta y(x_1) - \frac{d}{dx}F_{y''}\Big|_{x=x_1}\delta y(x_1) + F_{y''}\Big|_{x=x_1}\delta y'(x_1) = 0 \qquad (10)$$

directly derived from (7) and the Poisson equation (9) when necessarily $\delta J = 0$. A simple algorithm for obtaining equalities of the type (9) and (10) was demonstrated on the simplest example in the monograph [8] for the Euler–Lagrange functional.

As can be seen on Fig. 1, for the variation $\delta y(x_1)$ we have (see [8])

$$\delta y(x_1) = \delta y_1 - y'(x_1)\delta x_1, \qquad (11)$$

where δy_1, δx_1 are independent displacements of x_1, y_1. it is not so simple with $\delta y'(x_1)$, to find it we turn to Fig. 1 again and get

$$\delta y'(x_1) = y'(x_1) - \bar{y}'(x_1) = \tan\alpha - \tan\bar{\alpha} = \tan\alpha - \tan(\alpha - \delta\alpha) \approx \frac{\delta\alpha}{\cos^2\alpha} = \left[1 + y'^2(x_1)\right]\delta\alpha. \quad (12)$$

Here we have used simple properties $y' = \tan\alpha$ and, as a consequence, $1 + y'^2 = \dfrac{1}{\cos^2\alpha}$. It is well known from differential geometry [10] (see also [11]) that the curvature of a plane curve K at a point M can be determined by the formula

$$K = \frac{1}{R} = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}}, \qquad (13)$$

where R is the radius of curvature at M. Substituting here $y' = \tan\alpha$ and $y'' = \dfrac{\alpha'}{\cos^2\alpha}$ we have $\dfrac{1}{R} = \alpha'\cos\alpha$, or

$$\delta\alpha = \frac{\delta x_1}{R(x_1)\cos\alpha(x_1)} = \frac{\delta x_1\left[1 + y'^2(x_1)\right]^{\frac{1}{2}}}{R(x_1)}. \qquad (14)$$

It follows from (12) that

$$\delta y'(x_1) = y''(x_1)\delta x_1. \tag{15}$$

Hence the condition (9) gives

$$\left[F - y'\left(F_{y'} - \frac{dF_{y''}}{dx}\right) + y''F_{y''}\right]\Bigg|_{x=x_1}\delta x_1 + \left(F_{y'} - \frac{dF_{y''}}{dx}\right)\Bigg|_{x=x_1}\delta y_1 = 0. \tag{16}$$

In the case if δx_1 and δy_1 are independent, it follows from the expression (16) for a functional of the type (3) that

$$\begin{cases} F - y'\left(F_{y'} - \dfrac{dF_{y''}}{dx}\right) + y''F_{y''}\Bigg|_{x=x_1} = 0, \\ F_{y'} - \dfrac{dF_{y''}}{dx}\Bigg|_{x=x_1} = 0. \end{cases} \tag{17}$$

If the point M_1 moves along the trajectory $\varphi(x_1)$, instead of independent conditions (17) we get only one transversality condition

$$F + (\varphi' - y')\left(F_{y'} - \frac{dF_{y''}}{dx}\right) + y''F_{y''}\Bigg|_{x=x_1} = 0. \tag{18}$$

This condition is the required transversality for the case when the functionality has the form (3). It should be noted that if we consider a more complex functional, i.e., of the type (1), the variation $\delta y''(x_1)$ should be sought not as $\delta y''(x_1) = y'''(x_1)\delta(x_1)$, as it might seem from (15), but according to the procedure described above using calculation of curvature. Namely, one should compute as follows

$$\delta y''(x_1) = y''(x_1) - \bar{y}''(x_1) = (\tan\alpha)' - (\tan\bar{\alpha})' = \frac{\alpha'}{\cos^2\alpha} - \frac{\bar{\alpha}'}{\cos^2\bar{\alpha}} \approx$$
$$\approx \frac{\alpha' - \bar{\alpha}'}{\cos^2\alpha} = (1+y'^2)(\alpha' - \bar{\alpha}') = (1+y'^2)\left(\frac{1}{R\cos\alpha} - \frac{1}{\bar{R}\cos\bar{\alpha}}\right) = \frac{1+y'^2}{R\cos\alpha}(\cos\bar{\alpha} - \cos\alpha) =$$
$$= \frac{1+y'^2}{R\cos\alpha}\left[\cos(\alpha - \delta\alpha) - \cos\alpha\right] \approx \frac{1+y'^2}{R\cos\alpha}\sin\alpha\delta\alpha = \frac{(1+y'^2)y'}{R}\delta\alpha.$$

According to (14), from here we have

$$\delta y''(x_1) \approx \frac{(1+y'^2)y'}{R}\frac{\delta x_1}{R\cos\alpha} = \frac{\delta x_1\left(1+y'^2\right)^{\frac{3}{2}}}{R^2} = \frac{y'y''^2\left(1+y'^2\right)^{\frac{3}{2}}}{(1+y'^2)^3}\delta x_1,$$

and finally

$$\delta y''(x_1) = \frac{y'(x_1)y''^2(x_1)}{(1+y'^2(x_1))^{\frac{3}{2}}}\delta x_1. \tag{19}$$

As we can see, the difference between expression $\delta y''(x_1) = y'''(x_1)\delta x_1$ and (19) is significant. Following the proposed algorithm one can find any necessary variation of derivatives $\delta y^{(n)}$ and obtain appropriate transversality conditions. Indeed, taking into account the expression (19), instead of (18) we arrive at the following transversality condition

$$F + (\varphi' - y')\left(F_{y'} - \frac{dF_{y''}}{dx} + \frac{d^2F_{y'''}}{dx^2}\right) + \left(F_{y''} - \frac{dF_{y'''}}{dx}\right)y'' + \frac{y'(x_1)y''^2(x_1)}{(1+y'^2(x_1))^{\frac{3}{2}}}F_{y''}\Bigg|_{x=x_1} = 0. \tag{20}$$

Conclusion

1. An algorithm is proposed for finding transversality conditions for variational problems with moving boundary when a functional has the form (1).

2. A detailed calculation methodology based on the use of geometric properties of curvature is proposed.

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Об условии трансверсальности для одной вариационной задачи с подвижной границей

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Дан алгоритм получения условия трансверсальности для вариационной задачи с подвижной границей в случае функционала, содержащего производные n-го порядка от функции одной переменной. Приведено математическое обоснование этого подхода.

Ключевые слова: вариация, кривизна, подвижная граница, функционал, условие трансверсальности.