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Pseudospectral Methods for Nonlinear Pendulum Equations

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The article searched on mathematics and numerical solutions for the nonlinear pendulum (Chaotic pendulum). The numerical solution that was used for our research suitably the pseudospectral methods. With these equations, we studied and calculated on the interval $[-1, 1]$*, with boundary conditions already known. We used the software Mathematica 10.4 to calculate the results of the problems.*

Keywords: chaotic pendulum, Chebyshev, pseudospectral methods, differentiation matrices, collocation method, nonlinear equations.
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The differential equation of a single pendulum is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta(t) = -v^2 \sin\theta(t), \tag{1}$$

here v is the time, θ is value of the angular displacement, the number value $v^2 = g/l$, the clarify the value g is the acceleration due to gravity and l is the length of the pendulum.

The fact that the pendulum is placed in a friction environment, the drag force on the pendulum is calculated accordance with the Stokes law and is propertied to the instantaneous velocity of the pendulum because of the environment has value friction the equation of motion (1) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta(t) = -v^2 \sin\theta(t) - \alpha\frac{\mathrm{d}}{\mathrm{d}t}\theta(t), \tag{2}$$

and here by α is the coefficient of friction of the environment and show that is $\alpha > 0$.

To resist the friction, we need to add some external force, to simplified we choose the external periodic force and so the equation of motion becomes the equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta(t) = -v^2 \sin\theta(t) - \alpha\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) + \beta\cos(wt), \tag{3}$$

here β is amplitude and w is frequency.

When $\alpha = 0$ and then (3) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta(t) = -v^2 \sin\theta(t) + \beta\cos(wt), \tag{4}$$

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all the above equations are nonlinear second order differential equations. The nonlinear equation (3) has no analytic solution. However, if $\theta \to 0$ then $\sin(\theta) \approx \theta$, so the nonlinear differential equations above can be transformed into linear differential equations. This has got a lot of solutions, so we do not study.

With these nonlinear equations, we will be study and compute by the pseudospectral method.

1. Pseudospectral method (PSM)

Let $p(x)$ is a polynomial of degree n, and we know that it is values at the points $p(x_0)$, $p(x_1)$, $\ldots$, $p(x_n)$, then the first and second derivatives $p(x)$ at the same points are expressed in matrix form:

$$\begin{pmatrix} p'(x_0) \\ \vdots \\ p'(x_n) \end{pmatrix} = D \begin{pmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{pmatrix}, \qquad \begin{pmatrix} p''(x_0) \\ \vdots \\ p''(x_n) \end{pmatrix} = D^2 \begin{pmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{pmatrix}, \tag{5}$$

where $D = \{d_{i,j}\}$ is the so-called differentiation matrix [4].

In case when the Chebyshev-Gauss-Lobatto points are chose as the collocation points $y_k = = \cos(k\pi/n)$ [5],

$$D_{i,j} = \begin{cases} \dfrac{2n^2+1}{6} & i = j = 0 \\ \dfrac{c_i}{2c_j} \dfrac{(-1)^{i+j}}{\sin[\pi(i+j)/(2n)]\sin[\pi(i-j)/(2n)]} & i \neq j \\ \dfrac{\cos(\pi j/n)}{2\sin(\pi j/n)} & 0 < i = j < n \\ -\dfrac{2n^2+1}{6} & i = j = n \end{cases} \tag{6}$$

here $c_k = \begin{cases} 2 & \text{if } k = 0, n \\ 1 & \text{if } k = 1, 2, \ldots, n-1. \end{cases}$

The application of differential algebra in ordinary differential equations can also extend to nonlinear differential equations, so we transformed the matrix D into matrices [3]:

$$\begin{aligned} E^{(1)} &= \{d_{i,j}\}, \quad 1 \leqslant i,\ j \leqslant n-1, \\ e_0 = \{d_{i,0}\},\ e_n &= \{d_{i,n}\}, \quad 0 < i < n \end{aligned} \tag{7}$$

for a first-order differential element, the form $u'(x_i) = E^{(1)}u(x_i) + be_0 + ae_n$ here a and b are the two-point boundary-value on the range [–1; 1] of problem.

For a second-order differential element, we use $D^2 = \{d^{(2)}_{i,j}\}$ and define the matrices:

$$\begin{aligned} E^{(2)} &= \{d^{(2)}_{i,j}\}, \quad 1 \leqslant i,\ j \leqslant n-1, \\ e^2_0 = \{d^{(2)}_{i,0}\},\ e^2_n &= \{d^{(2)}_{i,n}\}, \quad 0 < i < n \end{aligned} \tag{8}$$

has the form $u''(x_i) = E^{(2)}u(x_i) + be^{(2)}_0 + ae^{(2)}_n$.

When we applied Pseudospectral method to solve nonlinear differential equations, nonlinear equations may or may not have a unique solution.

If we have identified a solution, we will proceed with an iterative procedure. Therefore, it is important to determine the iterative equation. The iterative procedure is simple, we assume $u^{(0)} = const$, then find $u^{(1)}$, $u^{(2)}$, $\ldots$, stop it until the error $\varepsilon = |u^{(k)} - u^{(k-1)}| < \varepsilon_0$.

2. Results

Equation (1), we shall consider the two-point boundary-value of problem on the range $[-1, 1]$, $\theta(-1) = a$, $\theta(1) = b$. We can find and transfer to form of method:

$$E^{(2)}\theta(t_j) = -v^2 \sin\theta(t_j) - be_0^{(2)} - ae_n^{(2)}, \quad j = \overline{1, n-1}. \tag{9}$$

Equation (9) are iterative equation. We give the error $\varepsilon \leqslant 10^{-12}$. In the Fig. 1 and Tab. 1 are the result are calculated based on the program by pseudospectral method, and then the solid line show the result calculated by Mathematica 10.4.

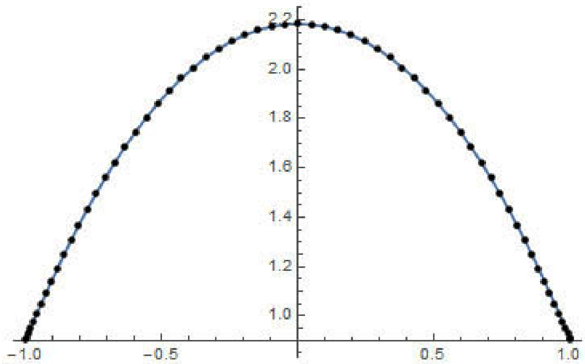

Fig. 1. Graphic of equation with $v = 1.7$, $n = 64$ in the case $a = b = 0.9$

Table 1. Competition the numerical results and error with Mathematica's calculations of the equation (1) with $v = 1.7$, $n = 64$ in the case $a = b = 0.9$

k	$y(k)$	PSM	Mathematica 10.4	Error
1	0.998795	0.903167	0.903167	1.53307×10^{-7}
10	0.881921	1.19371	1.19371	1.33523×10^{-7}
20	0.55557	1.80554	1.80554	7.05996×10^{-8}
30	0.0980171	2.17294	2.17294	4.01061×10^{-8}
40	–0.382683	2.00793	2.00793	2.20135×10^{-8}
50	–0.77301	1.43174	1.43174	1.19427×10^{-8}
60	–0.980785	0.950125	0.950125	1.65847×10^{-9}

Remasks: by this case, when $0 < v < 1.8$, the results are correct and convergence. When $v < 1.8$, means that the length of the string is at least g/v^2. When $v \geqslant 1.8$, equation (1) non-compliance boundary conditions $\theta(-1) = a$, $\theta(1) = b$.

Equation (2), we will be consider on the range $[-1, 1]$ and the boundary conditions $\theta(-1)=a$, $\theta(1) = b$. We can find iterative equation and transfer to form of method:

$$(E^{(2)} + \alpha E^{(1)})\theta = f - b(e_0^{(2)} + \alpha e_0) - a(e_n^{(2)} + \alpha e_n), \quad j = \overline{1, n-1} \tag{10}$$

here θ and f denote the vectors with the elements $\{t_j\}$ and $\{-v^2 \sin\theta(t_j)\}$.

In this problem, we give the error $\varepsilon < 10^{-8}$. In the Fig. 2, points of value are the calculated by result of the program by pseudospectral method, and then the solid line show the result

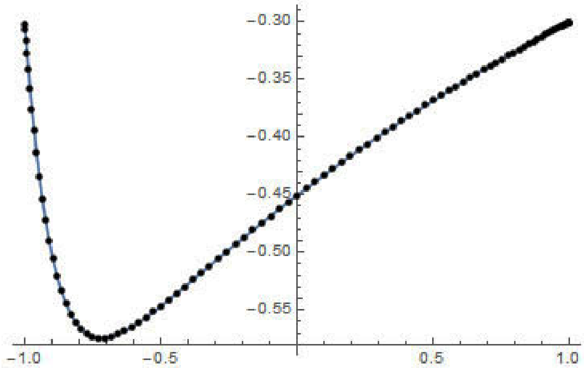

Fig. 2. Graphic of equation (2) with $\alpha = 10$, $v = 1.7$, $n = 96$ in the case $a = b = -0.3$

Table 2. Competition the numerical results and error with Mathematica's calculations of the equation (2) with $\alpha = 10$, $v = 1.7$, $n = 96$ in the case $a = b = -0.3$

k	$y(k)$	PSM	Mathematica 10.4	Error
5	0.986643	–0.301648	–0.301648	6.69452×10^{-10}
20	0.793353	–0.326501	–0.326501	3.08531×10^{-9}
35	0.412707	–0.381289	–0.381289	5.35524×10^{-9}
50	–0.0654031	–0.462363	–0.462363	6.22867×10^{-9}
65	–0.528068	–0.551825	–0.551825	6.90365×10^{-9}
80	–0.866025	–0.533131	–0.533131	1.19349×10^{-7}
95	–0.999465	–0.301757	–0.301757	7.68573×10^{-7}

calculated by Mathematica. In the Tab. 2 are the competition the numerical results and error with Mathematica's calculations with $\alpha = 10,\ v = 1.7,\ n = 96$ in the case $a = b = -0.3$.

Remasks: The fixed point, $\theta(t) = d\theta/dt = 0$ is linearly stable, i.e., small perturbations from this point will decay in time. The fixed point $\theta(t) = \pi, d\theta/dt = 0$ is linearly unstable, which means that small perturbations from this point will grow exponentially in time [2]. When $0 < v < 2.7$, the results are correct and convergence; when $v \geqslant 2.7$ method for big error and not convergence. When v fixed, α increase, then graph are left-leaning.

Equation (3), we will be consider on the range $[-1, 1]$ and in the case of boundary conditions $\theta(-1) = a,\ \theta(1) = b$. Equation (3) is non-Hamiltonian and it does not have an analytical solution. This is a nonlinear equation and has three dynamic variables. We found iterative equation and transferred to form of the method:

$$(E^{(2)} + \alpha E^{(1)})\theta = f - b(e_0^{(2)} + \alpha e_0) - a(e_n^{(2)} + \alpha e_n), \quad j = \overline{1, n-1} \tag{11}$$

here θ and f denote the vectors with the elements $\{t_j\}$ and $\{\beta \cos wt_j - v^2 \sin \theta(t_j)\}$.

With $v, \alpha, \beta, w > 0$, we give the error $\varepsilon \leqslant 10^{-12}$. In the Fig. 3, points of value are calculated the result of the program by the PSM, and then the solid line show the result calculated by Mathematica. Numerical results are given in Tab. 3.

Remasks: When $0 < v < 1.7$ the results are correct and convergence; when $\beta \cos(wt)$ descended then $\theta(t)$ ascended; when α ascended then $\theta(t)$ descended; the complex motion one would expect when the three forces are comparable; from here we can orient the pendulum control.

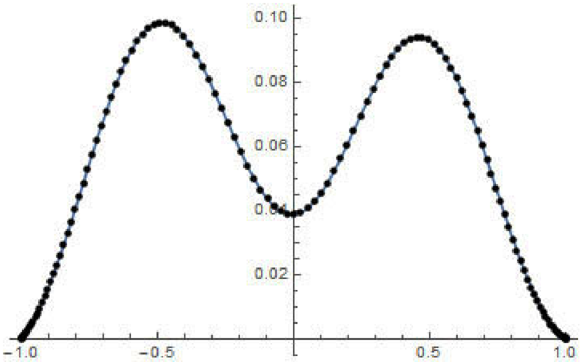

Fig. 3. Graphic of equation (3) with $\alpha = 0.5$, $v = 1.1$, $w = 2\pi$, $\beta = 1.3$, $n = 128$ in the case $a = b = 0$

Table 3. Competition the numerical results and error with Mathematica's calculations of the equation (3) with $\alpha = 0.5$, $v = 1.1$, $w = 2\pi$, $\beta = 1.3$, $n = 128$ in the case $a = b = 0$

k	$y(k)$	PSM	Mathematica 10.4	Error
1	0.999699	0.0000121827	0.0000121743	8.4617×10^{-10}
25	0.817585	0.027557	0.027557	1.50768×10^{-8}
50	0.33689	0.0854559	0.0854558	2.98413×10^{-8}
75	−0.266713	0.0722918	0.0722917	3.26543×10^{-8}
100	−0.77301	0.0487747	0.0487747	4.7381×10^{-9}
125	−0.99729	0.000283445	0.000283453	7.51629×10^{-9}

Equation (4), we will be consider on the range $[-1, 1]$ and in the case of boundary conditions $\theta(-1) = a,\ \theta(1) = b$. We found iterative equation and transferred to form of method:

$$E^{(2)}\theta = f - be_0^{(2)} - ae_n^{(2)}, \quad j = \overline{1, n-1} \tag{12}$$

here θ and f denote the vectors with the elements $\{t_j\}$ and $\{\beta \cos wt_j - v^2 \sin\theta(t_j)\}$ with $v, \beta, w > 0$, we give the error $\varepsilon \leqslant 10^{-12}$. In the Fig. 4, the value are the calculated base on the program by the PSM, and then the solid line show the result calculated by Mathematica. The calculation results are shown in Tab. 4.

Remasks: When $0 < v < 1.9$ the results is show correct and convergence; when $\beta\cos(wt)$ ascended then $\theta(t)$ ascended; when w ascended then $\theta(t)$ descended.

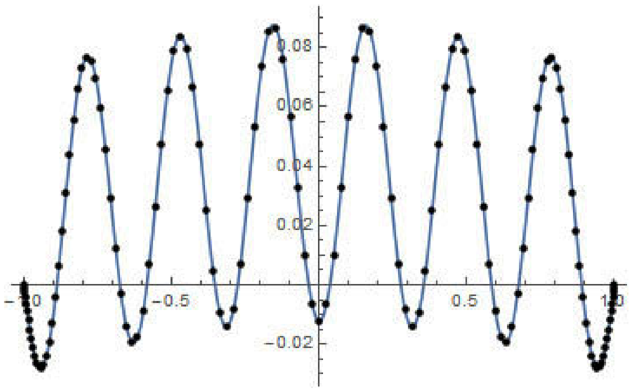

Fig. 4. Graphic of equation (4) with $v = 1.0$, $\beta = 20$, $w = 20$, $n = 128$ in the case $a = b = 0$

Table 4. Competition the numerical results and error with Mathematica's calculations of the equation (4) with $v = 1.0$, $\beta = 20$, $w = 20$, $n = 128$ in the case $a = b = 0$

k	$y(k)$	PSM	Mathematica 10.4	Error
1	0.999699	–0.000265697	–0.000265549	1.47924×10^{-7}
30	0.740951	0.0595138	0.059514	1.47715×10^{-7}
60	0.0980171	0.0566888	0.0566889	1.26391×10^{-7}
90	–0.595699	–0.0085085	–0.00850845	5.6271×10^{-8}
127	–0.999699	–0.000265697	–0.000265696	9.03764×10^{-10}

3. Summary

We have monitored four equations of the nonlinear pendulum with the range $[-1, 1]$ and the know boundary conditions by the pseudospectral methods. The method has issued good results. And then we can survey the variation of the function through our parameters and design of force acting on the pendulum to control the operation the pendulum.

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Псевдоспектральные методы нелинейных уравнений маятника

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В статье исследованы математические и численные решения для нелинейного маятника (хаотический маятник). Решение, которое использовалось для наших исследований, соответствовало псевдоспектральным методам. Вычисления проводили на интервале [–1, 1] с уже известными граничными условиями. Для расчета использовалось программное обеспечение Mathematica 10.4.

Ключевые слова: хаотический маятник, Чебышев, псевдоспектральные методы, дифференцирующие матрицы, метод коллокации, нелинейные уравнения.