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Permanents as formulas of summation over an algebra with a unique n -ary operation

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We give a new general definition for permanents over an algebra with a unique n -ary operation and study their properties. In particular, it is shown that properties of these permanents coincide with the basic properties of the classical Binet-Cauchy permanent (1812).

Keywords: permanents, noncommutative and multioperator algebras, the polarization theorem, polynomial identities.

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Introduction

The following concept of the permanent was first introduced independently by J. Binet and A. Cauchy in 1812. The permanent of a square $n \times n$ matrix $A = (a_{ij})$ over fields \mathbb{R} or \mathbb{C} (a commutative ring K) is defined to be the following sum: $Per(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \times \dots \times a_{n\sigma(n)}$, where the sum is taken over all permutations of the set $\{1, 2, \dots, n\}$. In other words,

$$Per(A) = \frac{1}{n!} \sum_{\tau, \sigma \in S_n} a_{\tau(1)\sigma(1)} \times \dots \times a_{\tau(n)\sigma(n)} = \sum_{\sigma \in S_n} Sym\{a_{1\sigma(1)} \times \dots \times a_{n\sigma(n)}\}, \quad (1)$$

where $Sym\{a_{1\sigma(1)} \times \dots \times a_{n\sigma(n)}\} = \frac{1}{n!} \sum_{\tau \in S_n} a_{\tau(1)\sigma(1)} \times \dots \times a_{\tau(n)\sigma(n)}$.

Let G be a groupoid with division by integers, Ψ an algebra with a unique n -ary operation $f(\mathbf{x}) = f(x_1, x_2, \dots, x_n) : \Psi^n \rightarrow G$. By analogy with (1) we introduce a more general definition for permanents (compare with [1, 2] and many others).

Definition. The e -permanent $ePer(A, f)$ of a square $n \times n$ matrix $A = (a_{ij})$ over the algebra Ψ is defined to be the following sum

$$ePer(A, f) = \frac{1}{n!} \sum_{\tau, \sigma \in S_n} f(a_{\tau(1)\sigma(1)}, \dots, a_{\tau(n)\sigma(n)}). \quad (2)$$

In other words,

$$ePer(A, f) := \sum_{\sigma \in S_n} Sym\{f(a_{1\sigma(1)}, \dots, a_{n\sigma(n)})\} = \sum_{\tau \in S_n} Sym\{f(a_{\tau(1)1}, \dots, a_{\tau(n)n})\} \quad (3)$$

where the symmetrization operator

$$Sym f(x_1, x_2, \dots, x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}),$$

and $Sym f(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n)$, if $f(x_1, x_2, \dots, x_n)$ is a symmetric function.

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1. Results

Properties of permanents $ePer(A, f)$ follow directly from properties of a general term of sums (1)–(2). For example, a general term of sum (3) is a symmetric function of its arguments.

In particular, we have obtained several combinatorial formulas (polynomial identities) for e-permanents that contain several free elements from algebra G . As a special case, one of them includes the known Ryser-Wilf formulas for $Per(A)$ (H. Wilf, 1968; H. Ryser, 1963), which gives the fastest algorithm for computing $Per(A)$. All computation formulas for e-permanents obtained here by means of the known polarization theorem for recovering a polyadditive function from its values on a diagonal [3, 4].

Theorem 1.

- (a) $ePer_f(A) = ePer_f(A^T)$.
 - (b) $ePer_f(A)$ is a symmetric function of rows and columns of the matrix A .
 - (c) If the n -ary operation $f(x_1, x_2, \dots, x_n): \Psi^n \rightarrow G$ is polyadditive then $ePer_f(A)$ is a polyadditive function of rows and columns of the matrix A .
 - (d) The e -permanent over the algebra Ψ_0 it is equal to 0, if the square $n \times n$ matrix A contains (up to shifts of rows and columns) a proper $r \times k$ zero-submatrix, where $r + k > n$. In this case $ePer(A) = 0$, if the matrix A contains a zero-row or a zero-column.
 - (e) If the algebra Ψ_0 contains the unit e , and I_n is the identity matrix then the equality $ePer_f(A) = e$ for the e -permanent over Ψ_0 is valid.
- In general case the Laplace formulas for permanent are not valid.

Theorem 2. If an n -ary operation $f(x_1, x_2, \dots, x_n): \Psi^n \rightarrow G$ is polyadditive then the following formula for $ePer(A)$ over the algebra Ψ is valid:

$$ePer(A, f) := \sum_{i=0}^n (-1)^k \sum_{1 \leq j_1 < \dots < j_k \leq n} \text{Sym} f(\gamma_1 - a_{1j_1} - a_{1j_2} - \dots - a_{1j_k}, \dots, \gamma_n - a_{nj_1} - a_{nj_2} - \dots - a_{nj_k}),$$

where $\gamma_1, \gamma_2, \dots, \gamma_n$ are elements from Ψ . In a special case, if an n -ary operation $f(x_1, x_2, \dots, x_n): \Psi^n \rightarrow G$ is symmetric and polyadditive then

$$ePer(A, f) := \sum_{i=0}^n (-1)^k \sum_{1 \leq j_1 < \dots < j_k \leq n} f(\gamma_1 - a_{1j_1} - a_{1j_2} - \dots - a_{1j_k}, \dots, \gamma_n - a_{nj_1} - a_{nj_2} - \dots - a_{nj_k}) \quad (4)$$

Introduce the operation

$$F(x) := f(x, x, \dots, x) : \Psi \rightarrow G.$$

Theorem 3. The following formula for $ePer_f(A)$ over the algebra Ψ containing $n!$ any elements $\gamma_\sigma \in \Psi_0, \sigma \in S_n$, is valid:

$$ePer(A, f) := \left\{ \sum_{\sigma \in S_n} \sum_{k=0}^n (-1)^{n-k} \sum_{1 \leq j_1 < \dots < j_k \leq n} F\left(\gamma_\sigma + \sum_{s=1}^k a_{j_s \sigma(j_s)}\right) \right\} / n!, \quad (5)$$

If in (5) $\gamma_\sigma = \gamma$ for each $\sigma \in S$ then

$$ePer(A, f) := \left\{ \sum_{k=0}^n (-1)^{n-k} (n-k)! \sum_{1 \leq j_1 < \dots < j_k \leq n, 1 \leq i_1 < \dots < i_k \leq n} F\left(\gamma + \sum_{s=1}^k a_{j_s i_s}\right) \right\} / n!,$$

which for $\gamma = 0$ gives

$$ePer(A, f) := \left\{ \sum_{k=1}^n (-1)^{n-k} (n-k)! \sum_{1 \leq j_1 < \dots < j_k \leq n, 1 \leq i_1 < \dots < i_k \leq n} F \sum_{s=1}^k a_{j_s i_s} \right\} / n!$$

Remark. In the simplest case, if in (4) we put $f(x_1, x_2, \dots, x_n) = \prod_{i=1}^n x_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $F(x) = x^n$ then we get the first polynomial identity for permanents over the field \mathbb{R} (G. Egorychev, 1979). In its turn, this polynomial identity for special values of free parametres gives the known Ryser-Wilf formulas (computing algorithm) for the Binet-Cauchy permanents.

It is common knowledge that the determinant $\det(A)$ can be computed in poly (n) time. On the other hand, the fastest Ryser-Wilf algorithm known for computing $Per(A)$ runs in $n2^{n-1}$ time. Moreover, L. Valiant (1979) has shown that the problem of computation $Per(A)$ even for $(0,1)$ -matrices is P -complete. The formulas (4) – (5) for $ePer(A, f)$ is the same as the Ryser-Wilf formulas for $ePer(A)$ (up to replacing an n -ary operation of multiplication for elements of a commutative ring by an n -ary operation $f(x_1, x_2, \dots, x_n) : \Psi^n \rightarrow G$.) Thus, the complexity of computation of $ePer(A, f)$ by formulas (4) – (5) directly depends on the complexity of the Ryser-Wilf algorithm for $Per(A)$ and the complexity of (fastest) algorithms for computation of the concrete polyadditive n -ary operation $f(x_1, x_2, \dots, x_n) : \Psi^n \rightarrow G$ and one-ary operation $F(x) := f(x, x, \dots, x) : \Psi \rightarrow G$ (see, for example, [5]).

Illustrative examples Let K_{ij} , $i, j = 1, 2, \dots, n$, be non-empty convex compact sets in Euclidean space \mathbb{R}^n bodies in \mathbb{R}^n with the addition of bodies by Minkowski, A be any unimodular transformation in \mathbb{R}^n . Let $V(K_1, K_2, \dots, K_n)$ be the mixed volume of bodies K_1, \dots, K_n , $V(K)$ the volume of body K ([6], Chapter 4). Then the following formula of multiple summation is valid:

$$\sum_{\sigma \in S_n} V(AK_{1\sigma(1)}, \dots, AK_{n\sigma(n)}) = \left\{ \sum_{k=0}^n (-1)^{n-k} (n-k)! \sum_{1 \leq j_1 < \dots < j_k \leq n, 1 \leq i_1 < \dots < i_k \leq n} V(K_0 + \sum_{s=1}^k AK_{j_s i_s}) \right\} / n!$$

where K_0 is any body in \mathbb{R}^n .

Similarly, let A_{ij} , $i, j = 1, 2, \dots, n$ be square $n \times n$ matrices over the field \mathbb{R} , $D(A_1, A_2, \dots, A_n)$ be the mixed discriminants of matrices A_1, \dots, A_n , A be any unimodular $n \times n$ matrix over field \mathbb{R} ([6], § 25).

Then

$$\sum_{\sigma \in S_n} D(AK_{1\sigma(1)}, \dots, AK_{n\sigma(n)}) = |Det(A)| \left\{ \sum_{k=0}^n (-1)^{n-k} (n-k)! \sum_{1 \leq j_1 < \dots < j_k \leq n, 1 \leq i_1 < \dots < i_k \leq n} \det(A_0 + \sum_{s=1}^k AK_{j_s i_s} K_{j_s i_s}) \right\} / n!$$

where A_0 is any $n \times n$ matrix over the field \mathbb{R} .

It is of interest to obtain similar results for Schur fuctions, the mixed discriminants, the resultants, and many other planar and space matrix functions over different algebraic systems of various type and its applications (see [4], [7] – [10] and many others). In our view, an answer to the following question is particularly interesting: for which operations $f(x_1, x_2, \dots, x_n) : \Psi^n \rightarrow G$ a fast evaluation of $ePer(A, f)$ with the help of quantum computers is possible? (see, example [11]).

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Перманенты как формулы суммирования над алгеброй с единственной n -арной операцией

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Дано новое общее определение перманентов над алгеброй с единственной n -арной операцией и изучены его свойства. В частности, показано, что свойства этих перманентов совпадают с основными свойствами классического Бине-Коши перманента (1812).

Ключевые слова: перманенты, некоммутативные и мультиоператорные алгебры, теорема поляризации, полиномиальные тождества.