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Jacobian Conjecture for Mappings of a Special Type in $\mathbb{C}^2$

Maria A. Stepanova*
Faculty of Mathematics and Mechanics
Lomonosov Moscow State University
Leninskie Gory, GSP-2, Moscow, 119992
Russia

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We show that a polynomial mapping of the type $(x \to F[x+f(a(x)+b(y))],\ y \to G[y+g(c(x)+d(y))])$*, where* (a,b,c,d,f,g,F,G) *are polynomials with non-zero Jacobian is a composition of no more than 3 linear or triangular transformations. This result, however, leaves the possibility of existence of a counterexample of polynomial complexity two.*

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The celebrated Jacobian conjecture [1] states that if the Jacobian of a polynomial mapping in $\mathbb{C}^n$ is not equal to zero, then this mapping is bijective, and in the case $n = 2$, due to Jung's theorem, is a composition of affine and triangular mappings. A triangular mapping is a mapping of the type $(X = x,\ Y = y + p(x))$. The Jacobian of a polynomial mapping is not equal to zero only if it is equal to a nonzero constant, which we can assume to be 1. It is known that if $n = 2$ and the degree of a mapping does not exceed 100, then the conjecture holds [2]. Despite long-term attempts of solving the problem the question remains open, and hence it is interesting to estimate the complexity of a hypothetical counterexample.

The degree of a polynomial is a usual method of measuring its complexity. In the paper [3] for analytical functions another method was proposed. For polynomials the complexity (*polynomial complexity*) is defined as follows. Polynomials of one variable, x or y, are of complexity zero, i.e. Cl_0 consists of polynomials of one variable. Other classes are constructed inductively with the help of addition, namely: $Cl_{n+1} = f(a_n(x,y) + b_n(x,y))$, where f is a polynomial of one variable, and $a_n(x,y), b_n(x,y) \in Cl_n$. In particular, according to this definition, Cl_1 consists of polynomials of the type $P(x,y) = f(a(x)+b(y))$, where (a,b,f) are polynomials of one variable. It is easy to see that such operation as multiplication xy due to equivalence

$$xy = \frac{(x+y)^2 - (x^2+y^2)}{2}$$

is of polynomial complexity two, as well as the fact that every polynomial is of finite polynomial complexity. Herewith, the degree of a polynomial of complexity one can be arbitrarily high. It is natural to define complexity of a mapping as the maximum of complexities of its coordinates. In [4] it was shown that for a mapping of complexity one the conjecture ($n = 2$) holds. There was also formulated a conjecture that a counterexample can be of complexity two. In this paper it is shown that for a mapping of complexity two of a special type (a perturbation of the identity mapping with the help of polynomials of complexity one) the conjecture holds true. This result, however, does not prohibit the existence of a counterexample of complexity two.

Theorem. *Let the polynomial mapping* $\Phi(x,y) = (X,Y)$ *be of the type*

$$X(x,y) = F[x + f(a(x)+b(y))],$$
$$Y(x,y) = G[y + g(c(x)+d(y))],$$

*step_masha@mail.ru

and the Jacobian of the mapping $J_\Phi(x,y)$ is equal to a nonzero constant. Then this mapping can be written as a composition of three or less mappings, each is either linear or triangular.

Proof. Notice that the functions F, G have to be linear, since by the chain rule $J_\Phi(x,y) = J_\Psi(x,y)F'[x+f(a(x)+b(y))]G'[y+g(c(x)+d(y))]$, where $\Psi(x,y) = (\tilde{X}, \tilde{Y})$ is of the type

$$\tilde{X}(x,y) = x + f(a(x)+b(y)),$$
$$\tilde{Y}(x,y) = y + g(c(x)+d(y)).$$

Therefore without loss of generality in what follows we may consider the mapping $\Psi(x,y)$.

Write the condition of equality of the Jacobian to 1:

$$1 + g'(c(x)+d(y))d'(y) + f'(a(x)+b(y))a'(x) + g'(c(x)+d(y))f'(a(x)+ + b(y))a'(x)d'(y) - g'(c(x)+d(y))f'(a(x)+b(y))c'(x)b'(y) = 1. \qquad (1)$$

If at least one of the functions $g'(c(x)+d(y))$, $f'(a(x)+b(y))$ is identically zero (let $g'(c(x)+d(y)) = 0$, for $f'(a(x)+b(y))$ the reasoning is similar), then from (1) we obtain that $f'(a(x)+b(y))a'(x) = 0$, where we conclude that $f(a(x)+b(y))$ can depend only of y. From $g'(c(x)+d(y)) = 0$ we obtain $g(c(x)+d(y)) = const$, i.e. the mapping in that case is triangular. Therefore we can assume that the functions $g'(c(x)+d(y)), f'(a(x)+b(y))$ are not identically zero. Then, dividing the identity (1) by $g'(c(x)+d(y))f'(a(x)+b(y))$, we obtain

$$\frac{d'(y)}{f'(a(x)+b(y))} + \frac{a'(x)}{g'(c(x)+d(y))} = c'(x)b'(y) - a'(x)d'(y). \qquad (2)$$

Let us consider possible cases.

1. Let $g'(c(x)+d(y)), f'(a(x)+b(y))$ be constants. Then, differentiating (2) with respect to x and y, we obtain:

$$c''(x)b''(y) = a''(x)d''(y). \qquad (3)$$

1.1. $c''(x) \equiv a''(x) \equiv 0$.

In this case $c'(x), a'(x)$ are constants, then from (2) integrating with respect to y we obtain that $d(y) = Ab(y) + By + C$, where A, B, C are constants. Since $c(x), a(x), f(x), g(x)$ are linear (from $g'(c(x)+d(y)), f'(a(x)+b(y)), c'(x), a'(x) \equiv const$), the mapping can be written as a composition of a linear and a triangular mapping.

1.2. $b''(y) \equiv d''(y) \equiv 0$.

This case is analogous to the previous.

1.3. At least one of $a''(x), c''(x)$ is not identically zero (without loss of generality let $a''(x) \neq 0$), and at least one of $b''(y), d''(y)$ is not identically zero.

1.3.1. Let $b''(y) \neq 0$. Then we can divide both parts of (3) by $a''(x)b''(y)$:

$$\frac{c''(x)}{a''(x)} = \frac{d''(y)}{b''(y)},$$

and therefore

$$\frac{c''(x)}{a''(x)} = \frac{d''(y)}{b''(y)} = const.$$

Consequently $c(x) = c_1a(x) + c_2x + c_3$, $d(y) = c_1b(y) + c_4y + c_5$, where c_i are constants. Then we can write the mapping as follows:

$$X(x,y) = x + a(x) + b(y),$$

$$Y(x,y) = y + D(a(x)+b(y)) + Ax + By + C,$$

where A, B, C, D are constants. Writing the condition of equality of Jacobian to 1 for such functions, we obtain:

$$1 + Db'(y) + B + a'(x) + Da'(x)b'(y) + Ba'(x) - Da'(x)b'(y) - Ab'(y) = 1,$$

whence, differentiating with respect to y, we find either $A = D$ or $b''(y) = 0$. In both cases we obtain the necessary type of the mapping.

1.3.2. Let $d''(y) \neq 0$. Then from (2) we get (since by assumption $a''(x) \neq 0$) $b''(y) \neq 0$, and therefore we can apply reasonings from the clause 1.3.1.

2. Let one of $g'(c(x)+d(y))$, $f'(a(x)+b(y))$ be constant, and the other be nonconstant (let $f'(a(x)+b(y)) = const$ without loss of generality). Then from (2) we get $d(y) = const$, because the summand $\dfrac{a'(x)}{g'(c(x)+d(y))}$ has to be a polynomial, since the other summands (2) are polynomials. Taking into account that in this case $d'(y) = 0$, from (2) we obtain:

$$\frac{a'(x)}{g'(c(x)+const)} = c'(x)b'(y). \tag{4}$$

2.1. Let $b'(y) \neq const$. Then from (4) we get $c'(x) \equiv 0$, whence $a'(x) \equiv 0$, i.e. $a(x), c(x), d(y) = const$, and consequently the mapping is triangular.

2.2. Let $b'(y) = B = const$. Then $a'(x) = Bg'(c(x)+const)c'(x) = B(g(c(x)+const))'$, and therefore, writing the condition of equivalence of Jacobian to unit, we obtain that the mapping is linear.

3. Let both $g'(c(x)+d(y))$, $f'(a(x)+b(y))$ be nonconstants.

Lemma. *If $deg(a') < deg(c)$ and $deg(d') < deg(b)$, then the mapping can be written as a composition of a triangular and a linear mapping.*

Proof. If $deg(a') < deg(c)$ and $deg(d') < deg(b)$, then $deg(a) \leqslant deg(c)$ and $deg(d) \leqslant deg(b)$. Denote by DEG_x the degree of the quotient with respect to x in the Euclid algorithm. Since $deg(f') \geqslant 1$, $deg(g') \geqslant 1$, the following inequalities hold:

$$DEG_x\left(\frac{d'(y)}{f'(a(x)+b(y))} + \frac{a'(x)}{g'(c(x)+d(y))}\right) \leqq DEG_x\left(\frac{d'(y)}{f'(a(x)+d(y))} + \frac{a'(x)}{g'(a(x)+d(y))}\right) \leqq$$
$$\leqq DEG_x\left(\frac{d(y)}{f'(a(x)+d(y))} + \frac{a(x)}{g'(a(x)+d(y))}\right) \leqq DEG_x(\frac{d(y)}{a(x)+d(y)} + \frac{a(x)}{a(x)+d(y)}) = 0,$$

analogically

$$DEG_y\left(\frac{d'(y)}{f'(a(x)+b(y))} + \frac{a'(x)}{g'(c(x)+d(y))}\right) \leqq 0.$$

From (2) it ensues that $\dfrac{d'(y)}{f'(a(x)+b(y))} + \dfrac{a'(x)}{g'(c(x)+d(y))}$ is a polynomial. Since its degree $\leqq 0$, we have

$$\frac{d'(y)}{f'(a(x)+b(y))} + \frac{a'(x)}{g'(c(x)+d(y))} = K = const, \tag{5}$$

whence

$$d'(y)g'(c(x)+d(y)) + a'(x)f'(a(x)+b(y)) = Kf'(a(x)+b(y))g'(c(x)+d(y)). \tag{6}$$

Let us consider possible cases.

1. At least one of the functions f', g' is nonlinear.

Then in the chain of inequalities above the last is strict, whence the degree of the polynomial $\dfrac{d'(y)}{f'(a(x)+b(y))} + \dfrac{a'(x)}{g'(c(x)+d(y))}$ is less than 0, i.e. $K = 0$. Then from (5) we obtain:

$$\frac{d'(y)}{f'(a(x)+b(y))} = -\frac{a'(x)}{g'(c(x)+d(y))},$$

whence

$$\frac{d'(y)}{f'(a(x)+b(y))} = -\frac{a'(x)}{g'(c(x)+d(y))} = const.$$

Hence, $d(y), a(x) = const$, and therefore the mapping under consideration can be written as:

$$X(x,y) = x + F(y), \quad Y(x,y) = y + G(x),$$

where F, G are some functions.

The condition of equality of the Jacobian to zero in this case gives $F'(y)G'(x) \equiv 0$, which yields that the mapping is triangular as was required.

2. Both functions f' и g' are linear: $f'(x) = Ax + B$, $g'(y) = Cy + D$ (moreover, we can assume that $A, C, K \neq 0$, since other cases were considered earlier). Then from (6) we obtain:

$$d'(y)(C(c(x)+d(y))+D) + a'(x)(A(a(x)+b(y))+B) = \\ = K(A(a(x)+b(y))+B)(C(c(x)+d(y))+D). \tag{7}$$

Let us equate the degrees with respect to x of both sides of (7):

$$deg_x(Cd'(y)c(x) + Aa'(x)a(x)) = deg_x(ACa(x)c(x)),$$

which is not true if $deg(a') < deg(c)$. This contradiction finishes the proof of the lemma.

Thus, we may assume that $deg(a') \geqq deg(c)$ or $deg(d') \geqq deg(b)$. Without loss of generality we assume $deg(d') \geqslant deg(b)$.

Consider the degrees with respect to y of both sides of (2). If $a'(x) \neq 0$ identically, then the degree of the right hand side is equal to $deg(d')$ (since $deg(d') \geqslant deg(b)$ yields $deg(d) > deg(b)$), and the degree of the left hand side is strictly less than $deg(d')$ (if $b \neq const$, and in the other case from (2) we obtain that $d = const$, whence, equating the Jacobian to 1, we obtain that the transformation is linear as required). Therefore $a'(x) = 0$, i.e. $a(x) = const = a$, and (2) can be rewritten as:

$$\frac{d'(y)}{f'(a(x)+b(y))} = c'(x)b'(y).$$

Whence we obtain that $d'(y) = c'(x)(f(a+b(y)))'_y \Rightarrow c'(x) = const = c$, and $d(y) = c(f(a+b(y))) + const$. Thus also in this case the mapping is of required type. The theorem is proved.

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Гипотеза о якобиане для отображений $\mathbf{C}^2$ специального вида

Мария А. Степанова
Механико-математический факультет
МГУ им. М. В. Ломоносова
Ленинские горы, ГСП-2, Москва, 119992
Россия

Показано, что полиномиальное отображение вида $(x,y)\to(F[x+f(a(x)+b(y))],\, G[y+g(c(x)+d(y))])$, *где* (a,b,c,d,f,g,F,G) *— полиномы, с ненулевым якобианом — это композиция не более чем трех линейных или треугольных преобразований. Этот результат, однако, оставляет возможным существование контрпримера полиномиальной сложности два.*

Ключевые слова: аналитическая сложность.