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Global Solvability of the One-dimensional Inverse Problem for the Integro-differential Equation of Acoustics

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The hyperbolic integro-differential acoustic equation is considered. Direct problem is to find the acoustic pressure from the initial - boundary value problem for this equation with point source located on the boundary of the space domain. The inverse problem is studied. It consists in determining the one-dimensional kernel of the integral term using the solution of the direct problem at $x = 0, t > 0$. Inverse problem is reduced to the system of integral equations for unknown functions. The principle of contraction mappings is applied to this system in the space of continuous functions with weighted norms. The global unique solvability of the inverse problem is proved.

Keywords: integrodifferential equation, inverse problem, Dirac delta function, kernel, weight function.

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Introduction

Inverse problems for hyperbolic integro-differential equations have been studied by many authors [1–5]. The inverse problem for the second order hyperbolic equation was studied [1]. The integral term is the convolution of one dimensional memory function of the medium and the solution of the direct problem. The inverse problem is reduced with the use of the Fourier method to the system of integral equations of Volterra type with respect to unknown functions depending on time variable. Problems in the determination of multidimensional kernel of viscoelasticity equations were studied [2], [3] (see also the references there in). Problems of reconstructing a one-dimensional kernel of the viscoelasticity equation in a bounded domain with constant Lamé coefficients and density were studied [4]. A similar problem was studied when Lamé coefficients and density are functions of x [5]. In the present work as in [2–5] the source initiating physical process of wave transmission is localized on the boundary of considered space domain. Theorem on local solvability of the inverse problem for the integro-differential equation of acoustics was considered [6]. Global solvability of the inverse problem for integro-differential equation of acoustics is studied in the paper.

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1. Statement of the problem and the main result

Let us consider an initial-boundary problem for integro-differential hyperbolic equation of the type

$$\frac{1}{c^2(z)}v_{tt} = \varrho_z - \frac{\partial \ln \rho(z)}{\partial z} \varrho, \quad z > 0, \quad t > 0, \quad (1)$$

$$v|_{t \leq 0} \equiv 0, \quad \varrho(+0, t) = \delta'(t), \quad (2)$$

where $c(z) > 0$ is the wave speed, $\rho(z)$ is the medium density, $v(z, t)$ is the acoustic pressure, $\delta'(t)$ is the derivative of the Dirac function, and $\varrho(z, t)$ depends on $v(z, t)$ as

$$\varrho(z, t) = v_z(z, t) + \int_0^t k(t - \tau)v_z(z, \tau)d\tau. \quad (3)$$

The inverse problem is to determine the kernel $k(t)$, $t > 0$ in (3) if

$$v(+0, t) = g(t), \quad t > 0. \quad (4)$$

is known. Equation (1) takes into account the absorption of ideal-elastic medium, and it arises in geophysics when properties of the medium is studied by seismic waves. Actually, with assumption on smoothness Boltzmann system of equations (one of the most common for linear nonelastic medium) in one dimensional case is reduced to equation (1).

Let us introduce, as in [5], new variable

$$x = \psi(z) = \int_0^z \frac{d\xi}{c(\xi)}$$

and the following designations

$$\tilde{v}(x, t) := v(\psi^{-1}(x), t), \quad a(x) := c(\psi^{-1}(x)), \quad b(x) := \rho(\psi^{-1}(x)).$$

Function $\psi^{-1}(x)$ is the inverse of $\psi(z)$. It is assumed everywhere in this paper that $c(z) > 0$, $\rho(z) > 0$. The main result of this work is the following theorem on common unique solvability of the inverse problem.

Theorem. *Let us assume that function $g(t)$ is represented in the form*

$$g(t) = -c(+0)\delta(t) + \theta(t)g_0(t),$$

where $g_0 \in C^2[0, T]$, and $\theta(t)$ is the Heaviside function. Furthermore, $(c(z), \rho(z)t) \in C^3[0, \psi^{-1}(T)]$. Then there exists a unique solution of the inverse problem (1)–(4), $k(t) \in C^2[0, T]$ for any fixed $T > 0$.

2. Reduction of the problem to a system of integro-differential equations

Equalities (1)–(4) are written with respect to new functions \tilde{v} , a , b and variable x in the following form

$$\frac{\partial^2 \tilde{v}}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} - \frac{\lambda'(x)}{\lambda(x)} \frac{\partial}{\partial x} \right) \left[\tilde{v}(x, t) + \int_0^t k(t - \tau)\tilde{v}(x, \tau)d\tau \right], \quad x > 0, \quad t > 0 \quad (5)$$

$$\tilde{v}|_{t \leq 0} \equiv 0, \quad \tilde{v}_x(+0, t) + \int_0^t k(t - \tau)\tilde{v}_x(+0, \tau)d\tau = c(+0)\delta'(t), \quad (6)$$

$$\tilde{v}(+0, t) = g(t), \quad t > 0, \tag{7}$$

where $\lambda(x) := a(x)b(x)$. In what follows $c(0), h(0), \lambda(0)$ and so on mean values of corresponding functions when x tends to 0 from right.

Now we transform integro-differentail equation (5) to exclude derivatives of function \tilde{v} with respect to x under the integral and to set coefficients of \tilde{v}_t and \tilde{v}_x in terms that outside the integral to zero. These requirements can be satisfied by introducing the new function u as follows

$$\left[\tilde{v}(x, t) + \int_0^t k(t - \tau)\tilde{v}(x, \tau)d\tau \right] \exp\left(-\frac{k(0)t}{2}\right) \sqrt{\frac{\lambda(0)}{\lambda(x)}} = u(x, t).$$

It is easy to verify that function \tilde{v} is expressed in terms of u as follows

$$\tilde{v}(x, t) = \left[\exp(k(0)t/2) u(x, t) + \int_0^t h(t - \tau) \exp(k(0)\tau/2) u(x, \tau)d\tau \right] \sqrt{\frac{\lambda(x)}{\lambda(0)}},$$

where

$$h(t) = -k(t) - \int_0^t k(t - \tau)h(\tau)d\tau. \tag{8}$$

Taking into account $u(x, t)$ and $h(t)$, equations (5) – (7) are written in the form

$$u_{tt} = u_{xx} + \Lambda(x)u + \int_0^t \tilde{k}(t - \tau)u(x, \tau)d\tau, \quad x > 0, \quad t > 0, \tag{9}$$

$$u|_{t \leq 0} \equiv 0, \tag{10}$$

$$u_x|_{x=+0} = c(0)\delta'(t) + \frac{1}{2}c(0)k(0)\delta(t) - \frac{\lambda'(0)}{2\lambda(0)}u(0, t), \tag{11}$$

$$u|_{x=+0} = \tilde{g}_0(t) + \int_0^t h_0(t - \tau)\tilde{g}_0(\tau)d\tau, \tag{12}$$

where

$$\Lambda(x) = \frac{h^2(0)}{4} - h'(0) + \frac{2\lambda(x)\lambda''(x) - 3(\lambda'(x))^2}{4\lambda^2(x)},$$

$$\tilde{k}(t) := h''(t) \exp\left(\frac{h(0)t}{2}\right), \quad \tilde{g}_0(t) = g(t) \exp\left(\frac{h(0)t}{2}\right), \quad h_0(t) := k(t) \exp\left(\frac{h(0)t}{2}\right).$$

It follows from the theory of hyperbolic equation that function $u(x, t)$ as the solution of the direct problem (9)–(12) has the property $u \equiv 0, t < x, x > 0$, and in the neighbourhood of the characteristic line $t = x$ has the following structure

$$u(x, t) = \alpha(x)\delta(t - x) + \theta(t - x)\tilde{u}(x, t), \tag{13}$$

where $\tilde{u}(x, t)$ is the regular function.

We denote $\beta(x) := \tilde{u}(x, x + 0)$. Substituting function (13) into equations (9)–(12) and using the method of singularity isolation [7, pp. 611–629], we find $\alpha'(x) = 0, \alpha(0) = -c(0), 2\beta'(x) - \Lambda(x)\alpha(x) = 0, \alpha'(0) - \beta(0) + \alpha(0)\frac{\lambda'(0)}{2\lambda(0)} = \frac{1}{2}c(0)k(0)$. Solving these ordinary differential equations, we obtain

$$\alpha(x) = -c(0), \quad \beta(x) = -\frac{c(0)}{2} \left(k(0) + \frac{\lambda'(0)}{\lambda(0)} + \int_0^x \Lambda(\xi)d\xi \right).$$

Then it follows that function $\tilde{u}(x, t)$ in the domain $D := \{(x, t) : t > x > 0\}$ satisfies equations

$$\tilde{u}_{tt} = \tilde{u}_{xx} + \Lambda(x)\tilde{u} - c(+0)\tilde{k}(t-x) + \int_0^t \tilde{k}(t-\tau)\tilde{u}(x, \tau)d\tau, \quad x > 0, \quad t > 0, \quad (14)$$

$$\tilde{u}|_{t=x+0} = \beta(x), \quad (15)$$

$$\left[\tilde{u}_x + \frac{\lambda'(0)}{2\lambda(0)}\tilde{u}(x, t) \right]_{x=0} = 0, \quad (16)$$

$$\tilde{u}|_{x=0} = \tilde{g}_0(t) - c(0)h_0(t) + \int_0^t h_0(\tau)\tilde{g}_0(t-\tau)d\tau, \quad t > 0. \quad (17)$$

Requiring continuity of functions $\tilde{u}(x, t)$, $(\partial\tilde{u}/\partial x)(x, t)$, it is easy to express $h(0)$, $h'(0)$ ($x = t = 0$) from relations (15)–(17):

$$h(0) = -\frac{2}{c(0)}\tilde{g}_0(0) - \frac{\lambda'(0)}{\lambda(0)}, \quad h'(0) = \frac{1}{2}(2h^2(0) - \Lambda(0)) - \frac{1}{c(0)}(\tilde{g}'_0(0) + \tilde{g}_0(0)h(0))$$

In order to obtain the last equalities the following relations are used

$$h'(t) = -k'(t) - k(0)h(t) - \int_0^t k'(t-\tau)h(\tau)d\tau,$$

$$h'(0) = -k'(0) + k^2(0).$$

They follow from (8). Further we substitute these relations for $h(0)$, $h'(0)$ into the expression for $\Lambda(x)$.

Let us introduce the following designations

$$\lambda_0 = \frac{\lambda'(0)}{2\lambda(0)}, \quad L_0 := \tilde{g}_0(0) + c(0)\lambda_0, \quad L_1 := \tilde{g}_0(0) - c(0)\lambda_0,$$

$$L_2 := \frac{2}{c(0)}L_0, \quad L_3 := \frac{2}{c(0)}(\tilde{g}'_0(0) - \lambda_0\tilde{g}_0(0)).$$

The proof of the theorem is based on the following lemma.

Lemma. *Fulfilling the conditions of the theorem problem (14) – (17) for $(x, t) \in D_T$, $D_T = ((x, t)|0 \leq x \leq t \leq T - x)$ is equivalent to the problem of finding functions $\tilde{u}(x, t)$, $(\partial\tilde{u}/\partial t)(x, t)$, $\tilde{k}(t)$, $h_0(t)$, $h'_0(t)$, $h''_0(t)$ that satisfy the following system of equations:*

$$\begin{aligned} \tilde{u}(x, t) = & \tilde{u}_0(x, t) + \\ & + \frac{1}{2} \int_0^x \int_{t-x+\xi}^{t+x-\xi} \left[\Lambda(\xi)\tilde{u}(\xi, \tau) - c(0)\tilde{k}(\tau-\xi) + \int_0^{\tau-\xi} \tilde{k}(\alpha)\tilde{u}(\xi, \tau-\alpha)d\alpha \right] d\tau d\xi, \quad (18) \end{aligned}$$

$$\begin{aligned} \frac{\partial\tilde{u}}{\partial t}(x, t) = & \frac{\partial\tilde{u}_0}{\partial t}(x, t) + \frac{1}{2} \int_{-x}^x \left[\Lambda(x-|\xi|)\tilde{u}(x-|\xi|, \xi+t) - c(0)\tilde{k}(\xi+t-x+|\xi|) + \right. \\ & \left. + \int_0^{\xi+t-x+|\xi|} \tilde{k}(x-|\xi|)\tilde{u}(x-|\xi|, \xi+t-\alpha)d\alpha \right] sgn(\xi)d\xi, \quad (19) \end{aligned}$$

$$\tilde{k}(t) = \frac{1}{c(0)} \left[\Lambda\left(\frac{t}{2}\right)\beta\left(\frac{t}{2}\right) - \beta''\left(\frac{t}{2}\right) \right] + \frac{2}{c(0)} \left[\tilde{g}''_0(t) - \lambda_0\tilde{g}'_0(t) \right] - 2h''_0(t) + L_2h'_0(t) + L_3h_0(t) +$$

$$\begin{aligned}
 & + \frac{2}{c(0)} \int_0^t \tilde{g}_0''(t-\tau)h_0(\tau)d\tau - \frac{2\lambda_0}{c(0)} \int_0^t \tilde{g}_0'(t-\tau)h_0(\tau)d\tau + \\
 & + \frac{2}{c(0)} \int_0^{\frac{t}{2}} \left[\Lambda(\xi) \frac{\partial \tilde{u}}{\partial t}(\xi, t-\xi) + k(t-2\xi)\beta(\xi) + \int_0^{t-2\xi} \tilde{k}(\tau) \frac{\partial \tilde{u}}{\partial t}(\xi, t-\xi-\tau)d\tau \right] d\xi, \quad (20)
 \end{aligned}$$

$$h_0(t) = -h(0) + \left(\frac{h^2(0)}{2} - h'(0) \right) t + \int_0^t (t-\tau)h_0''(\tau)d\tau, \quad (21)$$

$$h_0'(t) = \frac{h^2(0)}{2} - h'(0) + \int_0^t h_0''(\tau)d\tau, \quad (22)$$

$$h_0''(t) = -\tilde{k}(t) + \left(\frac{h^2(0)}{4} - h'(0) \right) h_0(t) - \int_0^t \tilde{k}(t-\tau)h_0(\tau)d\tau. \quad (23)$$

Using the d'Alembert formula, we obtain equation (18) from equations (14)–(17), where

$$\begin{aligned}
 \tilde{u}_0(x, t) = & \frac{1}{2} \left[\tilde{g}_0(t+x) + \tilde{g}_0(t-x) - c(+0)(h_0(t+x) + h_0(t-x)) + \right. \\
 & + \int_0^{t+x} h_0(\tau)\tilde{g}_0(t+x-\tau)d\tau + \int_0^{t-x} h_0(\tau)\tilde{g}_0(t-x-\tau)d\tau \left. \right] - \\
 & - \frac{\lambda_0}{2} \int_{t-x}^{t+x} \left[\tilde{g}_0(\tau) - c(+0)h_0(\tau) + \int_0^\tau h_0(\alpha)\tilde{g}_0(\tau-\alpha)d\alpha \right] d\tau.
 \end{aligned}$$

In order to get the equation for $\tilde{u}_t(x, t)$ we use the equivalent description of the domain D_T in equation (18):

$$\begin{aligned}
 \tilde{u}(x, t) = & \tilde{u}_0(x, t) + \\
 & + \frac{1}{2} \int_{t-x}^{t+x} \int_0^{|t-\tau|} \left[\Lambda(\xi)\tilde{u}(\xi, \tau) - c(+0)\tilde{k}(\tau-\xi) + \int_0^{\tau-\xi} \tilde{k}(\alpha)\tilde{u}(\xi, \tau-\alpha)d\alpha \right] d\xi d\tau.
 \end{aligned}$$

Differentiating this equation with respect to t , after some simplifications we obtain equation (19).

Taking the limit $t \rightarrow x + 0$ in equation (18), we obtain

$$\begin{aligned}
 \beta(x) = & \tilde{u}_0(x, x+0) + \\
 & + \frac{1}{2} \int_0^x \int_\xi^{2x-\xi} \left[\Lambda(\xi)\tilde{u}(\xi, \tau) - c(+0)\tilde{k}(\tau-\xi) + \int_0^{\tau-\xi} \tilde{k}(\alpha)\tilde{u}(\xi, \tau-\alpha)d\alpha \right] d\tau d\xi.
 \end{aligned}$$

Differentiating this equation two times with respect to x , after simple simplifications we obtain equation (20).

To close the system of integral equations (18)–(20) equalities (21)–(23) are used [5]. They follow from the definition of function $h_0(t)$ and equality (8). In fulfilling conditions the theorem, equivalence of system of integral equations (18)–(23) and inverse problem (14)–(17) is established by ordinary way [8]. Thus the lemma is proved.

3. Proof of the theorem

We present system of equations (18)–(23) in the form of operator equations

$$\varphi = A\varphi, \tag{24}$$

where

$$\begin{aligned} \varphi &= \left[\varphi_1(x, t), \varphi_2(x, t), \varphi_3(t), \varphi_4(t), \varphi_5(t), \varphi_6(t) \right] = \\ &= \left[\tilde{u}(x, t) + \frac{c(0)}{2}(h_0(t+x) + h_0(t-x)), \frac{\partial \tilde{u}}{\partial t}(x, t) + \frac{c(0)}{2}(\tilde{k}(t-x)x - \frac{1}{2}L_0(h_0(t+x) + L_1h_0(t-x)) + \right. \\ &\left. + \frac{c(0)}{2}(h'_0(t+x) + h'_0(t-x)), \tilde{k}(t) + 2h''_0(t) + \frac{2L_0}{c(0)}h'_0(t) + L_3h_0(t), h_0(t), h'_0(t), h''_0(t) + \tilde{k}(t) \right] \end{aligned}$$

is the vector function with components φ_i ($i = 1, 2, 3, 4, 5, 6$),

$$r_0 := \frac{h^2(0)}{2} - h'(0), \quad r_{00} := \frac{h^2(0)}{4} - h'(0).$$

Operator A is defined on the set of functions $\varphi \in C[D_T]$. Taking into account equations (18) – (23), it has the form

$$A = (A_1, A_2, A_3, A_4, A_5, A_6) :$$

$$\begin{aligned} A_1\varphi &= \varphi_{01} + \frac{1}{2} \left[\int_0^{t+x} \varphi_4(\tau)\tilde{g}_0(t+x-\tau)d\tau + \int_0^{t-x} \varphi_4(\tau)\tilde{g}_0(t-x-\tau)d\tau \right] - \\ &\quad + \frac{\lambda_0}{2} \int_{t-x}^{t+x} \left[c(+0)\varphi_4(\tau) - \int_0^\tau \varphi_4(\alpha)\tilde{g}_0(\tau-\alpha)d\alpha \right] d\tau + \\ &\quad + \frac{1}{2} \int_0^x \int_{t-x+\xi}^{t+x-\xi} \left[\Lambda(\xi) \left(\varphi_1(\xi, 2x-\xi) - \frac{c(0)}{2}(\varphi_4(2x) + \varphi_4(2x-2\xi)) \right) - c(0) \left(2\varphi_6(\tau-\xi) - \right. \right. \\ &\quad \left. \left. - \varphi_3(\tau-\xi) + \frac{2L_0}{c(0)}\varphi_5(\tau-\xi) + L_3\varphi_4(\tau-\xi) \right) + \int_0^{\tau-\xi} \left(2\varphi_6(\alpha) - \varphi_3(\alpha) + \frac{2L_0}{c(0)}\varphi_5(\alpha) + \right. \right. \\ &\quad \left. \left. + L_3\varphi_4(\alpha) \right) \left(\varphi_1(\xi, \tau-\alpha) - \frac{c(0)}{2}(\varphi_4(\tau-\alpha+\xi) + \varphi_4(\tau-\alpha-\xi)) \right) \right] d\alpha d\tau d\xi, \tag{25} \end{aligned}$$

$$\begin{aligned} A_2\varphi &= \varphi_{02} + \frac{1}{2} \int_0^{t-x} \varphi_4(\tau)\tilde{g}'_0(t-x-\tau)d\tau + \frac{1}{2} \int_0^{t+x} \varphi_4(\tau)\tilde{g}'_0(t+x-\tau)d\tau - \\ &- \frac{\lambda_0}{2} \int_{t-x}^{t+x} \varphi_4(\tau)\tilde{g}'_0(t+x-\tau)d\tau + \frac{1}{2} \int_{-x}^x \left[\Lambda(x-|\xi|) \left(\varphi_1(x-|\xi|, \xi+t) - \frac{c(0)}{2}(\varphi_4(\xi+t+x-|\xi|) + \right. \right. \\ &\left. \left. + \varphi_4(\xi+t-x+|\xi|)) \right) - c(0) \left(2\varphi_6(\xi+t-x+|\xi|) - \varphi_3(\xi+t-x+|\xi|) + \frac{2L_0}{c(0)}\varphi_5(\xi+t-x+|\xi|) + \right. \right. \\ &\quad \left. \left. + L_3\varphi_4(\xi+t-x+|\xi|) \right) + \int_0^{\xi+t-x+|\xi|} \left(2\varphi_6(\alpha) - \varphi_3(\alpha) + \frac{2L_0}{c(0)}\varphi_5(\alpha) + L_3\varphi_4(\alpha) \right) \times \right. \\ &\quad \left. \times \left(\varphi_1(x-|\xi|, \xi+t-\alpha) - \frac{c(0)}{2}(\varphi_4(\xi+t-\alpha+x-|\xi|) + \varphi_4(\xi+t-\alpha-x+|\xi|)) \right) d\alpha \right] \operatorname{sgn}(\xi)d\xi, \tag{26} \end{aligned}$$

$$\begin{aligned} A_3\varphi &= \varphi_{03} + \frac{2}{c(0)} \int_0^t \varphi_4(\tau)\tilde{g}''_0(t-\tau)d\tau - \frac{2\lambda_0}{c(0)} \int_0^t \varphi_4(\tau)\tilde{g}'_0(t-\tau)d\tau + \\ &\quad + \frac{2}{c(0)} \int_0^{t/2} \Lambda(\xi) \left(\varphi_2(\xi, t-\xi) + \frac{1}{2} \left(L_0\varphi_4(t) + L_1\varphi_4(t-2\xi) - c(0)(\varphi_5(t) + \varphi_5(t-2\xi)) \right) \right) - \end{aligned}$$

$$\begin{aligned}
 & -\frac{c(0)}{2}\xi\left(2\varphi_6(t-2\xi-\alpha)-\varphi_3(t-2\xi-\alpha)+\frac{2L_0}{c(0)}\varphi_5(t-2\xi-\alpha)+L_3\varphi_4(t-2\xi-\alpha)\right)d\xi+ \\
 & +\frac{2}{c(0)}\int_0^{t/2}\left(2\varphi_6(t-2\xi)-\varphi_3(t-2\xi)+\frac{2L_0}{c(0)}\varphi_5(t-2\xi)+L_3\varphi_4(t-2\xi)\right)\beta(\xi)d\xi \\
 & +\frac{2}{c(0)}\int_0^{t/2}\int_0^{t-2\xi}\left(2\varphi_6(\alpha)-\varphi_3(\alpha)+\frac{2L_0}{c(0)}\varphi_5(\alpha)+L_3\varphi_4(\alpha)\right)\times \\
 & \times\left(\varphi_2(\xi,t-\xi-\alpha)+\frac{1}{2}\left(\left(L_0\varphi_4(t-\alpha)+L_1\varphi_4(t-2\xi-\alpha)-c(0)(\varphi_5(t-\alpha)+\varphi_5(t-2\xi-\alpha))\right)-\right.\right. \\
 & \left.\left.-\frac{c(0)}{2}\xi\left(2\varphi_6(t-2\xi-\alpha)-\varphi_3(t-2\xi-\alpha)+\frac{2L_0}{c(0)}\varphi_5(t-2\xi-\alpha)+L_3\varphi_4(t-2\xi-\alpha)\right)\right)\right)d\alpha d\xi, \quad (27)
 \end{aligned}$$

$$A_4\varphi = \varphi_{04} + \int_0^t (t-\tau)\left(2\varphi_6(\tau)-\varphi_3(\tau)+\frac{2L_0}{c(0)}\varphi_5(\tau)+L_3\varphi_4(\tau)\right)d\tau, \quad (28)$$

$$A_5\varphi = \varphi_{05} + \int_0^t \left(2\varphi_6(\tau)-\varphi_3(\tau)+\frac{2L_0}{c(0)}\varphi_5(\tau)+L_3\varphi_4(\tau)\right)d\tau, \quad (29)$$

$$A_6\varphi = \varphi_{06} - \int_0^t \left(2\varphi_6(t-\tau)-\varphi_3(t-\tau)+\frac{2L_0}{c(0)}\varphi_5(t-\tau)+L_3\varphi_4(t-\tau)\right)\varphi_4(\tau)d\tau, \quad (30)$$

where the following notations are introduced

$$\begin{aligned}
 \varphi_0(x,t) & = (\varphi_{01}, \varphi_{02}, \varphi_{03}, \varphi_{04}, \varphi_{05}, \varphi_{06}) := \\
 & = \left[\frac{1}{2}\left(\tilde{g}_0(t+x)+\tilde{g}_0(t-x)\right)-\frac{\lambda_0}{2}\int_{t-x}^{t+x}\tilde{g}_0(\tau)d\tau, \frac{1}{2}\left(\tilde{g}'_0(t+x)+\tilde{g}'_0(t-x)+\lambda_0(\tilde{g}_0(t-x)-\right.\right. \\
 & \left.\left.-\tilde{g}_0(t+x)\right), \frac{1}{c(0)}\left(2(\tilde{g}''_0(t)-\lambda_0\tilde{g}'_0(t))+\Lambda\left(\frac{t}{2}\right)\beta\left(\frac{t}{2}\right)-\beta''\left(\frac{t}{2}\right)\right), -h(0)+r_0t, r_{00}, 0 \right]. \quad (31)
 \end{aligned}$$

Let C_σ be the Banach space of continuous functions. It is induced by family of weighted norms

$$\|\varphi\|_\sigma = \max \left\{ \sup_{(x,t)\in D_T} \left| \varphi_i(x,t)e^{-\sigma(t+(1+\rho)x)} \right|, i = 1, 2, \sup_{t\in[0,T]} \left| \varphi_j(t)e^{-\sigma t} \right|, j = 3, 4, 5, 6 \right\},$$

$\sigma \geq 0$, $\rho \in (0, 1)$ is some fixed number.

It is clear that when $\sigma = 0$ this space is the space of continuous functions with ordinary norm. Further we denote this norm by $\|\varphi\|$. By virtue of

$$e^{-\sigma T}\|\varphi\| \leq \|\varphi\|_\sigma \leq \|\varphi\|$$

norms $\|\varphi\|_\sigma$ and $\|\varphi\|$ are equivalent for any fixed $T \in (0, \infty)$. The number σ will be chosen later. Let $Q_\sigma(\varphi_0, \|\varphi_0\|) =: \{\varphi \mid \|\varphi - \varphi_0\|_\sigma \leq \|\varphi_0\|\}$ be the radius of the ball $\|\varphi_0\|$ with centre at the point φ_0 of some weighted space $C_\sigma(\sigma \geq 0)$, where function φ_0 is defined by (33) and

$$\|\varphi_0\| = \max \{ \|\varphi_{01}\|, \|\varphi_{02}\|, \|\varphi_{03}\|, \|\varphi_{04}\|, \|\varphi_{05}\|, \|\varphi_{06}\|, \}.$$

It is easy to see that inequality $\|\varphi\|_\sigma \leq \|\varphi_0\|_\sigma + \|\varphi_0\| \leq 2\|\varphi_0\|$ takes place for $\varphi \in Q_\sigma(\varphi_0, \|\varphi_0\|)$.

We prove that operator A is a contracting operator on the set $Q_\sigma(\varphi_0, \|\varphi_0\|)$ if the number $\sigma > 0$ is appropriately chosen. Let $\varphi(x, t) \in Q_\sigma(\varphi_0, \|\varphi_0\|)$. First we show that if $\sigma > 0$ is

appropriately chosen operator A translates ball to the ball, i.e., $A\varphi \in Q_\sigma(\varphi_0, \|\varphi_0\|)$. Actually, with the help of equalities (27)–(32) we have

$$\begin{aligned} \|A_1\varphi - \varphi_{01}\|_\sigma &= \sup_{(x,t) \in D_T} |(A_1\varphi - \varphi_{01})e^{-\sigma(t+(1+\rho)x)}| = \\ &= \sup_{(x,t) \in D_T} \left| \frac{1}{2} \left[\int_0^{t+x} \varphi_4(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau+(1+\rho)x)}\tilde{g}_0(t+x-\tau)d\tau + \right. \right. \\ &\quad \left. \left. + \int_0^{t-x} \varphi_4(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau+(1+\rho)x)}\tilde{g}_0(t-x-\tau)d\tau \right] - \right. \\ &+ \frac{\lambda_0}{2} \int_{t-x}^{t+x} \left[c(0)\varphi_4(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau+(1+\rho)x)} - \int_0^\tau \varphi_4(\alpha)e^{-\sigma\alpha}e^{-\sigma(t-\alpha+(1+\rho)x)}\tilde{g}_0(\tau-\alpha)d\alpha \right] d\tau + \\ &\quad + \frac{1}{2} \int_0^x \int_{t-x+\xi}^{t+x-\xi} \left[\Lambda(\xi) \left(\varphi_1(\xi, \tau)e^{-\sigma(\tau+(1+\rho)\xi)}e^{-\sigma(t-\tau-(1+\rho)\xi+(1+p)x)} - \right. \right. \\ &\quad \left. \left. - \frac{c(0)}{2} \left(\varphi_4(\tau-\xi)e^{-\sigma(\tau-\xi)}e^{-\sigma(t-\tau+\xi+(1+p)x)} + \varphi_4(\tau+\xi)e^{-\sigma(\tau+\xi)}e^{-\sigma(t-\tau-\xi+(1+p)x)} \right) - \right. \right. \\ &\quad \left. \left. - c(0) \left(2\varphi_6(\tau-\xi) - \varphi_3(\tau-\xi) + \frac{2L_0}{c(0)}\varphi_5(\tau-\xi) + L_3\varphi_4(\tau-\xi) \right) e^{-\sigma(\tau-\xi)}e^{-\sigma(t-\tau+\xi+(1+p)x)} + \right. \right. \\ &\quad \left. \left. + \int_0^{\tau-\xi} \left(2\varphi_6(\alpha) - \varphi_3(\alpha) + \frac{2L_0}{c(0)}\varphi_5(\alpha) + L_3\varphi_4(\alpha) \right) \times \right. \right. \\ &\quad \left. \left. \times e^{-\sigma\alpha} \left(\varphi_1(\xi, \tau-\alpha)e^{-\sigma(\tau-\alpha+(1+p)\xi)}e^{-\sigma(t-\tau-(1+p)\xi+(1+p)x)} - \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{c(0)}{2} \left(\varphi_4(\tau-\alpha+\xi)e^{-\sigma(\tau-\alpha+\xi)}e^{-\sigma(t-\tau-\xi+(1+p)x)} + \right. \right. \right. \\ &\quad \left. \left. \left. + \varphi_4(\tau-\alpha-\xi)e^{-\sigma(\tau-\alpha-\xi)}e^{-\sigma(t-\tau+\xi+(1+p)x)} \right) \right) d\alpha \right] d\tau d\xi \Big| \leq \\ &\leq \frac{\|\varphi_0\|}{\sigma} \left[2G_0(1 + \lambda_0T) + \lambda_0c(0) + \Lambda_0T(1 + c(0)) + \right. \\ &\quad \left. + 2L_{00}T(c(0) + 2T\|\varphi_0\|(1 + c(0))) \right] = \frac{\|\varphi_0\|}{\sigma} \alpha_1, \quad (32) \end{aligned}$$

$$\begin{aligned} \|A_2\varphi - \varphi_{02}\|_\sigma &= \sup_{(x,t) \in D_T} |(A_2\varphi - \varphi_{02})e^{-\sigma(t+(1+\rho)x)}| \leq \\ &\leq \frac{\|\varphi_0\|}{\sigma} \left[2G_1 + \lambda_0 + \Lambda_0(1 + c(0)) + L_{00} \left(c(0)\Lambda_0 + 2\|\varphi_0\|(T + c(0)) \right) \right] = \frac{\|\varphi_0\|}{\sigma} \alpha_2, \quad (33) \end{aligned}$$

$$\begin{aligned} \|A_3\varphi - \varphi_{03}\|_\sigma &= \sup_{t \in [0, T]} |(A_3\varphi - \varphi_{03})e^{-\sigma t}| \leq \\ &\leq \frac{\|\varphi_0\|}{\sigma} \left[\frac{1}{c(0)} \left(\Lambda_0(L_1 + c(0)) + L_{00} \left(B_0 + \|\varphi_0\| \left(4T + 3(L_0 + c(0)) + \frac{T^2}{2} \right) \right) + \right. \right. \\ &\quad \left. \left. + 4 \left(G_2 + \lambda_0G_1 + \frac{\Lambda_0}{p} + \frac{L_0}{2} \right) \right) + c(0)L_{00}\Lambda_0T + 2 \right] = \frac{\|\varphi_0\|}{\sigma} \alpha_3, \quad (34) \end{aligned}$$

$$\|A_4\varphi - \varphi_{04}\|_\sigma = \sup_{t \in [0, T]} |(A_4\varphi - \varphi_{04})e^{-\sigma t}| \leq 2 \frac{\|\varphi_0\|}{\sigma} L_{00}T = \frac{\|\varphi_0\|}{\sigma} \alpha_4, \quad (35)$$

$$\|A_5\varphi - \varphi_{05}\|_\sigma = \sup_{t \in [0, T]} |(A_5\varphi - \varphi_{05})e^{-\sigma t}| \leq 2 \frac{\|\varphi_0\|}{\sigma} L_{00} = \frac{\|\varphi_0\|}{\sigma} \alpha_5, \quad (36)$$

$$\|A_6\varphi - \varphi_{06}\|_\sigma = \sup_{t \in [0, T]} |(A_6\varphi - \varphi_{06})e^{-\sigma t}| \leq 2 \frac{\|\varphi_0\|}{\sigma} L_{00} [h_0 + r_0 T + 2L_{00} T^2 \|\varphi_0\|] = \frac{\|\varphi_0\|}{\sigma} \alpha_6. \quad (37)$$

Let $\sigma \geq \alpha_0 := \max(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$, then A translates ball $Q_\sigma(\varphi_0, \|\varphi_0\|)$ to ball $Q_\sigma(\varphi_0, \|\varphi_0\|)$.

Let now φ^1, φ^2 be any two elements from $Q_\sigma(\varphi_0, \|\varphi_0\|)$. Then carry out some easy estimations we obtain

$$\begin{aligned} \|(A\varphi^1 - A\varphi^2)_1\|_\sigma &= \sup_{(x,t) \in D_2} |(A\varphi^1 - A\varphi^2)_1 e^{-\sigma(t+(1+p)x)}| = \\ &\sup_{(x,t) \in D_T} \left| \frac{1}{2} \left[\int_0^{t+x} (\varphi_4^1(\tau) - \varphi_4^2(\tau)) e^{-\sigma\tau} e^{-\sigma(t-\tau+(1+p)x)} \tilde{g}_0(t+x-\tau) d\tau + \right. \right. \\ &\quad \left. \left. + \int_0^{t-x} (\varphi_4^1(\tau) - \varphi_4^2(\tau)) e^{-\sigma\tau} e^{-\sigma(t-\tau+(1+p)x)} \tilde{g}_0(t-x-\tau) d\tau \right] - \right. \\ &\quad \left. + \frac{\lambda_0}{2} \int_{t-x}^{t+x} \left[c(0) (\varphi_4^1(\tau) - \varphi_4^2(\tau)) e^{-\sigma\tau} e^{-\sigma(t-\tau+(1+p)x)} - \right. \right. \\ &\quad \left. \left. - \int_0^\tau (\varphi_4^1(\alpha) - \varphi_4^2(\alpha)) e^{-\sigma\alpha} e^{-\sigma(t-\alpha+(1+p)x)} \tilde{g}_0(\tau-\alpha) d\alpha \right] d\tau + \right. \\ &\quad \left. + \frac{1}{2} \int_0^x \int_{t-x+\xi}^{t+x-\xi} \left[\Lambda(\xi) \left((\varphi_1^1(\xi, \tau) - \varphi_1^2(\xi, \tau)) e^{-\sigma(\tau+(1+p)\xi)} e^{-\sigma(t-\tau-(1+p)\xi+(\rho+1)x} - \right. \right. \right. \\ &\quad \left. \left. - \frac{c(0)}{2} \left((\varphi_4^1(\tau-\xi) - \varphi_4^2(\tau-\xi)) e^{-\sigma(\tau-\xi)} e^{-\sigma(t-\tau+\xi+(1+p)x} + \right. \right. \right. \\ &\quad \left. \left. \left. + (\varphi_4^1(\tau+\xi) - \varphi_4^2(\tau+\xi)) e^{-\sigma(\tau+\xi)} e^{-\sigma(t-\tau-\xi+(1+p)x)} \right) \right) - \right. \\ &\quad \left. - c(0) \left(2(\varphi_6^1(\tau-\xi) - \varphi_6^2(\tau-\xi)) - (\varphi_3^1(\tau-\xi) - \varphi_3^2(\tau-\xi)) + \frac{2L_0}{c(0)} (\varphi_5^1(\tau-\xi) - \varphi_5^2(\tau-\xi)) + \right. \right. \\ &\quad \left. \left. + L_3 (\varphi_4^1(\tau-\xi) - \varphi_4^2(\tau-\xi)) \right) e^{-\sigma(\tau-\xi)} e^{-\sigma(t-\tau+\xi+(1+p)x} + \right. \\ &\quad \left. + \int_0^{\tau-\xi} \left[\left(2(\varphi_6^1(\alpha) - \varphi_6^2(\alpha)) - (\varphi_3^1(\alpha) - \varphi_3^2(\alpha)) + \frac{2L_0}{c(0)} (\varphi_5^1(\alpha) - \varphi_5^2(\alpha)) + L_3 (\varphi_4^1(\alpha) - \varphi_4^2(\alpha)) \right) e^{-\sigma\alpha} \times \right. \right. \\ &\quad \left. \left. \times \left(\varphi_1^1(\xi, \tau - \alpha) e^{-\sigma(\tau-\alpha+(1+p)\xi)} e^{-\sigma(t-\tau-(1+p)\xi+(1+p)x} - \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{c(0)}{2} \left(\varphi_4^1(\tau-\alpha+\xi) e^{-\sigma(\tau-\alpha+\xi)} e^{-\sigma(t-\tau-\xi+(1+p)x} + \varphi_4^2(\tau-\alpha-\xi) e^{-\sigma(\tau-\alpha-\xi)} e^{-\sigma(t-\tau+\xi+(1+p)x)} \right) \right) \right) + \right. \\ &\quad \left. + \left(2\varphi_6(\alpha) - \varphi_3(\alpha) + \frac{2L_0}{c(0)} \varphi_5(\alpha) + L_3 \varphi_4(\alpha) \right) e^{-\sigma\alpha} \times \right. \\ &\quad \left. \times \left((\varphi_1^1(\xi, \tau - \alpha) - \varphi_1^2(\xi, \tau - \alpha)) e^{-\sigma(\tau-\alpha+(1+p)\xi)} e^{-\sigma(t-\tau-(1+p)\xi+(1+p)x} - \right. \right. \\ &\quad \left. \left. - \frac{c(0)}{2} \left((\varphi_4^1(\tau - \alpha + \xi) - \varphi_4^2(\tau - \alpha + \xi)) e^{-\sigma(\tau-\alpha+\xi)} e^{-\sigma(t-\tau-\xi+(1+p)x} + \right. \right. \right. \\ &\quad \left. \left. \left. + (\varphi_4^1(\tau - \alpha - \xi) - \varphi_4^2(\tau - \alpha - \xi)) e^{-\sigma(\tau-\alpha-\xi)} e^{-\sigma(t-\tau+\xi+(1+p)x)} \right) \right) \right] d\alpha \right] d\tau d\xi \Big| \leq \\ &\leq \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \left[G_0(1 + \lambda_0 T) + \frac{\lambda_0 c(0)}{2} + \frac{\Lambda_0}{2} T(1 + c(0)) + \right. \\ &\quad \left. + L_{00} T \left(c(0) + 4T \|\varphi_0\| (1 + c(0)) \right) \right] = \frac{\|\varphi_0\|}{\sigma} \beta_1, \quad (38) \end{aligned}$$

$$\begin{aligned} \|(A\varphi^1 - A\varphi^2)_2\|_\sigma &= \sup_{(x,t) \in D_2} |(A\varphi^1 - A\varphi^2)_2 e^{-\sigma(t+(1+p)x)}| \leq \\ &\leq \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \left[G_1 + \frac{\lambda_0 + \Lambda_0(1 + c(0))}{2} + L_{00} \left(\frac{c(0)}{2} \Lambda_0 + 2\|\varphi_0\|(T + c(0)) \right) \right] = \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \beta_2, \end{aligned} \quad (39)$$

$$\begin{aligned} \|(A\varphi^1 - A\varphi^2)_3\|_\sigma &= \sup_{t \in [0, T]} |(A\varphi^1 - A\varphi^2)_3 e^{-\sigma t}| \leq \\ &\leq \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \left[\frac{1}{2c(0)} \left(\Lambda_0(L_1 + c(0)) + L_{00} \left(B_0 + 2\|\varphi_0\| \left(4T + 3(L_0 + c(0)) + \frac{T^2}{2} \right) \right) + \right. \right. \\ &\quad \left. \left. + 4 \left(G_2 + \lambda_0 G_1 + \frac{\Lambda_0}{p} + \frac{L_0}{2} \right) \right) + \frac{c(0)}{2} L_{00} \Lambda_0 T + 1 \right] = \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \beta_3, \end{aligned} \quad (40)$$

$$\|(A\varphi^1 - A\varphi^2)_4\|_\sigma = \sup_{t \in [0, T]} |(A\varphi^1 - A\varphi^2)_4 e^{-\sigma t}| \leq \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} L_{00} T = \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \beta_4 \quad (41)$$

$$\|(A\varphi^1 - A\varphi^2)_5\|_\sigma = \sup_{t \in [0, T]} |(A\varphi^1 - A\varphi^2)_5 e^{-\sigma t}| \leq \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} L_{00} = \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \beta_5 \quad (42)$$

$$\begin{aligned} \|(A\varphi^1 - A\varphi^2)_6\|_\sigma &= \sup_{t \in [0, T]} |(A\varphi^1 - A\varphi^2)_6 e^{-\sigma t}| \leq \\ &\leq \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} L_{00} [h_0 + r_0 T + 4L_{00} T^2 \|\varphi_0\|] = \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma} \beta_6 \end{aligned} \quad (43)$$

where $\beta_0 := \max(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$.

Let us introduce the following designations in equalities (32)–(43) $L_{00} := 3 + \frac{2L_0}{c(0)} + L_3$, $\Lambda_0 := \max_{x \in [0, T/2]} |\Lambda(x)|$, $B_0 := \max_{x \in [0, T/2]} |\beta(x)|$, $G_0 := \max_{t \in [0, T]} |\tilde{g}_0(t)|$, $G_1 := \max_{t \in [0, T]} |\tilde{g}'_0(t)|$, $G_2 := \max_{t \in [0, T]} |\tilde{g}''_0(t)|$. Hence if $\sigma > \beta_0$, then operator A performs contracting mapping on elements of the set $Q_\sigma(\varphi_0, \|\varphi_0\|)$.

It follows from above estimations that if the number σ is chosen from the condition $\sigma > \max(\alpha_0, \beta_0)$, then operator A is contracting operator on $Q_\sigma(\varphi_0, \|\varphi_0\|)$. Then by Banach principle equality (24) has in addition unique solution in $Q_\sigma(\varphi_0, \|\varphi_0\|)$ for any fixed $T > 0$.

Since $h_0(t) = k(t) \exp(h(0)t/2)$ then function $k(t)$ is defined as follows

$$k(t) = h_0(t) \exp(-h(0)t/2).$$

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Глобальная разрешимость одномерной обратной задачи для интегродифференциального уравнения акустики

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Рассматривается гиперболическое интегродифференциальное уравнение акустики. Прямую задачу представляет задача о нахождении акустического давления из начально-краевой задачи для этого уравнения сосредоточенным источником возбуждения, расположенным на границе пространственной области. Для прямой задачи изучается обратная задача, состоящая в определении одномерного ядра интегрального члена по известной в точке $x = 0$ для $t > 0$ решению прямой задачи. Эта задача сводится к решению системы интегральных уравнений относительно неизвестных функций. К последней в пространстве непрерывных функций с весовой нормой применяется принцип сжатых отображений. Доказана глобальная однозначная разрешимость поставленной задачи.

Ключевые слова: интегро-дифференциальные уравнения, обратная задача, дельта-функция Дирака, ядро интеграла, весовая функция.