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Study of the Non-isothermal Coupled Problem with Mixed Boundary Conditions in a Thin Domain with Friction Law

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This paper deals with the asymptotic behavior of a coupled system involving of an incompressible Bingham fluid and the equation of the heat energy, in a three-dimensional bounded domain with Tresca free boundary friction conditions. First we prove the existence and uniqueness results for the weak solution. Second, we show the strong convergence of the velocity and the temperature. Then a specific Reynolds limit equation is obtained, and the uniqueness of the limit velocity and temperature are proved.

Keywords: asymptotic approach, boundary conditions, Coupled problem, Fourier law, non-isothermal Bingham fluid, Tresca law, Reynolds equation.
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Introduction

In many problems, which study the asymptotic behavior for a problem of continuum mechanics in a thin domain, we transform the problem into an equivalent problem on a domain Ω independent of the parameter ε. Specifically, the case of Bingham fluid has been studied by many authors, for example : the analysis of the Bingham fluid flow variational inequality was carried out in [9], where the authors investigated the existence, uniqueness and regularity of the solution for the steady and in-stationary flows in a reservoir. Existence and extra regularity results for the d-dimensional Bingham fluid flow problem with Dirichlet boundary conditions are also studied in [11, 12]. The numerical solution of the stationary Bingham fluid flow problem is studied in [6, 7, 13]. The study of the a nonlinear boundary value problem governed by partial differential equations which describe the evolution of nonlinear elastic materials has been considered in [1]. In [10], the author has given in the last chapter of his doctoral thesis the asymptotic behavior of a Bingham fluid in a thin domain. Unfortunately this work is not done due to the difficulty encountered in this study which resides on the choice of test functions because of the boundary conditions imposed. Then in [5], the authors studied the same problem, in which, only the Dirichlet conditions on the boundary have been considered. The authors in [8] have proved the asymptotic analysis of a isothermal Bingham fluid in a thin domain with non linear Tresca boundary conditions.

In this present paper, we further the research of [8] on the asymptotic behavior of a Bingham fluid in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with boundary $\Gamma^\varepsilon = \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon \cup \overline{\omega}$. However, this time we consider a coupled problem which describes the motion of an incompressible fluid in a thin domain,

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governed by a coupled system of the equation of motion and the equation of the heat energy, obtained by using Fourier's law and neglecting the dissipation term. We consider Dirichlet boundary conditions on $\overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon$, where $\overline{\Gamma}_L^\varepsilon$ is the lateral one, the Fourier boundary condition at the top surface $\overline{\Gamma}_1^\varepsilon$, finally, a nonlinear Tresca interface condition and homogeneous Neumann condition for the temperature at the bottom one ω. The weak form of the problem is a variational equality. The small change of variable $z=\dfrac{x_3}{\varepsilon}$, transforms the initial problem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε. We prove some estimates on the displacement and temperature independent of the small parameter. The passage to the limit on ε, permits us to obtain a weak form of the Reynolds equation, give a lower-dimensional Bingham law, prevalent in engineering literature and the uniqueness of the solution (u^*, T^*).

This article is organized as follows. In Section 1, we recall the weak formulation of our coupled problem considered. Some estimates and convergence theorem by using the Korn and Poincaré inequalities (developed recently in Refs [3, 4]) are given in Section 2. The limit problem with a specific weak form of the Reynolds equation, the uniqueness of the limit velocity and temperature are given in Section 3.

1. Problem statement and variational formulation

Let ω be fixed region in the plane $x' = (x_1, x_2) \in \mathbb{R}^2$. We suppose that ω has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x')$ where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0 < \underline{h} \leqslant h(x') \leqslant \overline{h}$ for all $(x', 0)$ in ω. We denote by Ω^ε the domain of the flow:

$$\Omega^\varepsilon = \left\{x = (x',\, x_3) \in \mathbb{R}^3 :\, (x', 0) \in \omega,\ 0 < x_3 < \varepsilon h(x')\right\}.$$

The boundary of Ω^ε is Γ^ε. We have $\Gamma^\varepsilon = \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon \cup \overline{\omega}$ where Γ_L^ε is the lateral boundary.

Let σ^ε denotes the total Cauchy stress tensor: $\sigma^\varepsilon = -p^\varepsilon I + \sigma^{D,\varepsilon}$, where $\sigma^{D,\varepsilon}$ denotes its deviatoric part, and p^ε the pressure. The fluid is supposed to be viscoplastic, and the relation between $\sigma^{D,\varepsilon}$ and $D(u^\varepsilon)$ is given by the Bingham model:

$$\begin{cases} \sigma^{D,\varepsilon} = \alpha^\varepsilon \dfrac{D(u^\varepsilon)}{|D(u^\varepsilon)|} + 2\mu D(u^\varepsilon), & \text{when } D(u^\varepsilon) \neq 0; \\ |\sigma^{D,\varepsilon}| \leqslant \alpha^\varepsilon, & \text{when } D(u^\varepsilon) = 0, \end{cases}$$

or equivalently:

$$D(u^\varepsilon) = \begin{cases} \dfrac{1}{2\mu}\left(1 - \dfrac{\alpha^\varepsilon}{|\sigma^{D,\varepsilon}|}\right)\sigma^{D,\varepsilon} & \text{when } \left|\sigma^{D,\varepsilon}\right| > \alpha^\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

here $\alpha^\varepsilon \geqslant 0$ is the yield stress, $\mu > 0$ is the constant viscosity, u^ε is the velocity field and $D(u^\varepsilon) = \dfrac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$. For any tensor $\tau = (\tau_{ij})$, the notation $|\tau|$ represents the matrix norm: $|\tau| = \dfrac{1}{\sqrt{2}}\left(\sum_{i,j}\tau_{ij}\right)^{\frac{1}{2}}$.

• The law of conservation of momentum

$$-div\,(\sigma^\varepsilon) = f^\varepsilon \quad \text{in } \ \Omega^\varepsilon. \tag{2.1}$$

where $f^\varepsilon = (f_i^\varepsilon)_{1\leqslant i\leqslant 3}$, denotes the body forces.

• The equation of the heat energy

$$-\frac{\partial}{\partial x_i}\left(G^{\varepsilon}\frac{\partial T^{\varepsilon}}{\partial x_i}\right)=2\mu^{\varepsilon}\left(T^{\varepsilon}\right)d_{ij}\left(u^{\varepsilon}\right)d_{ij}\left(u^{\varepsilon}\right)+\sqrt{2}\alpha^{\varepsilon}\left|D\left(u^{\varepsilon}\right)\right|+r^{\varepsilon}\left(T^{\varepsilon}\right)\quad \text{in } \Omega^{\varepsilon} \tag{2.2}$$

obtained by using Fourier's law in which we neglect the dissipation term, where $G^{\varepsilon}=G^{\varepsilon}(x)$ is the thermal conductivity and $r^{\varepsilon}\left(T^{\varepsilon}\right)$ is the heat sources (see [9]).
• The incompressibility equation

$$div(u^{\varepsilon})=0 \quad \text{in } \Omega^{\varepsilon}. \tag{2.3}$$

Our boundary conditions are describe as
• At the surface Γ_1^{ε} we assume

$$\left.\begin{array}{c}\sigma_{\tau}(u^{\varepsilon})+l^{\varepsilon}u^{\varepsilon}=0\\ u^{\varepsilon}\cdot n=0\end{array}\right\}\text{on } \Gamma_1^{\varepsilon}, \tag{2.4}$$

where $l^{\varepsilon}>0$ on which we will bring precisions.
• On Γ_L^{ε}, the velocity is known and is parallel to the ω-plane

$$u^{\varepsilon}=0, \text{ on } \Gamma_L^{\varepsilon}. \tag{2.5}$$

• On ω, there is no-flux condition across ω so that

$$u^{\varepsilon}\cdot n=0, \tag{2.6}$$

the tangential velocity on ω is unknown and satisfies Tresca boundary conditions with friction coefficient k^{ε} (as [9]):

$$\left.\begin{array}{l}|\sigma_{\tau}^{\varepsilon}|<k^{\varepsilon}\Rightarrow u_{\tau}^{\varepsilon}=0\\ |\sigma_{\tau}^{\varepsilon}|=k^{\varepsilon}\Rightarrow \exists\lambda\geqslant 0 \ u_{\tau}^{\varepsilon}=-\lambda\sigma_{\tau}^{\varepsilon}\end{array}\right\}\text{ on } \omega, \tag{2.7}$$

Here $n=(n_1,n_2,n_3)$ is the unit outward normal to Γ^{ε}, and

$$u_n^{\varepsilon}=u^{\varepsilon}.n,\ \ u_{\tau}^{\varepsilon}=u^{\varepsilon}-u_n^{\varepsilon}.n,\ \ \sigma_n^{\varepsilon}=\left(\sigma^{\varepsilon}.n\right).n \ \text{ and } \sigma_{\tau}^{\varepsilon}=\sigma^{\varepsilon}.n-\left(\sigma_n^{\varepsilon}\right).n$$

are, respectively, the normal and the tangential velocity on ω, and the components of the normal and the tangential stress tensor on ω.

For the temperature, we suppose that

$$T^{\varepsilon}=0 \text{ on } \Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}, \tag{2.8}$$

$$\frac{\partial T^{\varepsilon}}{\partial n}=0 \ \text{ on } \omega. \tag{2.9}$$

To get a weak formulation, we introduce :

$$K^{\varepsilon}=\left\{\phi\in H^1(\Omega^{\varepsilon})^3:\phi=0 \text{ on } \Gamma_L^{\varepsilon},\ \phi.n=0 \text{ on } \omega\cup\ \Gamma_1^{\varepsilon}\right\}, \tag{2.10}$$

$$K_{div}^{\varepsilon}=\left\{\phi\in K^{\varepsilon}:div(\phi)=0\right\},$$

$$L_0^2(\Omega^{\varepsilon})=\left\{q\in L^2(\Omega^{\varepsilon}):\int_{\Omega^{\varepsilon}}q\,dx=0\right\},$$

$$H^1_{\Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}}\left(\Omega^{\varepsilon}\right)=\left\{\varphi\in H^1\left(\Omega^{\varepsilon}\right):\varphi=0 \ \text{ on } \Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}\right\}.$$

A formal application of Green's formula, using (2.1)–(2.9) leads to the weak formulation:

Find $u^\varepsilon \in K^\varepsilon_{div}$, $p^\varepsilon \in L^2_0(\Omega^\varepsilon)$ and $T^\varepsilon \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega^\varepsilon)$ such that

$$a\left(T^\varepsilon; u^\varepsilon, \varphi - u^\varepsilon\right) - \left(p^\varepsilon, \operatorname{div}\varphi\right) + l^\varepsilon \int_{\Gamma_1^\varepsilon} u^\varepsilon\left(\varphi - u^\varepsilon\right) d\tau + J\left(\varphi\right) - J\left(u^\varepsilon\right) \geqslant$$

$$\geqslant \left(f^\varepsilon, \varphi - u^\varepsilon\right), \quad \forall\varphi \in K^\varepsilon\left(\Omega^\varepsilon\right), \tag{2.11}$$

$$b\left(T^\varepsilon, \varphi\right) = c\left(u^\varepsilon; T^\varepsilon, \varphi\right), \quad \forall\varphi \in H^1_{\Gamma_1\cup\Gamma_L}\left(\Omega^\varepsilon\right), \tag{2.12}$$

where

$$a\left(T^\varepsilon; u, \varphi\right) = 2\int_{\Omega^\varepsilon} mu\left(T^\varepsilon\right) d_{ij}\left(u^\varepsilon\right) d_{ij}\left(\varphi\right) dx, \tag{2.13}$$

$$\left(p^\varepsilon, div\,\varphi\right) = \int_{\Omega^\varepsilon} p^\varepsilon\, div\,\varphi dx, \tag{2.14}$$

$$j(\varphi) = \int_\omega k^\varepsilon\left|\varphi\right|\ dx' + \sqrt{2}\alpha^\varepsilon \int_{\Omega^\varepsilon} \left|D\left(\varphi\right)\right| dx, \tag{2.15}$$

$$b\left(T^\varepsilon, \varphi\right) = \int_{\Omega^\varepsilon} G^\varepsilon\left(\nabla T^\varepsilon\right)\left(\nabla\varphi\right) dx, \tag{2.16}$$

$$c\left(u; T^\varepsilon, \varphi\right) = 2\int_{\Omega^\varepsilon} \mu\left(T^\varepsilon\right) d_{ij}\left(u\right) d_{ij}\left(u\right)\varphi dx + 2\alpha^\varepsilon \int_{\Omega^\varepsilon} \left|D\left(u\right)\right|\varphi dx + \int_{\Omega^\varepsilon} r^\varepsilon\left(T^\varepsilon\right)\varphi dx. \tag{2.17}$$

Theorem 1.1. *Assume that $f^\varepsilon \in L^2(\Omega^\varepsilon)^3$ and $k^\varepsilon \in L^\infty_+(\omega)$; then there exists a unique $u^\varepsilon \in K^\varepsilon_{div}$, $T^\varepsilon \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega^\varepsilon)$ and $p^\varepsilon \in L^2_0(\Omega^\varepsilon)$ (to an additive constant) solution to problem (2.11)–(2.12).*

Proof. • For the proof of the equality (2.11), see [8, 10].
• Now, for the equality (2.12), multiplying the equality (2.2) by $\varphi \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega^\varepsilon)$ and by using Green's formula, we find

$$\sum_{i=1}^3 \int_{\Omega^\varepsilon} \frac{\partial T^\varepsilon}{\partial x_i}\frac{\partial\varphi}{\partial x_i} dx = \sum_{i,j=1}^3 \int_{\Omega^\varepsilon} 2\mu^\varepsilon\left(T^\varepsilon\right) D^2\left(u^\varepsilon\right)\varphi dx + \sqrt{2}\alpha^\varepsilon \int_{\Omega^\varepsilon} \frac{D^2\left(u^\varepsilon\right)}{\left|D\left(u^\varepsilon\right)\right|}\varphi dx + \int_{\Omega^\varepsilon} r^\varepsilon\left(T^\varepsilon\right)\varphi dx$$

and as $D^2\left(u^\varepsilon\right) = \left|D\left(u^\varepsilon\right)\right|^2$, we obtain (2.12). □

2. Change of the domain and some estimates

We shall now focus our attention on the asymptotic analysis of problem (2.1)–(2.9). For this, we transform this problem into an equivalent one on a domain Ω independent of the parameter ε via the rescaling $z=\dfrac{x_3}{\varepsilon}$. So, for (x', x_3) in Ω^ε, we have (x', z) in

$$\Omega = \left\{(x', z) \in \mathbb{R}^3,\ (x', 0) \in \omega \text{ and } 0 < z < h\left(x'\right)\right\},$$

and we denote by $\Gamma = \bar{\omega} \cup \Gamma_L \cup \Gamma_1$ its boundary, then we define the following functions in Ω

$$\hat{u}^\varepsilon_i(x', z) = u^\varepsilon_i(x', x_3), \quad i = 1, 2, \quad \hat{u}^\varepsilon_3(x', z) = \varepsilon^{-1}u^\varepsilon_3(x', x_3) \text{ and } \hat{p}^\varepsilon(x', z) = \varepsilon^2 p^\varepsilon(x', x_3).$$

Let us assume the following dependence (with respect of ε) of the data:

$$\hat{f}(x', z) = \varepsilon^2 f^\varepsilon(x', x_3),\ \hat{\mu} = \mu^\varepsilon,\ \hat{\alpha} = \varepsilon\alpha^\varepsilon,\ \ \hat{l} = \varepsilon l^\varepsilon \ \text{ and } \ \hat{k} = \varepsilon k^\varepsilon.$$

$$\widehat{G} = G^\varepsilon,\ \ \widehat{r} = \varepsilon^2 r^\varepsilon \ \text{ and } \ T^\varepsilon\left(x', x_3\right) = \hat{T}^\varepsilon\left(x', z\right).$$

Let

$$\begin{aligned}
K &= \left\{\hat{\phi} \in H^1(\Omega)^3 : \hat{\phi} = 0 \text{ on } \Gamma_L, \quad \hat{\phi}.n = 0 \text{ on } \omega \cup \ \Gamma_1\right\},\\
K_{div} &= \left\{\hat{\phi} \in K : div(\hat{\phi}) = 0\right\},\\
H^1_{\Gamma_1\cup\Gamma_L}(\Omega) &= \left\{\hat{\phi} \in H^1(\Omega) : \hat{\phi} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_L\right\},\\
V_z &= \left\{v = (v_1, v_2) \in L^2(\Omega)^2 : \frac{\partial v_i}{\partial z} \in L^2(\Omega);\ v = 0 \text{ on } \Gamma_L\right\}.
\end{aligned}$$

The norm of V_z is $\|v\|_{V_z} = \left(\sum_{i=1}^{2}\left(\|v_i\|^2_{0,\Omega} + \left\|\frac{\partial v_i}{\partial z}\right\|^2_{0,\Omega}\right)\right)^{1/2}$.

By injecting the new data and unknown factors in (2.11)–(2.12) and after multiplication by ε, we deduce

$$\begin{aligned}
&\sum_{i,j=1}^{2}\int_\Omega\left[\varepsilon^2\mu\left(\frac{\partial \hat{u}^\varepsilon_i}{\partial x_j}+\frac{\partial \hat{u}^\varepsilon_j}{\partial x_i}\right)-\hat{p}^\varepsilon\delta_{ij}\right]\frac{\partial(\hat{\phi}_i-\hat{u}^\varepsilon_i)}{\partial x_j}dx'dz+\sum_{i=1}^{2}\int_\Omega\mu\left(\frac{\partial \hat{u}^\varepsilon_i}{\partial z}+\varepsilon^2\frac{\partial \hat{u}^\varepsilon_3}{\partial x_i}\right)\frac{\partial(\hat{\phi}_i-\hat{u}^\varepsilon_i)}{\partial z}dx'dz+\\
&+\int_\Omega\left(2\mu\varepsilon^2\frac{\partial \hat{u}^\varepsilon_3}{\partial z}-\hat{p}^\varepsilon\right)\frac{\partial(\hat{\phi}_3-\hat{u}^\varepsilon_3)}{\partial z}dx'dz+\sum_{j=1}^{2}\int_\Omega\mu\varepsilon^2\left(\varepsilon^2\frac{\partial \hat{u}^\varepsilon_3}{\partial x_j}+\frac{\partial \hat{u}^\varepsilon_j}{\partial z}\right)\frac{\partial(\hat{\phi}_3-\hat{u}^\varepsilon_3)}{\partial x_j}dx'dz+\\
&+\sum_{i=1}^{2}\hat{l}\int_\omega \hat{u}^\varepsilon_i(x',h(x'))\left(\hat{\phi}_i(x',h(x'))-\hat{u}^\varepsilon_i(x',h(x'))\right)\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'+\\
&+\int_\omega \hat{l}\varepsilon^2\hat{u}^\varepsilon_3(x',h(x'))\left(\hat{\phi}_3(x',h(x'))-\hat{u}^\varepsilon_3(x',h(x'))\right)\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'+\\
&+\int_\omega\hat{k}\left(|\hat{\phi}-s|-|\hat{u}^\varepsilon-s|\right)dx'+\sqrt{2}\hat{\alpha}\int_{\Omega^\varepsilon}\left(\left|\widetilde{D}\left(\hat{\phi}\right)\right|-\left|\widetilde{D}\left(\hat{u}^\varepsilon\right)\right|\right)dx'dz \geqslant\\
&\geqslant \sum_{j=1}^{2}\int_{\Omega^\varepsilon}(\hat{f}_i,\hat{\phi}_i-\hat{u}^\varepsilon_i)dx'dz+\int_{\Omega^\varepsilon}\varepsilon(\hat{f}_3,\hat{\phi}_3-\hat{u}^\varepsilon_3)dx'dz. \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega\varepsilon^2\widehat{G}\left(\hat{T}^\varepsilon\right)\left(\nabla\hat{T}^\varepsilon\right)\left(\nabla\hat{\phi}\right)dx'dz = 2\int_\Omega\hat{\mu}\left(\hat{T}^\varepsilon\right)|D\left(\hat{u}^\varepsilon\right)|^2\hat{\phi}dx'dz+\\
&+\sqrt{2}\hat{\alpha}\int_\Omega\left|\tilde{D}\left(\hat{u}^\varepsilon\right)\right|\hat{\phi}dx'dz+\int_\Omega\widehat{r}\left(\hat{T}^\varepsilon\right)\hat{\phi}dx'dz, \quad \forall\hat{\phi}\in H^1_{\Gamma_1\cup\Gamma_L}(\Omega), \qquad (3.2)
\end{aligned}$$

where

$$\left|\widetilde{D}\left(v\right)\right| = \left[\frac{1}{4}\sum_{i,j=1}^{2}\varepsilon^2\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)^2+\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial v_i}{\partial z}+\varepsilon^2\frac{\partial v_3}{\partial x_i}\right)^2+\varepsilon^2\left(\frac{\partial v_3}{\partial z}\right)^2\right]^{\frac{1}{2}}.$$

2.1. A priori estimates on the velocity and the pressure

In this subsection, we will obtain a priori estimates for the velocity field $\hat{u}^\varepsilon$ and the pressure $\hat{p}^\varepsilon$ in the domain Ω. These estimates will be useful in proving the convergence of $\hat{u}^\varepsilon$ toward the expected function. However, it will not be enough to pass to the limit, and better estimates will be obtained in the next subsection.

Theorem 2.1. *Let the assumptions of Theorems* 2.1 *and* 2.3 *hold; then there exists a constant* C *independent of* ε *such that*

$$\varepsilon^2 \sum_{i,j=1}^{2} \|\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\|_{0,\Omega}^2 + \sum_{i=1}^{2} \|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\|_{0,\Omega}^2 + \varepsilon^2 \|\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\|_{0,\Omega}^2 + \varepsilon^4 \sum_{i=1}^{2} \|\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\|_{0,\Omega}^2 \leqslant C, \tag{3.3}$$

$$\|\hat{u}_i^\varepsilon\|_{0,\Omega} \leqslant C, \text{ for } i = 1,2, \tag{3.4}$$

$$\|\varepsilon \hat{u}_3^\varepsilon\|_{0,\Omega} \leqslant C, \tag{3.5}$$

$$\|\frac{\partial \hat{p}^\varepsilon}{\partial x_i}\|_{-1,\Omega} \leqslant C, \text{ for } i = 1,2, \tag{3.6}$$

$$\|\frac{\partial \hat{p}^\varepsilon}{\partial z}\|_{-1,\Omega} \leqslant C\varepsilon. \tag{3.7}$$

2.2. A priori estimates on the temperature

In this subsection, we look for a priori estimates on the temperature $\hat{T}^\varepsilon$, for this we need to establish the following Theorem:

Theorem 2.2. *Under the same assumptions as in Theorem 3.1, there exists a constant* C *independent of* ε *such that*

$$\sum_{i=1}^{2} \left\| \varepsilon \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \leqslant C, \tag{3.8}$$

$$\left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leqslant C. \tag{3.9}$$

Proof. We choose in (3.2), $\hat{\phi} = \hat{T}^\varepsilon$ and by Korn inequality, we find

$$\int_\Omega \varepsilon^2 \widehat{G}\left(\hat{T}^\varepsilon\right) \nabla \hat{T}^\varepsilon \nabla \hat{T}^\varepsilon dx'dz \geqslant \widehat{G}\varepsilon^2 \left\|\nabla \hat{T}^\varepsilon\right\|_{L^2(\Omega)}^2 \geqslant \widehat{G}\varepsilon^2 \sum_{i=1}^{2} \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 + \widehat{G} \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \tag{3.10}$$

Let

$$I_1 = 2\int_\Omega \hat{\mu}\left(\hat{T}^\varepsilon\right) \left|\widetilde{D}\left(\hat{u}^\varepsilon\right)\right|^2 \hat{T}^\varepsilon dx'dz,$$

$$I_2 = \sqrt{2}\hat{\alpha} \int_\Omega \left|\widetilde{D}\left(\hat{u}^\varepsilon\right)\right| \hat{T}^\varepsilon dx'dz \quad \text{and} \quad I_3 = \int_\Omega \hat{r}\left(\hat{T}^\varepsilon\right) \hat{T}^\varepsilon dx'dz.$$

By the Cauchy-Schwartz, Young inequalities and the compact injection $H^1(\Omega)$ in $L^4(\Omega)$, there exists a constant $C_1(\Omega)$ independent of ε, such that

$$|I_1| \leqslant 2\mu C_4(\Omega) \begin{bmatrix} \sum_{i,j=1}^{2} \varepsilon^2 \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{H^1(\Omega)}^2 + \sum_{i=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{H^1(\Omega)}^2 + \\ + \sum_{i=1}^{2} \varepsilon^4 \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{H^1(\Omega)}^2 + \varepsilon^2 \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{H^1(\Omega)}^2 \end{bmatrix} \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)},$$

So, using (3.3), we find: $|I_1| \leqslant 2\mu C_4(\Omega) C \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)}$.

Similarly,

$$|I_2| \leqslant \sqrt{2}\hat{\alpha} C \left\|\hat{T}^\varepsilon\right\|_{L^2(\Omega)} \quad \text{and} \quad |I_3| \leqslant \widehat{r}_{\max} \overline{h} \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}. \tag{3.11}$$

By injecting (3.11) in (3.10), it becomes

$$\widehat{G}\varepsilon^2\sum_{i=1}^{2}\left\|\frac{\partial\hat{T}^{\varepsilon}}{\partial x_i}\right\|^2_{L^2(\Omega)}+\widehat{G}\left\|\frac{\partial\hat{T}^{\varepsilon}}{\partial z}\right\|^2_{L^2(\Omega)}\leqslant\left(2\mu C_4\left(\Omega\right)C+\sqrt{2}\hat{\alpha}C+\widehat{r}_{\max}\overline{h}\right)\left\|\hat{T}^{\varepsilon}\right\|_{L^2(\Omega)}.$$

As $\left\|\hat{T}^{\varepsilon}\right\|_{L^2(\Omega)}\leqslant\overline{h}\left\|\frac{\partial\hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}$, we find:

$$\widehat{G}\varepsilon^2\sum_{i=1}^{2}\left\|\frac{\partial\hat{T}^{\varepsilon}}{\partial x_i}\right\|^2_{L^2(\Omega)}+\widehat{G}\left\|\frac{\partial\hat{T}^{\varepsilon}}{\partial z}\right\|^2_{L^2(\Omega)}\leqslant C_5\left\|\frac{\partial\hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)},\tag{3.12}$$

where: $C_5=\left(2\mu C_4\left(\Omega\right)C+\sqrt{2}\hat{\alpha}C+\widehat{r}_{\max}\overline{h}\right)\overline{h}$.

According to (3.12) we deduce that : $\left\|\frac{\partial\hat{T}^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}\leqslant C_5\widehat{G}^{-1}$.

By injecting this last estimate in (3.12), we deduce (3.8) and (3.9). □

Theorem 2.3. *Under the same assumptions as in Theorem 3.1, there exist* $u^\star=(u_1^\star,u_2^\star)\in V_z$, $T^*\in V_z$ *and* $p^\star\in L_0^2(\Omega)$ *such that:*

$$\hat{u}_i\rightharpoonup u_i^\star,\ i=1,2\quad\textit{weakly in } V_z,\tag{3.13}$$

$$\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\rightharpoonup 0,\ i,j=1,2\ \textit{weakly in } L^2(\Omega),\tag{3.14}$$

$$\varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\rightharpoonup 0,\quad\textit{weakly in } L^2(\Omega),\tag{3.15}$$

$$\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\rightharpoonup 0,\ i=1,2\ \textit{weakly in } L^2(\Omega),\tag{3.16}$$

$$\hat{p}^\varepsilon\rightharpoonup p^\star,\textit{weakly in }\ L^2(\Omega),\ p^\star\ \textit{depend only of } x'.\tag{3.17}$$

$$\varepsilon\hat{u}_3^\varepsilon\rightharpoonup 0,\quad\textit{weakly in } L^2(\Omega),\tag{3.18}$$

$$\hat{T}^\varepsilon\rightharpoonup T^*\quad\textit{weakly in } V_z\tag{3.19}$$

$$\varepsilon\frac{\partial\hat{T}^\varepsilon}{\partial x_i}\rightharpoonup 0\quad\textit{weakly in } L^2\left(\Omega\right).\tag{3.20}$$

Proof. From (3.3) and (3.4), we deduce (3.13). Also (3.14) follow from (3.3) and (3.13). As $div\left(\hat{u}^\varepsilon\right)=0$, by (3.14), we obtain (3.15). From (3.5) and (3.3), (3.16) hold. Using (3.6) and (3.7) we get (3.17). Because $div\left(\hat{u}^\varepsilon\right)=0$, by (3.5) and with a particular choice of test function, we get (3.18). Finally, the convergences (3.19) and (3.20) are deduced directly from estimates (3.8)–(3.9). □

3. Study of the limit problem

In this section, using the second equation of (2.4) on Γ_1^ε, passing all non linear terms on the right and the linear terms on the left in the variational inequalities (3.1) and (3.2). Then, we apply the $\liminf\limits_{\varepsilon\longrightarrow 0}$ on the left and the $\lim\limits_{\varepsilon\longrightarrow 0}$ on the right, using the convergence results of the Theorem 3.3, we deduce

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(T^{*}\right)\frac{\partial u_i^{\star}}{\partial z}\frac{\partial(\hat{\phi}_i-u_i^{\star})}{\partial z}dx'dz-\int_{\omega}p^{\star}(x')\left(\sum_{i=1}^{2}\hat{\phi}_i(x',h(x'))\frac{\partial h}{\partial x_i}\right)dx'-$$
$$-\int_{\Omega}p^{\star}(x')\left(\frac{\partial\hat{\phi}_1}{\partial x_1}+\frac{\partial\hat{\phi}_2}{\partial x_2}\right)dx'dz+\hat{l}\sum_{i=1}^{2}\int_{\omega}u_i^{\star}(x',h(x'))\left[(\hat{\phi}_i-u_i^{\star})(x',h(x'))\right]dx'+$$
$$+\hat{\alpha}\int_{\Omega}\left(\left|\frac{\partial\hat{\phi}}{\partial z}\right|-\left|\frac{\partial u^{\star}}{\partial z}\right|\right)dx'dz+\hat{k}\int_{\omega}(|\hat{\phi}|-|u^{\star}|)dx'\geqslant\sum_{j=1}^{2}(\hat{f},\hat{\phi}-u^{\star}),\quad\forall\hat{\phi}\in\Pi\left(K\right),\quad(4.1)$$

$$-\frac{\partial}{\partial z}\left(\widehat{G}\frac{\partial T^{*}}{\partial z}\right)=\sum_{i=1}^{2}\hat{\mu}\left(T^{*}\right)\left(\frac{\partial u_i^{*}}{\partial z}\right)^{2}+\sqrt{2}\hat{\alpha}\left|\frac{\partial u^{*}}{\partial z}\right|+\hat{r}\left(T^{*}\right),\quad in\ L^{2}\left(\Omega\right).\quad(4.2)$$

Moreover if

$$\int_{\Omega}\left(\hat{\phi}_1(x',z))\frac{\partial\theta}{\partial x_1}\left(x'\right)+\hat{\phi}_2(x',z))\frac{\partial\theta}{\partial x_2}\left(x'\right)\right)dx'dz=0,\quad\forall\theta\in C_0^{1}\left(\omega\right),\quad(4.3)$$

then

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(T^{*}\right)\frac{\partial u_i^{\star}}{\partial z}\frac{\partial(\hat{\phi}_i-u_i^{\star})}{\partial z}dx'dz+\hat{l}\sum_{i=1}^{2}\int_{\omega}u_i^{\star}(x',h(x'))\left[\hat{\phi}_i(x',h(x')-u_i^{\star}(x',h(x')\right]dx'+$$
$$+\hat{\alpha}\int_{\Omega}\left(\left|\frac{\partial\hat{\phi}}{\partial z}\right|-\left|\frac{\partial u^{\star}}{\partial z}\right|\right)dx'dz+\int_{\omega}\hat{k}(|\hat{\phi}|-|u^{\star}|)dx'\geqslant\sum_{j=1}^{2}(\hat{f},\hat{\phi}-u^{\star}),\quad(4.4)$$

where $\Pi\left(K\right)=\left\{\overline{\phi}=\left(\hat{\phi}_1,\hat{\phi}_2\right)\in H^{1}(\Omega)^{2}:\exists\hat{\phi}_3\text{ such that }\phi=\left(\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3\right)\in K\right\}$.

Theorem 3.1. *The variational inequality (4.4) is equivalent the following system*

$$\int_{\Omega}\hat{\mu}\left(T^{*}\right)\left|\frac{\partial u^{\star}}{\partial z}\right|^{2}dx'dz+\hat{l}\int_{\omega}\left|u^{\star}(x',h(x')\right|^{2}dx'+\int_{\omega}\hat{k}|u^{\star}|dx'+$$
$$+\hat{\alpha}\int_{\Omega}\left|\frac{\partial u^{\star}}{\partial z}\right|dx'dz-\int_{\Omega}\hat{f}u^{\star}dx'dz=0\quad(4.5)$$

and

$$\int_{\Omega}\hat{\mu}\left(T^{*}\right)\frac{\partial u^{\star}}{\partial z}\frac{\partial\hat{\phi}}{\partial z}dx'dz+\hat{l}\int_{\omega}\left(u^{\star}\ \hat{\phi}\right)(x',h(x')dx'+\int_{\omega}\hat{k}|\hat{\phi}|dx'+\hat{\alpha}\int_{\Omega}\left|\frac{\partial\hat{\phi}}{\partial z}\right|dx'dz\geqslant$$
$$\geqslant\int_{\Omega}\hat{f}\ \hat{\phi}\ dx'dz,\ \forall\hat{\phi}\in\Sigma\left(K\right),\quad(4.6)$$

where $\Sigma\left(K\right)=\left\{\bar{\phi}\in\Pi\left(K\right):\bar{\phi}\text{ satisfies condition }(4.3)\right\}$.

Proof. According to [3, *Lemma* 5.3] we can choose $\hat{\phi}=2u^{\star}$ and $\hat{\phi}=0$ respectively in (4.4), to obtain (4.5).

For (4.6), we choose $\hat{\phi}=\left(\widehat{\psi}-u^{\star}\right)$ for all $\widehat{\psi}\in\Sigma\left(K\right)$. □

Theorem 3.2. *Let us set*

$$\sigma^{\star}=-\nabla p^{\star}+\widetilde{\sigma}^{\star}\ \ and\ \ \widetilde{\sigma}^{\star}=\mu\left(T^{*}\right)\frac{\partial u^{\star}}{\partial z}+\hat{\alpha}\pi,\quad(4.7)$$

$$F\left(\hat{k}\hat{\phi},\frac{\partial\hat{\phi}}{\partial z}\right)=\int_{\Omega}\mu\left(T^{*}\right)\frac{\partial u^{\star}}{\partial z}\frac{\partial\hat{\phi}}{\partial z}dx'dz+\hat{l}\int_{\omega}\left(u^{\star}\hat{\phi}\right)\left(x',h(x')dx'-\int_{\Omega}\hat{f}\hat{\phi}\,dx'dz,\ \forall\hat{\phi}\in\Sigma\left(K\right).\right. \tag{4.8}$$

then

$$-\frac{\partial}{\partial z}\left[\mu\left(T^{*}\right)\frac{\partial u^{\star}}{\partial z}+\hat{\alpha}\frac{\partial u^{\star}/\partial z}{|\partial u^{\star}/\partial z|}\right]=\hat{f}-\nabla p^{\star}\ \ in\ L^{2}(\Omega)^{2}, \tag{4.9}$$

where π *obtained by the Hanh-Banach theorem, i.e.* $\exists\,(\chi,\pi)\in L^{\infty}(\omega)^{2}\times L^{\infty}(\Omega)^{2}$, *with* $\|\chi\|_{\omega,\infty}\leqslant 1$, $\|\pi\|_{\Omega,\infty}\leqslant 1$, *such that*

$$F\left(\hat{k}\hat{\psi},\frac{\partial\hat{\psi}}{\partial z}\right)=-\int_{\omega}\chi\hat{k}\hat{\psi}dx'-\hat{\alpha}\int_{\Omega}\pi\frac{\partial\hat{\psi}}{\partial z}dx'dz. \tag{4.10}$$

Proof. If $\dfrac{\partial u^{\star}}{\partial z}=0$, from (4.7), we get $|\widetilde{\sigma}^{\star}|\leqslant\hat{\alpha}$.
In particular, from (4.5) and (4.8), we get

$$\int_{\omega}\hat{k}|u^{\star}|dx'+\hat{\alpha}\int_{\Omega}\left|\frac{\partial u^{\star}}{\partial z}\right|dx'dz=\int_{\omega}\chi\hat{k}u^{\star}dx'+\hat{\alpha}\int_{\Omega}\pi\frac{\partial u^{\star}}{\partial z}dx'dz. \tag{4.11}$$

Also, from (4.8) and (4.10), we have

$$\begin{aligned}\int_{\Omega}\mu\left(T^{*}\right)\frac{\partial u^{\star}}{\partial z}\frac{\partial\hat{\psi}}{\partial z}dx'dz+\hat{l}\int_{\omega}\left(u^{\star}\hat{\psi}\right)\left(x',h(x')dx'+\int_{\omega}\chi\hat{k}\hat{\psi}dx'+\right.\\ +\hat{\alpha}\int_{\Omega}\pi\frac{\partial\hat{\psi}}{\partial z}dx'dz-\int_{\Omega}\hat{f}\hat{\psi}dx'dz=0,\forall\hat{\psi}\in\Sigma\left(K\right).\end{aligned} \tag{4.12}$$

Now using (4.11), we have

$$\hat{\alpha}\int_{\left|\frac{\partial u^{\star}}{\partial z}\right|\neq 0}\left(\left|\frac{\partial u^{\star}}{\partial z}\right|-\pi\frac{\partial u^{\star}}{\partial z}\right)dx'dz+\int_{\omega}\hat{k}\left(|u^{\star}|-\chi u^{\star}\right)dx'=0, \tag{4.13}$$

since $\|\chi\|_{\omega,\infty}\leqslant 1$ and $\|\pi\|_{\Omega,\infty}\leqslant 1$, we deduce

$$\left|\frac{\partial u^{\star}}{\partial z}\right|=\pi\frac{\partial u^{\star}}{\partial z}\ \text{and}\ \ |u^{\star}|=\chi u^{\star}. \tag{4.14}$$

Hence, if $\left|\dfrac{\partial u^{\star}}{\partial z}\right|\neq 0$ by (4.7), we obtain

$$\widetilde{\sigma}^{\star}=\mu\left(T^{*}\right)\frac{\partial u^{\star}}{\partial z}+\hat{\alpha}\frac{\partial u^{\star}/\partial z}{|\partial u^{\star}/\partial z|}. \tag{4.15}$$

In this case $|\widetilde{\sigma}^{\star}|=\left(\mu\left(T^{*}\right)+\dfrac{\hat{\alpha}}{|\partial u^{\star}/\partial z|}\right)\left|\dfrac{\partial u^{\star}}{\partial z}\right|=\mu\left(T^{*}\right)|\partial u^{\star}/\partial z|+\hat{\alpha}>\hat{\alpha}$.
Therefore, we can write

$$\mu\left(T^{*}\right)\frac{\partial u^{\star}}{\partial z}=\begin{cases}0 \quad \text{if } |\widetilde{\sigma}^{\star}|\leqslant\hat{\alpha},\\ \widetilde{\sigma}^{\star}-\hat{\alpha}\dfrac{\partial u^{\star}/\partial z}{|\partial u^{\star}/\partial z|}\quad\text{if}\ \ |\widetilde{\sigma}^{\star}|>\hat{\alpha},\end{cases}$$

which is a lower-dimensional Bingham law.

Besides, from (4.12) there exists $p^\star \in L^2(\Omega)^2$ (see [5, 13]) such that

$$\int_\Omega \mu\left(T^*\right) \frac{\partial u^\star}{\partial z}\frac{\partial \hat{\psi}}{\partial z} dx'dz + \hat{l}\int_\omega \left(u^\star \hat{\psi}\right)(x', h(x')dx' + \int_\omega \underline{n}\hat{k}\hat{\psi}dx' + \\ + \hat{\alpha}\int_\Omega \underline{m}\frac{\partial \hat{\psi}}{\partial z}dx'dz - \int_\Omega \hat{f}\hat{\psi}dx'dz = -\int_\Omega \nabla p^\star \hat{\psi}dx'dz, \quad \forall \hat{\psi} \in \Pi(K). \quad (4.16)$$

Using (4.15)–(4.16) becomes

$$\int_\Omega \mu\left(T^*\right) \widetilde{\sigma}^\star \frac{\partial \hat{\psi}}{\partial z} dx'dz + \hat{l}\int_\omega \left(u^\star \hat{\psi}\right)(x', h(x')dx' + \int_\omega \underline{n}\hat{k}\hat{\psi}dx' = \\ = \int_\Omega \hat{f}\hat{\psi}dx'dz - \int_\Omega \nabla p^\star \hat{\psi}dx'dz, \ \forall \hat{\psi} \in \Pi(K), \quad (4.17)$$

from which (4.9) follows if we take in (4.17) $\hat{\psi} \in H_0^1(\Omega)^2$. □

Theorem 3.3. *Under the assumptions of preceding theorems, the traces $s^\star, \tau^\star$ satisfy the following inequality*

$$\int_\omega \left(\frac{h^3}{12}\nabla p^* + \tilde{F}(x') + \int_0^h \int_0^y \mu\left(T^*(x',\xi)\right)\frac{\partial u^*(x',\xi)}{\partial \xi}\partial\xi dy + \right. \\ \left. + \hat{\alpha}\int_0^h \int_0^y \frac{\partial u^*/\partial\xi}{|\partial u^*/\partial\xi|}(x',\xi)\,d\xi dy\right)\nabla\varphi(x')\,dx' - \int_\omega \frac{h}{2}\left(\int_0^h \mu\left(T^*(x',\xi)\right)\frac{\partial u^*(x',\xi)}{\partial\xi}\partial\xi - \right. \\ \left. - \frac{\hat{\alpha}h}{2}\int_0^h \frac{\partial u^*/\partial\xi}{|\partial u^*/\partial\xi|}(x',\xi)\,d\xi\right)\nabla\varphi(x')\,dx' = 0, \quad \forall\varphi \in H^1(\omega), \quad (4.18)$$

where $\widetilde{F}(x') = \int_0^h F(x',y)\,dy - \frac{h}{2}F(x',h)$, $F(x',y) = \int_0^y\int_0^\xi \widehat{f}^\varepsilon(x',t)\,dtd\xi.$

Proof. We integrate twice (4.9) between 0 and z, to obtain

$$-\hat{\mu}\left(T^*(x';z)\right)\frac{\partial u^*}{\partial z}(x';z) - \hat{\alpha}\frac{\partial u^*/\partial z}{|\partial u^*/\partial z|} + \hat{\mu}\left(\zeta^*(x')\right)\tau^*(x') + \hat{\alpha}\frac{\tau^*}{|\tau^*|} = \\ = \int_0^z \hat{f}(x',\xi)\,d\xi - z\nabla p^*, \quad (4.19)$$

where, $\tau^*(x') = \frac{\partial u^*}{\partial z}(x',0)$ and $\zeta^*(x') = T^*(x',0)$.
By integrating between 0 in z, we find:

$$-\int_0^z \hat{\mu}\left(T^*(x';\xi)\right)\frac{\partial u^*}{\partial\xi}(x';\xi)\,d\xi - \hat{\alpha}\int_0^z \frac{\partial u^*/\partial\xi}{|\partial u^*/\partial\xi|}d\xi + \\ + \hat{\mu}\left(\zeta^*(x')\right)\tau^*(x')\,z + \hat{\alpha}\frac{\tau^*}{|\tau^*|}z = \int_0^z\int_0^\xi \hat{f}(x',y)\,dyd\xi - \frac{z^2}{2}\nabla p^* \quad (4.20)$$

in particular for $z = h$ we obtain,

$$\left(\hat{\mu}\left(\zeta^*(x')\right)\tau^*(x')\,h + \hat{\alpha}\frac{\tau^*}{|\tau^*|}h\right)\frac{h}{2} = \frac{h}{2}\int_0^h \hat{\mu}\left(T^*(x';\xi)\right)\frac{\partial u^*}{\partial\xi}(x';\xi)\,\xi + \\ + \hat{\alpha}\frac{h}{2}\int_0^h \frac{\partial u^*/\partial\xi}{|\partial u^*/\partial\xi|}d\xi + \frac{h}{2}\int_0^h\int_0^\xi \hat{f}(x',y)\,dyd\xi - \frac{h^3}{4}\nabla p^* \quad (4.21)$$

integrating (4.20) between 0 and h, we obtain:

$$\left(\hat{\mu}\left(\zeta^*\left(x'\right)\right)\tau^*\left(x'\right)h+\hat{\alpha}\frac{\tau^*}{|\tau^*|}h\right)\frac{h}{2}=\int_0^h\int_0^y\hat{\mu}\left(T^*\left(x';\xi\right)\right)\frac{\partial u^*}{\partial z}\left(x';\xi\right)d\xi dy+$$
$$+\hat{\alpha}\int_0^h\int_0^y\frac{\partial u^*/\partial\xi}{|\partial u^*/\partial\xi|}d\xi dy+\int_0^h\int_0^y\int_0^\xi\hat{f}\left(x',t\right)dtd\xi dy-\frac{h^3}{6}\nabla p^*. \quad (4.22)$$

Substituting (4.21) into (4.22) and for all $\varphi\in H^1(\omega)$ we deduce (4.18). □

For the uniqueness of the limit velocity and temperature, we put:

$$W_z=\left\{u\in V_z:\frac{\partial^2u}{\partial z^2}\in L^2\left(\Omega\right)\right\},$$
$$B_c=\left\{u\in W_z\times W_z:\left\|\frac{\partial u}{\partial z}\right\|_{V_z}\leqslant c\right\},$$
$$\tilde{W}_z=\left\{u\in W_z\times W_z:u\text{ satisfies condition }(4.3)\right\}.$$

Theorem 3.4. *The solution (u^*,T^*) of the limit problem (4.2) and (4.5)–(4.6) is unique in $\left(\tilde{W}_z\cap B_c\right)\times W_z$, for all*

$$0<c<c_0=\left(2C_{\hat{\mu}}\beta^4\right)^{-\frac{1}{2}}\left[\underline{G}\left[1+\left(\overline{h}\right)^2\right]^{-1}-C_{\hat{r}}\right]^{\frac{1}{2}},$$

where $\beta>0$, $C_{\hat{\mu}}>0$, $C_{\hat{r}}>0$ and $\underline{G}$ are determined in the proof.

Proof. For the proof of this theorem, we follow the same steps as in [2]. Let $\left(u^{\star,1},T^{*,1}\right)$, $\left(u^{\star,2},T^{*,2}\right)$ be two solutions of (4.2) and (4.5)–(4.6).

$$\sum_{i=1}^2\int_\Omega\hat{\mu}\left(T^{1,*}\right)\frac{\partial u_i^{*,1}}{\partial z}\frac{\partial(\hat{\varphi}_i-u_i^{*,1})}{\partial z}dx'dz+$$
$$+\sum_{i=1}^2\hat{l}\int_\omega u_i^{*,1}(x',h(x'))\left(\hat{\varphi}_i(x',h(x'))-u_i^{*,1}(x',h(x'))\right)dx'+\hat{\alpha}\int_\Omega\left(|D_z\left(\hat{\varphi}\right)|-\left|D_z\left(u^{*,1}\right)\right|\right)dx'dz+$$
$$+\int_\omega\hat{k}\left(|\hat{\varphi}-s|-|u^{*,1}-s|\right)dx'\geqslant\sum_{i=1}^2\int_{\Omega^\varepsilon}(\hat{f}_i,\hat{\varphi}_i-u_i^{*,1})dx'dz \quad (4.23)$$

$$\sum_{i=1}^2\int_\Omega\hat{\mu}\left(T^{*,2}\right)\frac{\partial u_i^{*,2}}{\partial z}\frac{\partial(\hat{\varphi}_i-u_i^{*,2})}{\partial z}dx'dz+$$
$$+\sum_{i=1}^2\hat{l}\int_\omega u_i^{*,2}(x',h(x'))\left(\hat{\varphi}_i(x',h(x'))-u_i^{*,2}(x',h(x'))\right)dx'+\hat{\alpha}\int_\Omega\left(D_z\left(\hat{\varphi}\right)-D_z\left(u^{*,2}\right)\right)dx'dz+$$
$$+\int_\omega\hat{k}\left(|\hat{\varphi}-s|-|u^{*,2}-s|\right)dx'\geqslant\sum_{i=1}^2\int_{\Omega^\varepsilon}(\hat{f}_i,\hat{\varphi}_i-u_i^{*,2})dx'dz \quad (4.24)$$

where $D_z\left(\hat{\varphi}\right)=\left(\sum_{i=1}^2\left(\frac{\partial\hat{\varphi}_i}{\partial z}\right)^2\right)^{\frac{1}{2}}$. Let us put $\hat{\varphi}=u^{*,2}$ in (4.23) and $\hat{\varphi}=u^{*,1}$ in (4.24), then adding two new equations, we find

$$\sum_{i=1}^2\int_\Omega\left(\hat{\mu}\left(T^{*,1}\right)\frac{\partial u_i^{*,1}}{\partial z}\frac{\partial\left(u_i^{*,2}-u_i^{*,1}\right)}{\partial z}+\hat{\mu}\left(T^{*,2}\right)\frac{\partial u_i^{*,2}}{\partial z}\frac{\partial\left(u_i^{*,1}-u_i^{*,2}\right)}{\partial z}\right)dx'dz\geqslant\hat{l}\sum_{i=1}^2\left\|u_i^1-u_i^2\right\|_{L^2(\omega)}^2.$$

As

$$\sum_{i=1}^{2}\int_{\Omega}\left(\hat{\mu}\left(T^{*,1}\right)\frac{\partial u_i^{*,1}}{\partial z}\frac{\partial(u_i^{*,2}-u_i^{*,1})}{\partial z}+\hat{\mu}\left(T^{*,2}\right)\frac{\partial u_i^{*,2}}{\partial z}\frac{\partial(u_i^{*,1}-u_i^{*,2})}{\partial z}\right)dx'dz =$$
$$= -\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(T^{*,1}\right)\left|\frac{\partial}{\partial z}(u_i^{*,2}-u_i^{*,1})\right|^2 dx'dz+\sum_{i=1}^{2}\int_{\Omega}\left(\hat{\mu}\left(T^{*,1}\right)-\hat{\mu}(T^{*,2})\right)\frac{\partial u_i^{*,2}}{\partial z}\frac{\partial(u_i^{*,1}-u_i^{*,2})}{\partial z}dx'dz.$$

then

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(T^{*,1}\right)\left|\frac{\partial}{\partial z}(u_i^{*,2}-u_i^{*,1})\right|^2 dx'dz \leqslant$$
$$\leqslant \sum_{i=1}^{2}\int_{\Omega}\left[\hat{\mu}\left(T^{*,1}\right)-\hat{\mu}\left(T^{*,2}\right)\right]\frac{\partial u_i^{*,2}}{\partial z}\frac{\partial(u_i^{*,1}-u_i^{*,2})}{\partial z}dx'dz. \quad (4.25)$$

As $\hat{\mu} \geqslant \underline{\mu} > 0$ and using Poincaré's inequality, we have

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(T^{*,1}\right)\left|\frac{\partial}{\partial z}(u_i^{*,2}-u_i^{*,1})\right|^2 dx'dz \geqslant \mu_*\left[1+\left(\overline{h}\right)^2\right]^{-1}\left\|u^{*,2}-u^{*,1}\right\|_{V_z}^2. \quad (4.26)$$

Now, the analogous results of [2], is given by

$$\left|\sum_{i=1}^{2}\int_{\Omega}\left(\hat{\mu}\left(T^{*,1}\right)-\hat{\mu}(T^{*,2})\right)\frac{\partial u_i^{*,2}}{\partial z}\frac{\partial(u_i^{*,1}-u_i^{*,2})}{\partial z}dx'dz\right| \leqslant \sqrt{2}\beta^2 C_{\hat{\mu}}c\left\|T^{*,1}-T^{*,2}\right\|_{V_z}\left\|u^{*,2}-u^{*,1}\right\|_{V_z},$$

where, $\beta > 0$, $C_{\hat{\mu}} > 0$ and $c > 0$ are respectively deduced from, the embedding of V_z in $L^4(\Omega)$, the assumption $\hat{\mu}$ is $C_{\hat{\mu}}$-Lipschitz continuous function on $\mathbb{R}$, and $u^{*,i} \in B_c$. Therefore

$$\left\|u^{*,2}-u^{*,1}\right\|_{V_z \times V_z} \leqslant \sqrt{2}\beta^2 C_{\hat{\mu}}\mu_*^{-1}\left[1+\left(\overline{h}\right)^2\right]c\left\|T^{*,1}-T^{*,2}\right\|_{V_z}. \quad (4.27)$$

On the other hand,

$$\int_{\Omega}\widehat{G}\frac{\partial T^{*,1}}{\partial z}\frac{\partial\hat{\psi}}{\partial z}dx'dz =$$
$$= \sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(T^{*,1}\right)\left(\frac{\partial u_i^{*,1}}{\partial z}\right)^2\hat{\psi}dx'dz+\sqrt{2}\hat{\alpha}\int_{\Omega}\left|D_z\left(u^{*,1}\right)\right|\hat{\psi}dx'dz+\int_{\Omega}\hat{r}\left(T^{*,1}\right)\hat{\psi}dx'dz \quad (4.28)$$

$$\int_{\Omega}\widehat{G}\frac{\partial T^{*,2}}{\partial z}\frac{\partial\hat{\psi}}{\partial z}dx'dz =$$
$$= \sum_{i=1}^{2}\int_{\Omega}\hat{\mu}\left(T^{*,2}\right)\left(\frac{\partial u_i^{*,2}}{\partial z}\right)^2\hat{\psi}dx'dz+\sqrt{2}\hat{\alpha}\int_{\Omega}\left|D_z\left(u^{*,2}\right)\right|\hat{\psi}dx'dz+\int_{\Omega}\hat{r}\left(T^{*,2}\right)\hat{\psi}dx'dz \quad (4.29)$$

By subtraction and choosing $\psi = \left(T^{*,1}-T^{*,2}\right) \in H^1_{\Gamma_L\cup\Gamma_1}(\Omega)$, we find

$$\int_{\Omega}\widehat{G}\left|\frac{\partial}{\partial z}\left(T^{*,1}-T^{*,2}\right)\right|^2 dx'dz = \sum_{k=1}^{4} I_k, \quad (4.30)$$

where

$$I_1 = \sum_{j=1}^{2} I_1^j,\ I_1^j = \int_\Omega \hat{\mu}\left(T^{*,1}\right) \frac{\partial}{\partial z}\left(u_i^{*,1} + u_i^{*,2}\right) \frac{\partial}{\partial z}\left(u_i^{*,1} - u_i^{*,2}\right)\left(T^{*,1} - T^{*,2}\right) dx'dz,$$
$$I_2 = \sum_{j=1}^{2} I_2^j,\quad I_2^j = \int_\Omega \left[\hat{\mu}\left(T^{*,1}\right) - \hat{\mu}\left(T^{*,2}\right)\right]\left(\frac{\partial u_i^{*,2}}{\partial z}\right)^2 \left(T^{*,1} - T^{*,2}\right) dx'dz,$$
$$I_3 = \int_\Omega \left(\hat{r}\left(T^{*,1}\right) - \hat{r}\left(T^{*,2}\right)\right)\left(T^{*,1} - T^{*,2}\right) dx'dz,$$
$$I_4 = \sqrt{2}\hat{\alpha} \int_\Omega \left(D_z\left(u^{*,1}\right) - D_z\left(u^{*,2}\right)\right)\left(T^{*,1} - T^{*,2}\right) dx'dz.$$

The increases of I_k, $k = 1, 2, 3$ are given by [2] as follows

$$I_1 = \left|\sum_{j=1}^{2} I_1^j\right| \leqslant 2\sqrt{2}\mu^* \beta^2 c \left\|u_i^{*,1} - u_i^{*,2}\right\|_{V_z \times V_z} \left\|T^{*,1} - T^{*,2}\right\|_{V_z}, \tag{4.31}$$
$$\left|I_2^i\right| \leqslant C_{\hat{\mu}} \beta^4 \left\|T^{*,1} - T^{*,2}\right\|_{V_z}^2 \left\|u_i^2\right\|_{W_z}^2, \tag{4.32}$$
$$\left|I_3\right| \leqslant C_{\hat{r}} \left\|T^{*,1} - T^{*,2}\right\|_{V_z}^2, \tag{4.33}$$

where $C_{\hat{r}} > 0$ deducted from the assumption $\hat{r}$ is $C_{\hat{r}}$-Lipschitz continuous function on $\mathbb{R}$. Using the Cauchy-Schwartz inequality, we obtain:

$$\left|I_4\right| \leqslant 2\hat{\alpha} \left\|T^{*,1} - T^{*,2}\right\|_{V_z}^2 \left\|u^{*,2} - u^{*,1}\right\|_{V_z \times V_z}. \tag{4.34}$$

Injecting (4.31)–(4.34) in (4.30), we find:

$$\underline{G}\left[1 + \left(\overline{h}\right)^2\right]^{-1} \left\|T^{*,1} - T^{*,2}\right\|_{V_z}^2 \leqslant 2\sqrt{2}\mu^* \beta^2 c \left\|u_i^{*,1} - u_i^{*,2}\right\|_{V_z \times V_z} \left\|T^{*,1} - T^{*,2}\right\|_{V_z} +$$
$$+ C_{\hat{\mu}} \beta^4 c^2 \left\|T^{*,1} - T^{*,2}\right\|_{V_z}^2 + C_{\hat{r}} \left\|T^{*,1} - T^{*,2}\right\|_{V_z}^2 + 2\hat{\alpha} \left\|T^{*,1} - T^{*,2}\right\|_{V_z}^2 \left\|u^{*,2} - u^{*,1}\right\|_{V_z \times V_z}$$

so

$$\left\|T^{*,1} - T^{*,2}\right\|_{V_z} \leqslant \left[\underline{G}\left[1 + \left(\overline{h}\right)^2\right]^{-1} - 2C_{\hat{\mu}} \beta^4 c^2 - C_{\hat{r}}\right]^{-1} \left[2\sqrt{2}\overline{\mu}\beta^2 c + 2\hat{\alpha}\right] \left\|u^{*,2} - u^{*,1}\right\|_{V_z \times V_z}.$$

where, $\underline{\hat{\mu}} < \hat{\mu} < \overline{\hat{\mu}}$, and $\underline{G} = \min \widehat{G}$. It is assumed that:

$$0 < c < c_0 = \left(2C_{\hat{\mu}} \beta^4\right)^{-\frac{1}{2}} \left[\underline{G}\left[1 + \left(\overline{h}\right)^2\right]^{-1} - C_{\hat{r}}\right]^{\frac{1}{2}},$$
$$\underline{G} > \left[1 + \left(\overline{h}\right)^2\right] C_{\hat{r}}.$$

Therefore,

$$\left\|T^{*,1} - T^{*,2}\right\|_{V_z} \leqslant \left(2\sqrt{2}\overline{\mu}\beta^2 c + 2\hat{\alpha}\right)\left(c_0^2 - c^2\right)^{-1} \left\|u^{*,2} - u^{*,1}\right\|_{V_z \times V_z}. \tag{4.35}$$

Now, Injecting (4.27) in (4.35), we obtain:

$$\left(1 - \left(2\sqrt{2}\overline{\mu}\beta^2 c + 2\hat{\alpha}\right)\left(c_0^2 - c^2\right)^{-1} \sqrt{2}\beta^2 C_{\hat{\mu}} \left(\underline{\mu}^{-1}\right)\left[1 + \left(\overline{h}\right)^2\right] c\right) \left\|T^{*,1} - T^{*,2}\right\|_{V_z} \leqslant 0$$

assuming that

$$\left(1-\left(2\sqrt{2}\overline{\mu}\beta^2 c+2\hat{\alpha}\right)\left(c_0^2-c^2\right)^{-1}\sqrt{2}\beta^2 C_{\hat{\mu}}\left(\underline{\mu}^{-1}\right)\left[1+\left(\overline{h}\right)^2\right]c\right)>0,$$

we have

$$\left\|T^{*,1}-T^{*,2}\right\|_{V_z}=0.$$

According to (4.27), we deduce

$$\left\|u^{*,2}-u^{*,1}\right\|^2_{V_z\times V_z}\leqslant 0.$$

Then $u^{*,2}=u^{*,1}$ almost everywhere in $V_z\times V_z$. This completes the proof. □

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Исследование проблемы неизотермической связи со смешанными граничными условиями в тонком домене с законом трения

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В настоящей работе рассматривается асимптотическое поведение связанной системы с несжимаемой жидкостью Бингхэма и уравнения тепловой энергии в трехмерной ограниченной области с условиями свободного трения Треска. Во-первых, мы доказываеме результаты существования и единственности для слабого решения. Во-вторых, мы показываем сильную сходимость скорости и температуры. Затем получаем конкретное предельное уравнение Рейнольдса и доказываем единственность предельной скорости и температуры.

Ключевые слова: асимптотический подход, граничные условия, сопряженная задача, закон Фурье, неизотермическая жидкость Бингама, закон Треска, уравнение Рейнольдса.