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Construction of Szegő and Poisson Kernels in Convex Domains

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In this paper, we construct Szegő and Poisson kernels in convex domains in $\mathbb{C}^n$ and study their properties.

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This paper contains some results related to the construction of Szegő and Poisson kernels in convex domains which are of significant importance for integral representations in such domains.

1. Construction the Szegő kernel

Let D be a bounded domain in $\mathbb{C}^n$ with a smooth boundary. Let $\mathcal{H}(D)$ be the space of holomorphic functions in D with the topology of uniform convergence on compact subsets of D, and $\mathcal{H}(\overline{D})$ be the space of holomorphic functions in a neighborhood of $\overline{D}$ with the corresponding topology. The space $\mathcal{H}(\overline{D})$ is the subspace in $\mathcal{L}^2(\partial D)$ with respect to the measure $d\mu$ on ∂D, where $d\mu = g(\zeta)d\sigma$, $g(\zeta) \in \mathcal{C}^1(\partial D)$, $g(\zeta) > 0$, and $d\sigma$ is the Lebesgue measure on ∂D. By the Maximum Modulus Theorem the mapping $\mathcal{H}(\overline{D}) \longrightarrow \mathcal{L}^2(\partial D)$ is injective. By $\mathcal{H}^2 = \mathcal{H}^2(\partial D)$ we denote the closure of $\mathcal{H}(\overline{D})$ in $\mathcal{L}^2$.

Consider a restriction mapping $r : \mathcal{H}(\overline{D}) \longrightarrow \mathcal{H}(D)$. The mapping r continues until continuous from $\mathcal{H}^2$ in $\mathcal{H}(D)$.

Lemma 1 (Lemma 4.1. [1])**.** *The restriction mapping $r : \mathcal{H}(\overline{D}) \longrightarrow \mathcal{H}(D)$ is continuous, if $\mathcal{H}(\overline{D})$ is considered with topology induced by the space $\mathcal{L}^2$.*

Therefore, the mapping r continues until a continuous map $i : \mathcal{H}^2 \longrightarrow \mathcal{H}(D)$. In this case, we say that for functions $f \in \mathcal{H}^2$ there is a holomorphic continuation $\tilde{f} = i(f)$ in D. Further, this continuation will be denoted by the same symbol f.

In [1] ther was considered the Lebesgue measure $d\sigma$ on the boundary of the domain, in our case for the measure $d\mu = g(\zeta)d\sigma$ the proof is similar.

Since the space $\mathcal{H}^2$ is a Hilbert separable space, there exists an orthonormal basis

$$\{\varphi_k\}_{k=1}^{\infty} \tag{1}$$

in the metric $\mathcal{L}^2$. Therefore, any function $f \in \mathcal{H}^2$ expands in a Fourier series:

$$f(\zeta) = \sum_{k=1}^{\infty} c_k \varphi_k(\zeta) \tag{2}$$

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with respect to the basis (1), which converges in the topology of $\mathcal{L}^2$, where $c_k = (f, \varphi_k) = = \int\limits_{\partial D} f(u)\bar{\varphi}_k(u)\, d\mu(u)$. Then

$$f(\zeta) = \sum_{k=1}^{\infty}\left(\int_{\partial D} f(u)\bar{\varphi}_k(u)\, d\mu(u)\varphi_k(\zeta)\right) = \int_{\partial D} f(u)\sum_{k=1}^{\infty}\bar{\varphi}_k(u)\varphi_k(\zeta)\, d\mu(u).$$

Denote $K(\zeta, \bar{u}) = \sum\limits_{k=1}^{\infty}\varphi_k(\zeta)\bar{\varphi}_k(u)$ and $K(\zeta, \bar{u}) \in \mathcal{H}(\overline{D})$ on $\zeta \in \overline{D}$ for a fixed $u \in D$.

Lemma 2. *We can choose an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ in $\mathcal{H}^2$, which consists of functions φ_k in $\mathcal{H}(\overline{D})$.*

Lemma 3. *If D is a bounded strictly convex domain with a smooth boundary, then we can choose a polynomials basis $\{\varphi_k\}_{k=1}^{\infty}$.*

Further on, we assume that the basis is chosen in accordance with Theorem 5.1 [1]. According to this theorem the continuation of the kernel $K(\zeta, \bar{u})$ has the property:

$$i(f)(z) = \int_{\partial D} f(\zeta)K(z, \bar{\zeta})\, d\mu(\zeta), \quad z \in D,$$

where $K(z, \bar{\zeta}) = \sum\limits_{k=1}^{\infty} i(\varphi_k)(z)i(\bar{\varphi}_k)(\zeta)$ and the series converges uniformly on compact subsets of $D \times D$. This kernel we call *the Szegő kernel*. Then

$$f(z) = \int_{\partial D} f(\zeta)K(z, \bar{\zeta})\, d\mu(\zeta), \tag{3}$$

where $f(z)$ is identified with $\tilde{f}(z) = i(f)(z)$ and $f \in \mathcal{H}^2$.

We define the Poisson kernel

$$P(z, \zeta) = \frac{K(z, \bar{\zeta}) \cdot K(\zeta, \bar{z})}{K(z, \bar{z})} = \frac{K(z, \bar{\zeta}) \cdot \overline{K}(z, \bar{\zeta})}{K(z, \bar{z})} = \frac{|K(z, \bar{\zeta})|^2}{K(z, \bar{z})},$$

and $K(z, \bar{z}) = \sum\limits_{k=1}^{\infty}\varphi_k(z)\bar{\varphi}_k(z) = \sum\limits_{k=1}^{\infty}|\varphi_k(z)|^2 \geqslant 0$.

Lemma 4. *The kernel $K(z, \bar{z}) > 0$ for any $z \in D$.*

Lemma 5. *A function $f \in \mathcal{H}(\overline{D})$ satisfies the integral representation*

$$f(z) = \int_{\partial D} f(\zeta)P(z, \zeta)\, d\mu(\zeta), \tag{4}$$

for $z \in D$.

Corollary 1. *If the space $\mathcal{H}(\overline{D})$ is dense in the space $\mathcal{H}(D) \cap \mathcal{C}(\partial D) = \mathcal{A}(D)$, then a function $f \in \mathcal{A}(D)$ satisfies the integral representation* (4).

Suppose that the domain D satisfies the condition (A)*: for any point $\zeta \in \partial D$ and any neighborhood $U(\zeta)$ the Szegö kernel $K(z, \bar{\zeta})$ is uniformly bounded in $z \in D$ and $z \notin U(\zeta)$*. Further on, we assume that the domain D satisfies the condition (A).

Theorem 1. *Let D be a strictly convex domain in $\mathbb{C}^n$ and the kernel $K(z, \bar{\zeta})$ satisfies the Hölder condition with exponent $\frac{1}{2} < \alpha \leqslant 1$ for $\zeta \in \partial D$ and a fixed $z \in D$. Then the domain D and the kernel $K(z, \bar{\zeta})$ satisfy the condition (A).*

Consider the restriction of the form

$$L(z,\zeta,\bar\zeta)=\frac{\sum_{k=1}^{\infty}\delta_k\,d\bar\zeta[k]\wedge d\zeta}{\left[\rho'_{\zeta_1}(\zeta_1-z_1)+\ldots+\rho'_{\zeta_n}(\zeta_n-z_n)\right]^n}$$

to ∂D, it is

$$L(z,\zeta,\bar\zeta)=$$

$$=\frac{\psi(\zeta,\bar\zeta)\,d\sigma(\zeta)}{\left[\rho'_{\zeta_1}(\zeta_1-z_1)+\ldots+\rho'_{\zeta_n}(\zeta_n-z_n)\right]^n}=\frac{\psi(\zeta,\bar\zeta)\,d\mu(\zeta)}{g(\zeta)\left[\rho'_{\zeta_1}(\zeta_1-z_1)+\ldots+\rho'_{\zeta_n}(\zeta_n-z_n)\right]^n}=$$

$$=\frac{\psi_1(\zeta,\bar\zeta)\,d\mu(\zeta)}{\left[\rho'_{\zeta_1}(\zeta_1-z_1)+\ldots+\rho'_{\zeta_n}(\zeta_n-z_n)\right]^n}=\widetilde{L}(z,\zeta,\bar\zeta)\,d\mu(\zeta).$$

The proof of Theorem 1 shows that

$$K(z,\bar\zeta)=\widetilde{L}(z,\zeta,\bar\zeta) \tag{5}$$

for $\zeta\in\partial D$.

Lemma 6. *The function $K(z,\zeta)$ is unbounded as $z\to\zeta$ and $\zeta\in\partial D$, $z\in D$.*

2. The Poisson kernel and its properties

For a function $f\in\mathcal{C}(\partial D)$ we define the Poisson integral:

$$P[f](z)=F(z)=\int_{\partial D}f(\zeta)P(z,\zeta)\,d\mu(\zeta).$$

In strictly convex domain that satisfy the condition (A), from Equality (5) and the form of the kernel $P(z,\zeta)$, it follows that this kernel is a continuous function for $z\in D$ and then the function $F(z)$ is continuous in D.

Theorem 2. *Let D be a bounded strictly convex domain in $\mathbb{C}^n$ satisfying the condition (A), and $f\in\mathcal{C}(\partial D)$, then the function $F(z)$ continuously extends onto $\overline{D}$ and $F(z)\big|_{\partial D}=f(z)$.*

Consider the differential form

$$\omega=c\sum_{k=1}^{n}(-1)^{k-1}\bar\zeta_k\,d\bar\zeta[k]\wedge d\zeta,$$

where $c=\dfrac{(n-1)!}{(2\pi i)^n}$. Find the restriction of this form to ∂D for the domain D of the form $D=\{z\in\mathbb{C}^n:\ \rho(z)<0\}$. Then by Lemma 3.5 [5], we get

$$d\bar\zeta[k]\wedge d\zeta=(-1)^{k-1}2^{n-1}i^n\frac{\partial\rho}{\partial\bar\zeta_k}\cdot\frac{d\sigma}{|\operatorname{grad}\rho|}.$$

Therefore, the restriction of ω to ∂D is equal to

$$d\mu=\omega\big|_{\partial D}=\frac{(n-1)!}{2\pi^n}\sum_{k=1}^{n}\bar\zeta_k\frac{\partial\rho}{\partial\bar\zeta_k}\cdot\frac{d\sigma}{|\operatorname{grad}\rho|}.$$

We denote

$$g(\zeta)=\frac{(n-1)!}{2\pi^n}\sum_{k=1}^{n}\bar\zeta_k\frac{\partial\rho}{\partial\bar\zeta_k}\cdot\frac{1}{|\operatorname{grad}\rho|}.$$

Proposition 1. *If D is a strictly convex circular domain, then $g(\zeta)$ is a real-valued function that does not vanish on ∂D.*

Therefore, we can assume that $g(\zeta) > 0$ on ∂D. Therefore, $d\mu = gd\sigma$ is a measure and for it all previous constructions are true.

Proposition 2. *Let D be a strictly convex $(p_1, \ldots, p_n)$-circular domain, i.e.*

$$\rho(\zeta_1, \ldots, \zeta_n) = \rho(\zeta_1 e^{ip_1\theta}, \ldots, \zeta_n e^{ip_n\theta}), \quad 0 \leqslant \theta \leqslant 2\pi,$$

where $p_1, \ldots, p_n$ are positive rational numbers. Then the function

$$\sum_{k=1}^{\infty} \bar{\zeta}_k p_k \frac{\partial \rho}{\partial \bar{\zeta}_k}$$

is real-valued and not zero.

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Построение ядер Сегё и Пуассона в выпуклых областях

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В этой статье строятся ядра Сеге и Пуассона в выпуклых областях пространства $\mathbb{C}^n$ и изучаются их свойства.

Ключевые слова: выпуклые области, ядра Сегё и Пуассона.